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Quantum Information and Computing

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Modul No.03

Lecture No.15

Measurement Postulates-II

In the last lecture we had introduced move to the concept of measurement. We talked first about general measurements in and then we said that a special case measurement operators or the projection operators so this is called projection measurement or one other measurement and today we will do two things first thing is to talk about.

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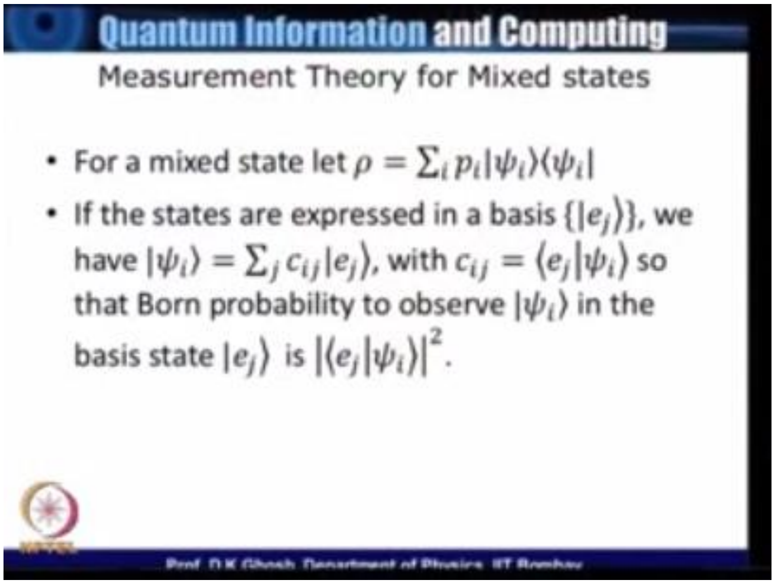

What happens if your state is mixed with how do you extend what is the corresponding measurement postulate so if you recall.

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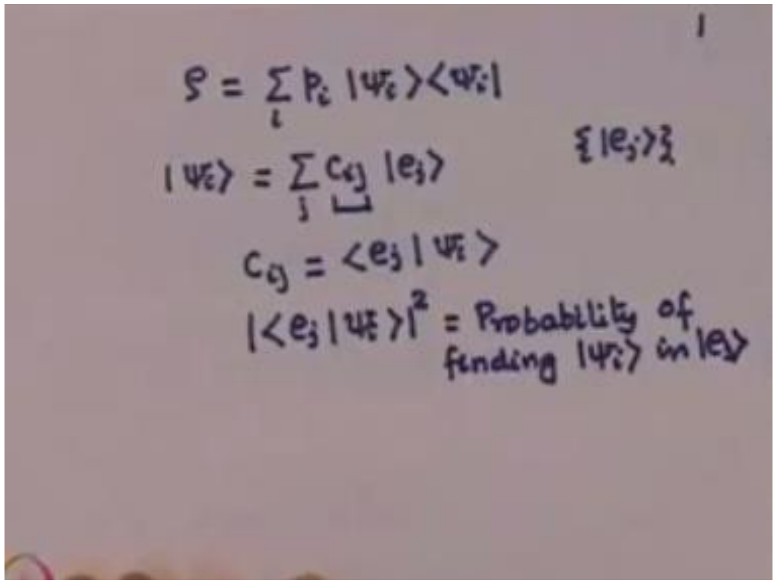

That we had defined our density matrix ϱ/ Σ_i P_i does the classical probability with which the state ψ_i is there in the mixture so ϱ/ Σ_i P_i ψ_i ψ_i and let us suppose I am still talking about expressing the state ψ_i in a basics and let us say that ψ_i is written as Σ_i c_{ij} e_j where my e_j are the basics states then by born rule my c_{ij} the coefficients there given by the product of e_j with the state ψ suppose I am observing the state ψ_i and the probability of finding the state ψ_i in the basics e_j is the given by simply absolute square of this quantity. So what we know want to do is to see how this how this.

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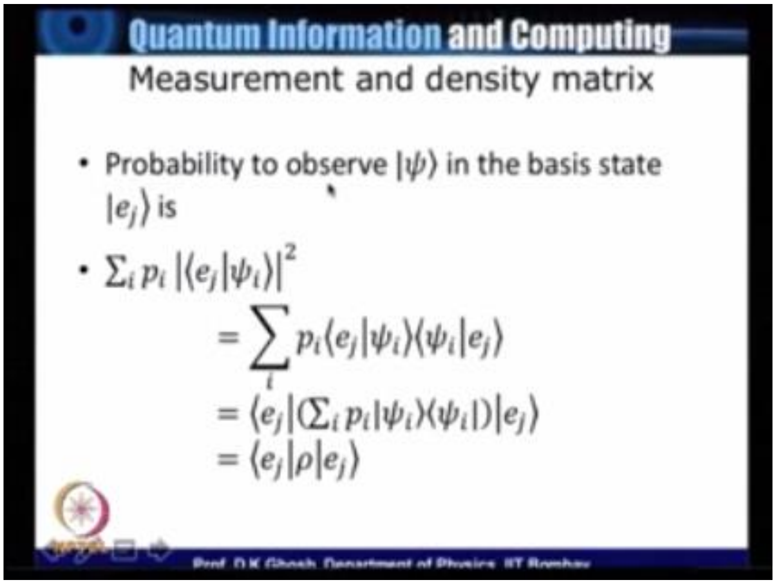

Translate into the corresponding expression for the density matrix now this is a fairly straight forward thing as a we can observe from here they from the slide Σ_i P_i $\langle e_j\ \psi_j\rangle^2$ so notice that if this is what we are observing the state ψ so the question that I am trying to answer here is I know what is the probability that the state ψ_i will appear in the basis state e_j but since my total state ψ is Σ_i P_i ψ_i supposing I am looking at this mixed state then the probability of an obituary member of this assembly in the basis state e_j is then given by first the probability of picking up ψ_i which is P_i where would the probability that a measurement of ψ_i gives me the state $e_{j.}$

so which is e_j ψ_i absolute square and if Σ that over all states i then I get the probability to observe the state ψ in the basis state e_j now this absolute square you can simply rewrite as e_j ψ_i ψ_i e_j and since these are numbers basically I rearrange them little bit by wetting this e_j $_{\Sigma i}$ this Σ_i bring it inside $\Sigma_i P_i$ Σ_i cat Σ_i bar okay and then e_j but this is my definition of ϱ so therefore the probability to observe the state ψ in the basic state e_j is skimpily given by the matrix element of the density matrix in the basics state e_j ϱ ψ e_j.

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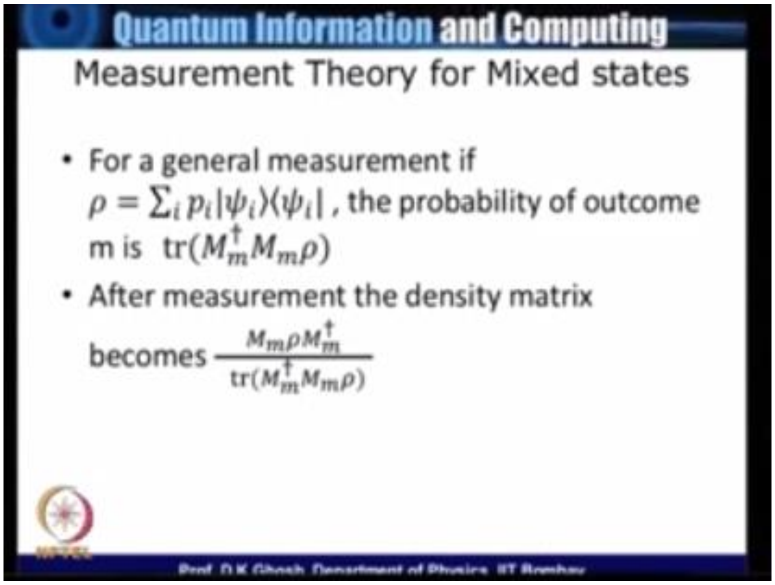

Now so the question is this that suppose I have made general measurement then we have seen what we what we want to find out what is the probability of the outcome now the according to the quantum postulate it is given by trace of $M_m{}^{\dagger}$ $M_m{}^{\varrho}$ I will tell you how and after the measurement the density matrix becomes $M_m{}^{\varrho}$ $M_m{}^{\dagger}$ / by this probability of the outcome so this is regarding density matrix remember there is no square root in the denominator of that because we are talking now about a density matrix which is a product of a cat ψ. So in this case let us look at what is the trace.

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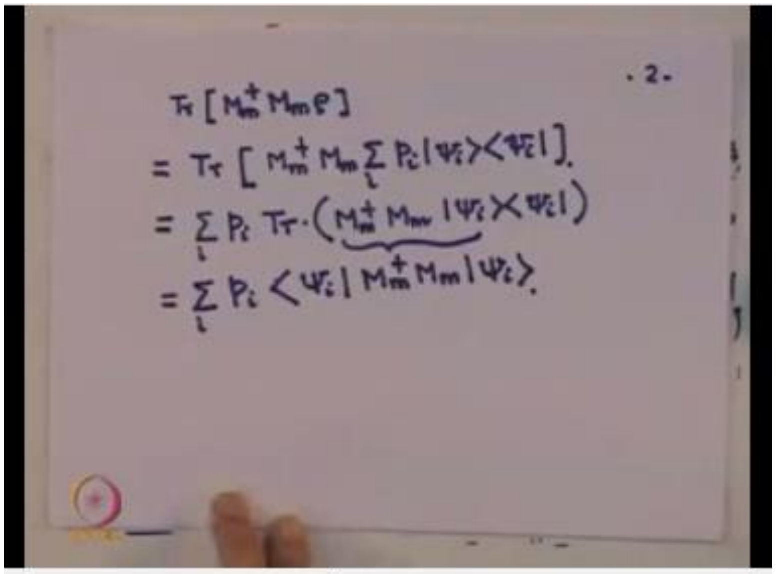

Of $M_m{}^{\dagger} M_m{}^{\varrho}$ so this quantity is let us just rewrite it trace of $M_m{}^{\dagger}$ M_m let me write ϱ in It is full form that P_i ψ_i cat ψ_i bra so I rewrite this as Σ P_i and then it is a trace of $M_m{}^{\dagger} M_m$ ψ_i ψ_i the remember the trace of A we have talked about it earlier the trace of a cat with a bra so this is a cat and operator acting on ψ_i the trace of cat with the bar is simply the scalar product of the bar with the cat so the fore this is equal to $\Sigma_i P_i$ trace of so this trace evaluates to $<\psi_i$ | $M_m{}^{+}$ M_m | acting on ψ_i again. Now this is the what the trace stands for and after measurement then the state becomes.

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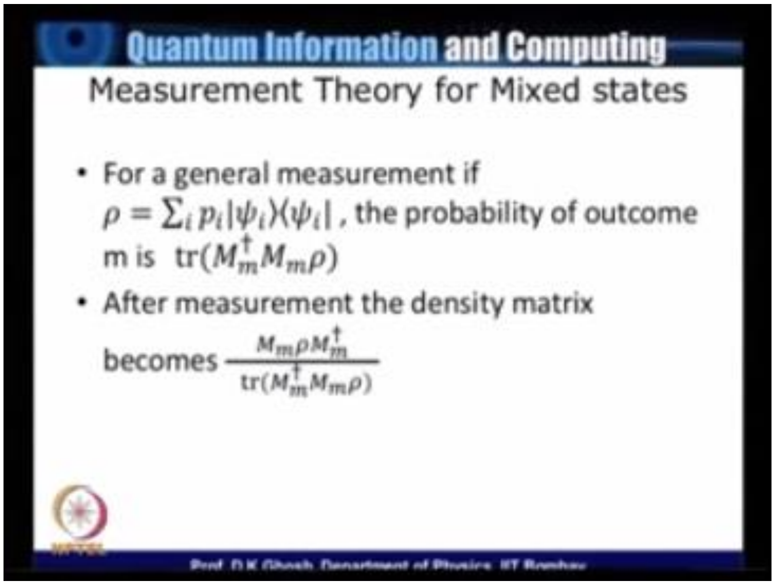

Go to the slide the after measurement that state becomes $M_m \rho M_m^+/Tr(M_m + M_m \rho)$ this is of this is the part of the postulate it takes a bit of an algebra to prove this is identical to the case where we take the definition of ρ in terms of Σ_i Pi | ψi>< ψi| and then we know what the postulate for the state is.

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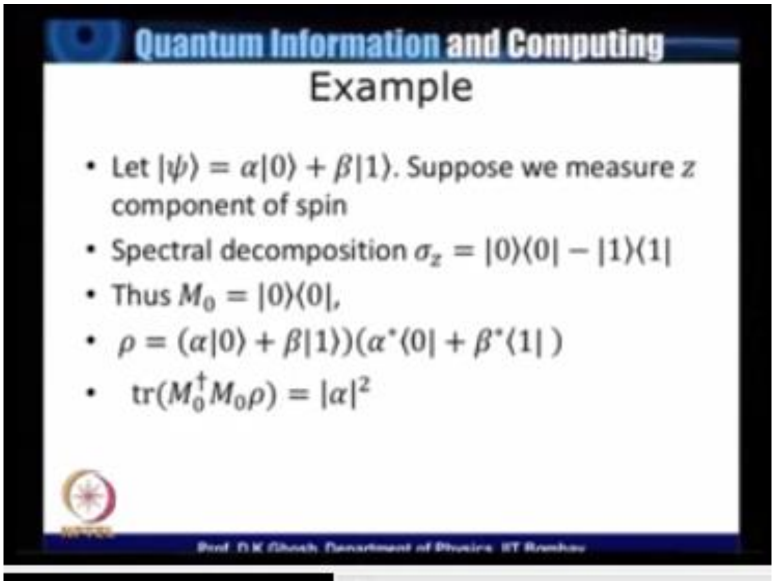

So let me give you an example on how this works; now suppose I start with a same example as before that is I take a 1 qubit state ψ.

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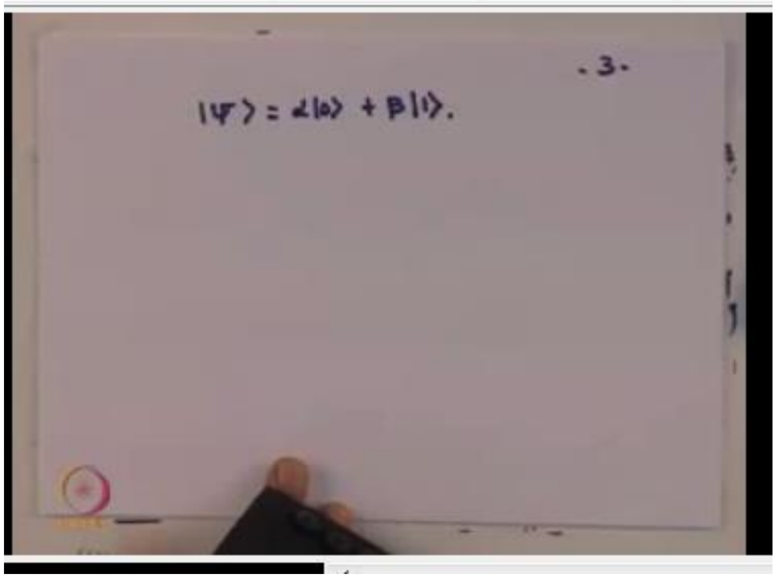

Which is equal to α|0> + β|1> and let us suppose we are measuring the z component of this thing.

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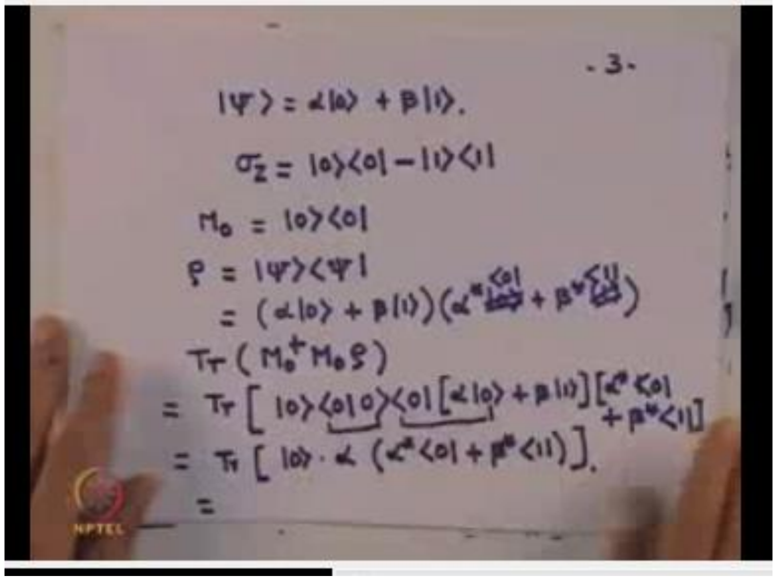

So that my corresponding operator is σz so σz operator if you recall is $|0\rangle\langle 0| - |1\rangle\langle 1|$ this is nothing but the spectral decomposition because I know σz has Eigen value either +1 or -1 so this is the state corresponding to +1, this is the operator corresponding to the Eigen value equal to -1, so therefore my M0 which corresponding to the Eigen value +1 is simply $|0\rangle\langle 0|$ what you mean by ρ**?** My ρ is $|\psi\rangle\langle\psi|$.

So this is equal to $\alpha|0\rangle + \beta|1\rangle$ and the cat will the corresponding bra's is $\alpha^{*} |0\rangle + \beta^{*} |1\rangle$ now you can easily calculate now by the previous formula that i gave you what is Tr (M0+ M0) I have written down already M0 there times **ϱ** it trivializes the bra, when Galion this is equal to $|\alpha|^2$ we can see how, so this is Tr of let me just do this algebra a little bit, so this is M0+ so I have got $|0\rangle\langle 0|$.

Again M0 so that $|0\rangle\langle 0|$ again, **ϱ** is what i have written down here in this form so let me just re-write it $[\alpha|0\rangle + \beta|1\rangle]\ [\alpha^{*} |0\rangle + \beta^{*} |1\rangle]$ is a error there so let us look at this, this is equal to 1. Now again so I have got here $\alpha|0\rangle\ \alpha^{*}$ etc, so I get this is Tr($|0\rangle$ now this $|0\rangle\langle 0|$ is 1, this $|0\rangle\langle 1|$ is 0 so therefore I am left with a α from here and then on the bra's side I got $\alpha^{*} |0\rangle + \beta^{*} |1\rangle$.

Now since I am taking a trace so I have got a cat and the bra I know taking at tracing is multiplying the corresponding with a cat so this is equal to.

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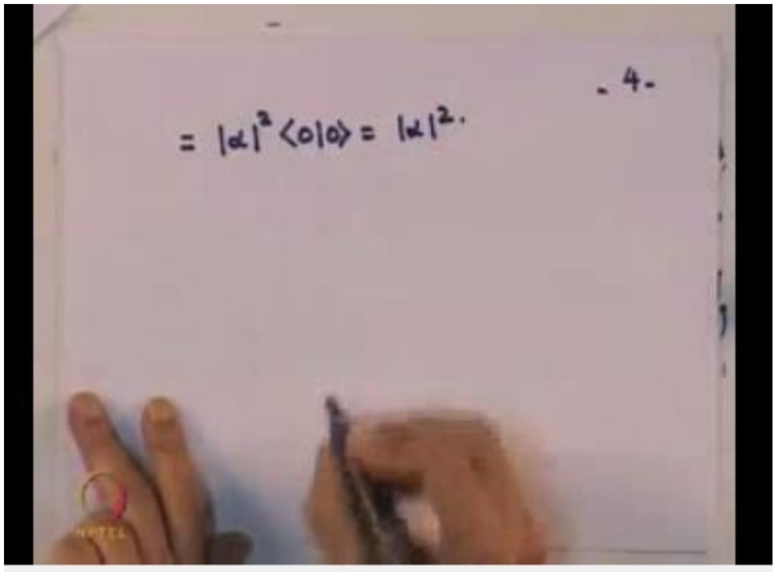

So this is equal to simply $|\alpha|^2$ times |0><0| which is equal to of course $|\alpha|^2$.

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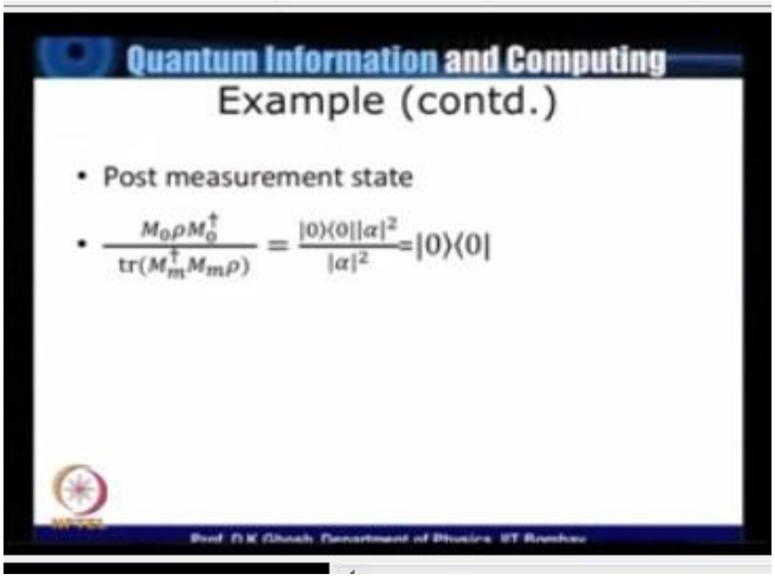

So as a result.

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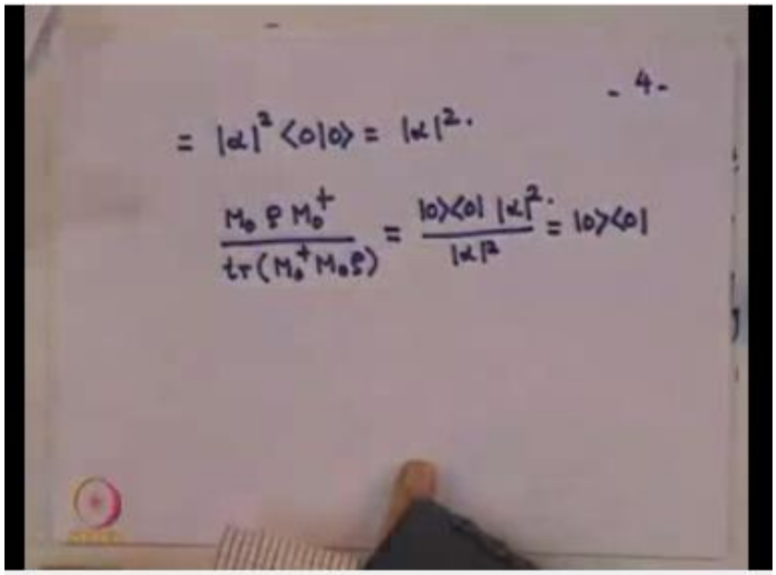

According to this my post measurement state will be M0 ρ M0+/ Tr(M0+ M0 ρ) and that is equal to |0><0| the I have got $|\alpha|^2 / |\alpha|^2$ which is nothing but |0><0| , basically the corresponding density matrix. Another interesting thing that comes out is to what happen if you repeat a density matrix? Now as I told you earlier.

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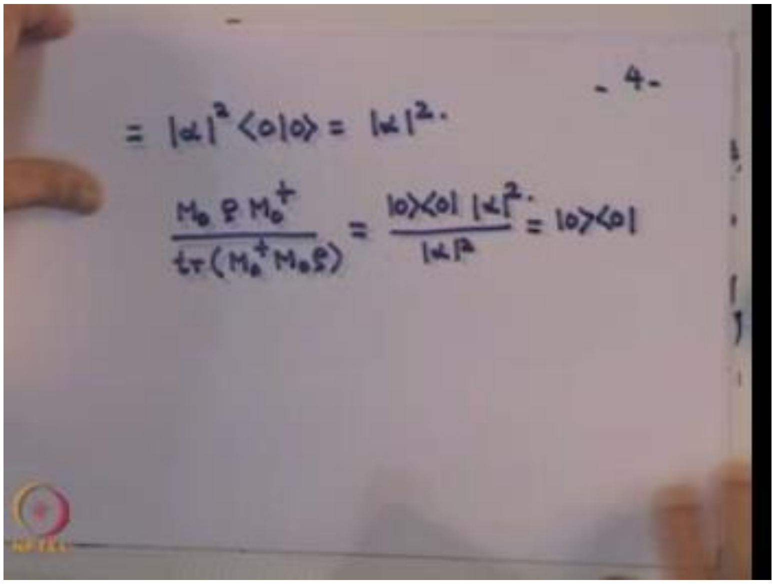

That since on making a measurement the state collapses to a particular state supposing I am doing it in bases and the state collapses to zero, now if I repeat the measurement. The state of course could remain in the same state but let us do the following, supposing I have a system.

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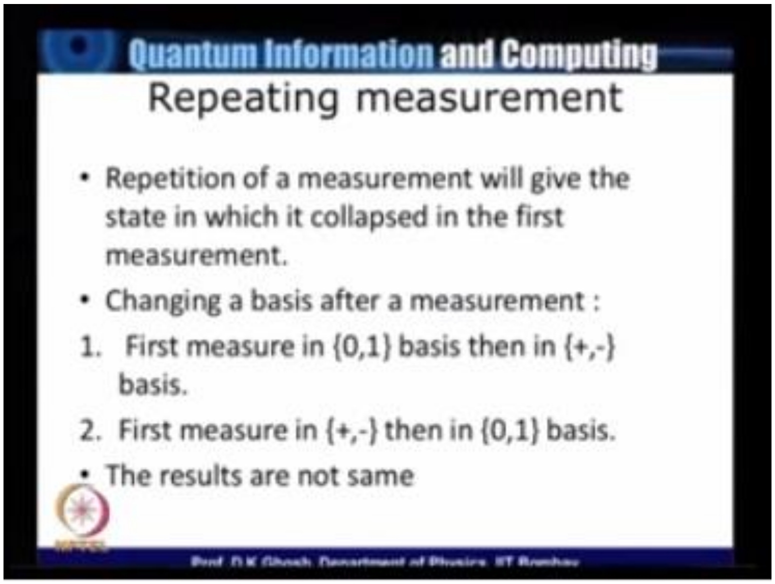

\We could start with the same α|0> + β|1> but this time I consider measuring in either the |0><1| basis which is the computation of bases or in the diagonal bases, so if you now work out what be the result that I get, if I make a measurement first in the computational bases and then in the diagonal bases and in the second cases what i do is, first make the measurement in the diagonal bases and the n make the measurement in the computational bases.

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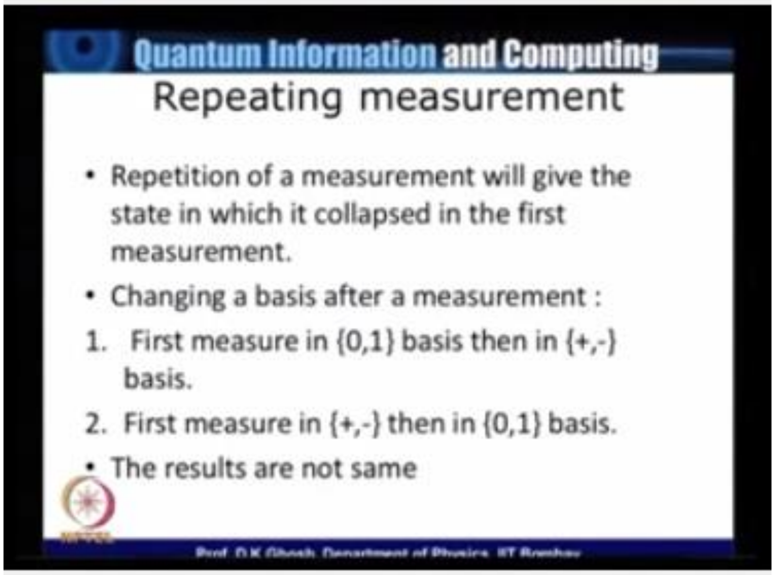

Now you can immediately see that the results are not the same, so I am doing a reputation but what I am doing is I am changing the order in which the bases chosen is alive, first computational then diagonal.

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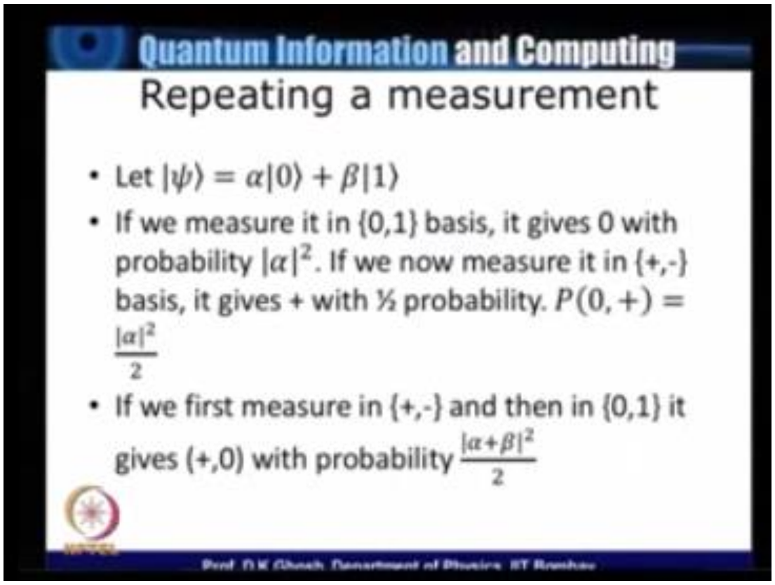

Or first diagonal then computational, so let me come back to my same old state $\psi = \alpha|0\rangle + \beta|1\rangle$.

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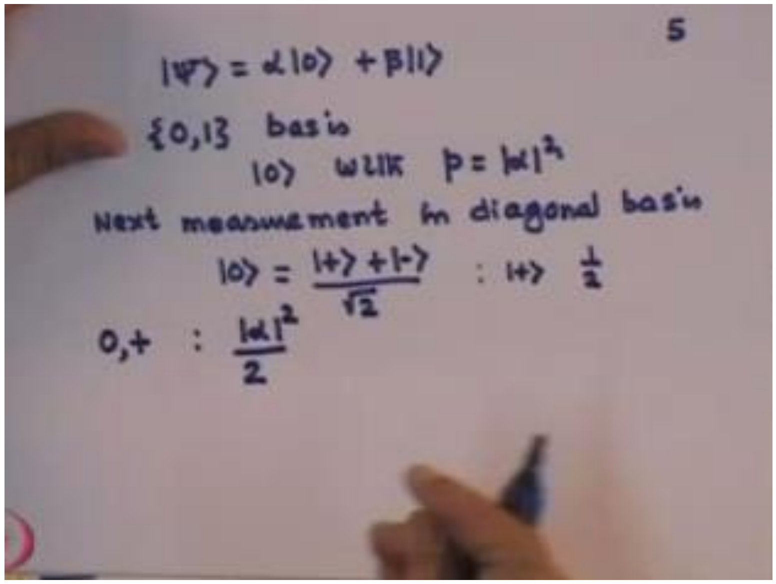

So if I come to that I am not going to repeat the calculation that I made out here supposing I first measure if {0,1} basis I know that I will get the state |0> with probability $|\alpha|^2$ and after I have got this result my state has become 0, now the question is this, that what do I get. Suppose I now make a measurement in the diagonal basis so next measurement in diagonal basis.

Remember the state |0> to which my system at collapsed after making my first measurement can be written as |+>+|->/√2, so if I now make a measurement in the diagonal basis it would give me |+> with the probability ½ and |-> with the probability 1/2. So therefore, the probability of getting 0,+ as my result is given by $|\alpha|^2/2$ and 0,- similarly is also $|\alpha|^2/2$. Now suppose instead I decided to first measure in diagonal basis. Now recall what do I do, my state is still.

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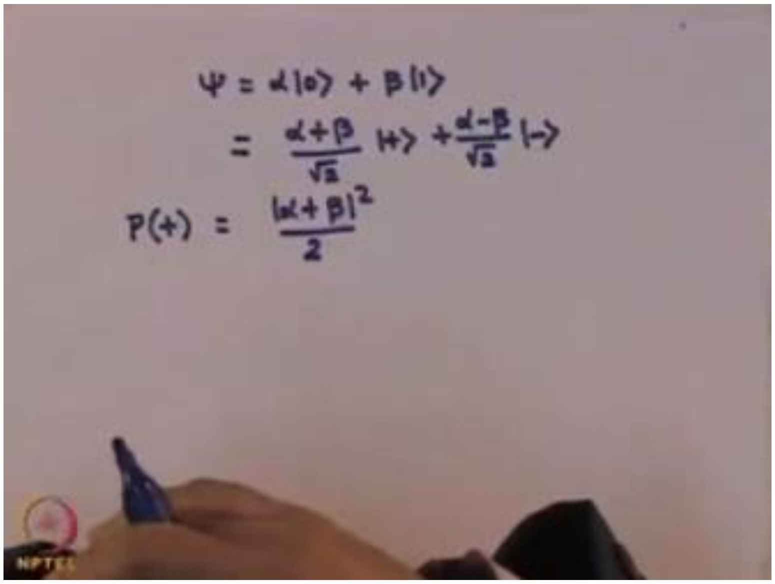

$\Psi=\alpha|0>+\beta|1>$ as I said earlier you have to first express it in the diagonal basis that gives you $\alpha+\beta/\sqrt{2}\ |+>+\ \alpha-\beta/\sqrt{2}\ |->$ since my first measurement is in the diagonal basis the probability of getting + is given by $|\alpha+\beta|^2/2$, so at that stage suppose I have got a + and I now measure the state in the computational basis. So if I measure the state in the computational basis, now my state had collapsed to the state |+ > so therefore I will re-express the state |+ > in terms of 0 and 1 and find out what result do I get.

Now obviously this probability is not the same as the probability for getting 0+, so +0 probability is not the same as the 0+ probability.

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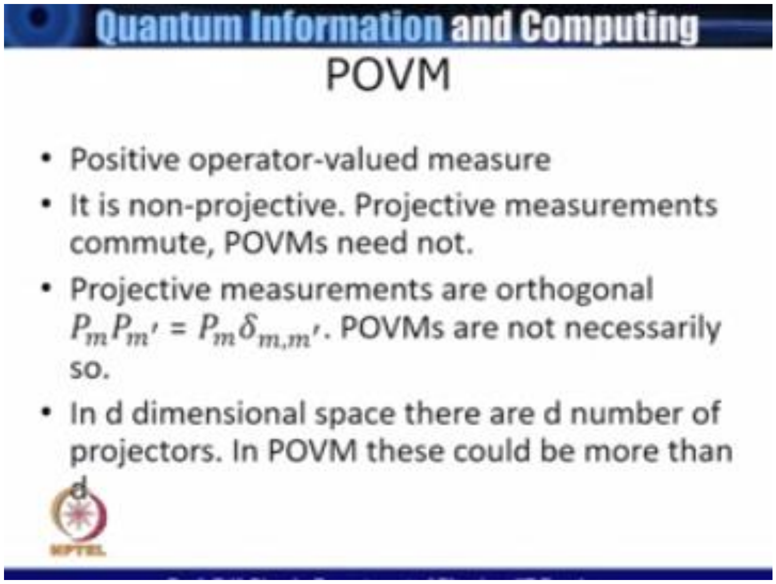

So far we have been talking about projective or vernal element, there are another special type of measurement known as POVM which is a short form for positive operator valued measure. We need not go into the Nomenclature, but this will non projective. I mean projective operators are special cases on this but these are non projective. The projective operators, projective measurement commute POVMs need not, the other thing about be projectors where they were orthogonal projectors in the sense $P_mP_{m'}$ was equal to $P_m\delta_{m,m'}$ the prime is the prolong place in the slide. But POVMs are not necessarily so.

The other thing is if I look at it d dimensional space there exactly d number of projectors. In POVMs these could be more than d. so POVM that I am talking about is a very special class of non projective measurement.

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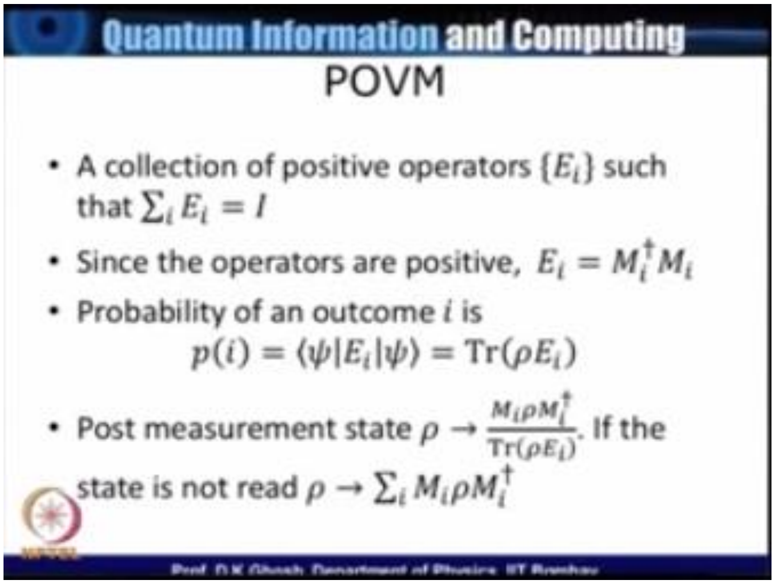

And is a general measurement, so let me define POVM. A POVM is basically a collection of positive operators.

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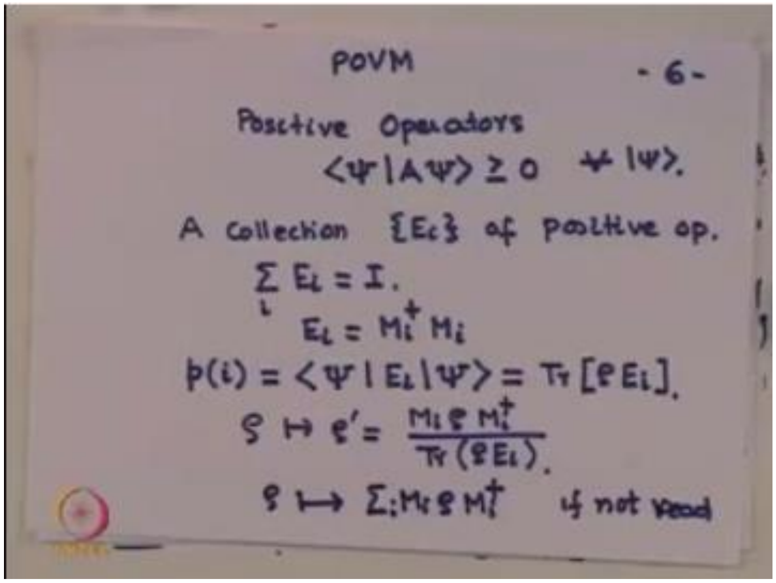

You might recall that the definition of projective operators is, an operator is a positive operator if $<\psi|A\psi> \geq 0$ for every $|\psi>$ but we will not go into this aspect of a, so first thing we say we talk about a collection of positive operator. And of course I need completeness so sum over $\sum_I E_i=I$, now since this is a positive operators we are finite them in small space you can show it that this also happens to be [indiscernible][00:19:59] and I can find out a representation of E_i as equal to $M_i^+M_i$. My postulates that tell me the probability of an outcome i is given by $<\psi|E_i|\psi>$ or if you want to talk in the language of density matrix this is simply equal to $Tr[\rho E_i]$.

The post measurement state of this is to is ρ going to some ρ' which is equal to $M_i\rho M_i^+$ divided by trace of the same thing which is the probability ρE_i. now this is what the result will be after a measurement has been made and if I or if we read that state which has come but supposing we do not read it then it remains in the linear combination states that is ρ remains us $\sum i$ M_i ρ $M_{i\,+}$ if not read remember the reading is important because for the reading does is to collapse it in to one of the possibilities so therefore this is what would happen if you do not read if you do not read the all the possibilities excess if you read one particular state concept.

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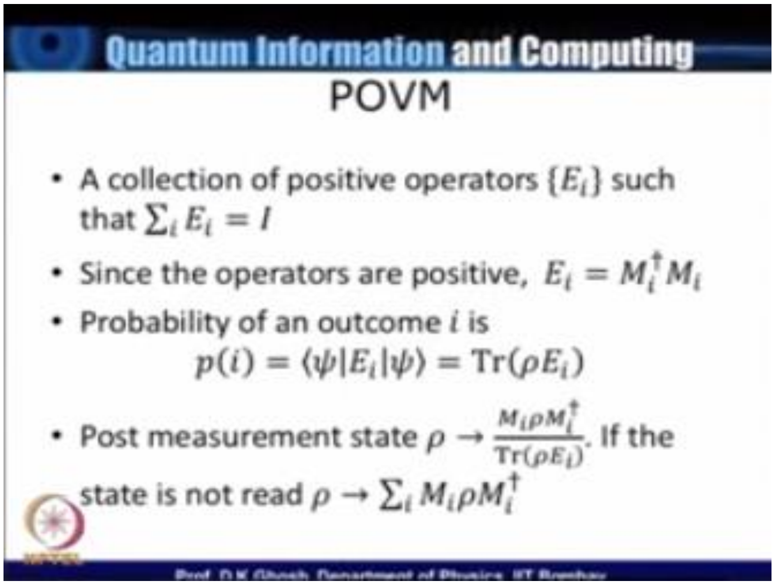

So that was quote POVM serve lent.

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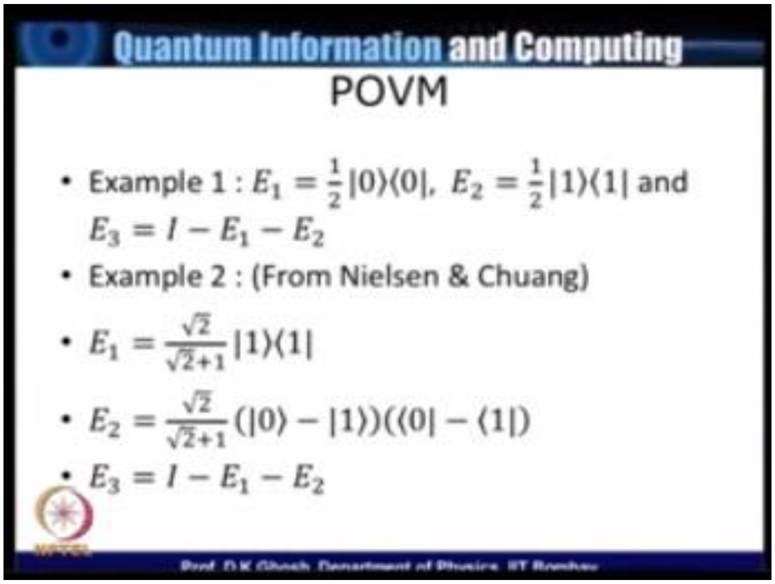

I will give you couple of examples the first one is parley straight forward actually this is also projection operators but suppose I define instead E_1 equal to not 00 but it is half of those and E_2 half of 1, 1 now since I want completeness I define E_3 Identity - E_1 - E2 you can prove that this is positive operators and the completeness is by definition of E_3 actually.

So this is an example of POVM now a very interesting example I will pick up from Nielsen & Chuang the exact construction method is not important but it will also see that POVM is are useful in trying to distinguish non ortho willing states the point that I want to make use.

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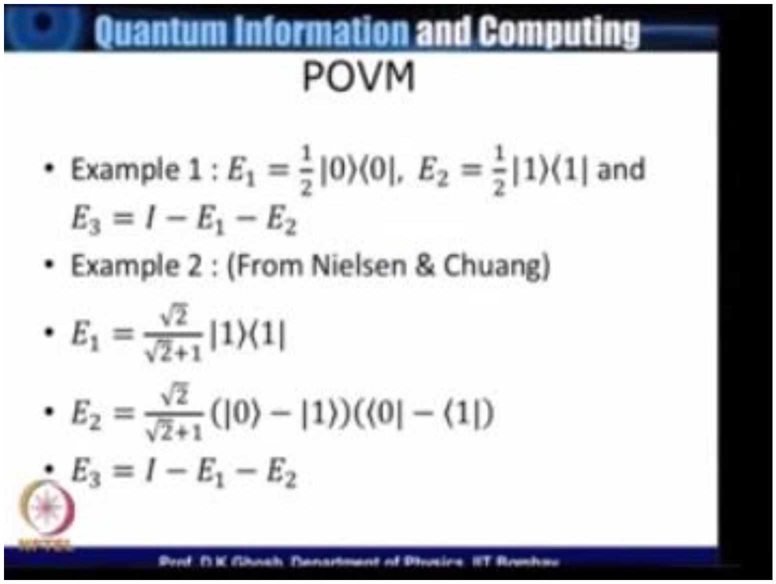

Suppose okay we will come to that question but supposing I define $E_{1=}$ √2/√2 +1, 1 1, E2 = √2/√2 +1 you can check that these are require for both normalize and hand for making them complete so this is 0 – 1 get x 0 -1 bra and E_3 in order to ensure completeness a simple equal to I- E_1- E2. So these are my three elements of Peoria of course the movement you see this definition you can immediately conclude that these are not orthogonal projectors.

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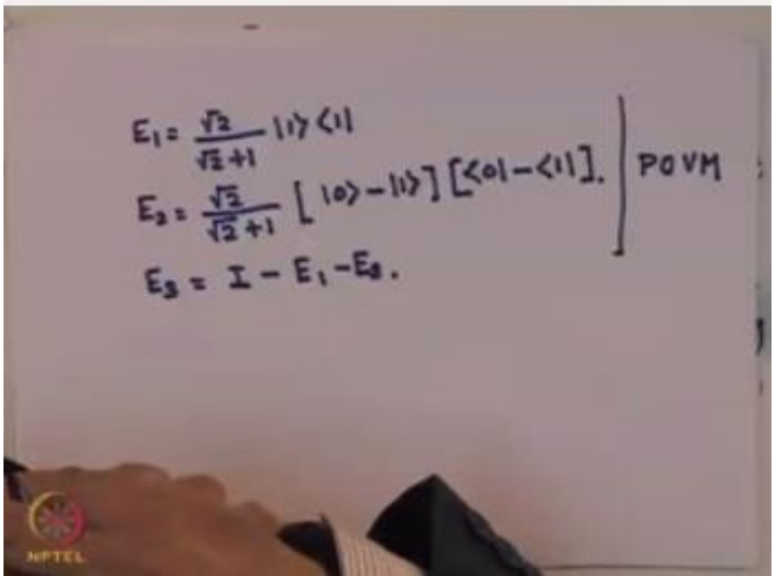

Because I am in two times in much space but have three elements now the interesting thing about this POVM is that suppose I am given two states one is a silent which is simply the state 0 that other one is state ψ2which is the state 0 + 1 /√2 now you Cannot distinguish these states by a orthogonal projector projected measurements. The reason is that 0 +1/√2 has a projection both along the state 0 and along the state 1. But let us look at what does this POVM that have written down does now look at it.

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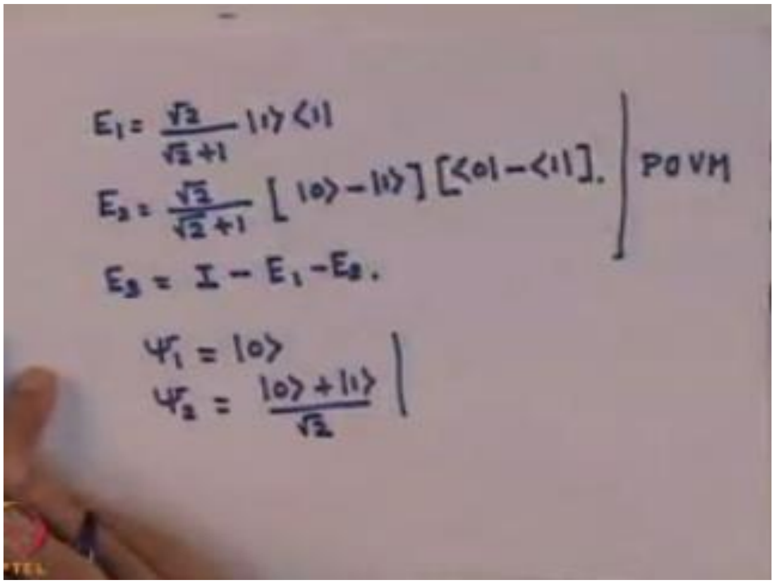

Supposing I take I get one state because I do not know which one whether it is I 1 of ψ2 and I get the result you want now if I get the result you want this state could not have been ψ 1 because it is orthogonal one now same time only two possibilities ψ 1 and ψ2 if I get the result you want the state must have been sighted. On the other hand if my measurement gives me the result E2 you notice that this is orthogonal to this operator active on ψ2 gives me 0.

So therefore the given state could not have ψ2 and since the only option is ψ1 it must have been ψ1 so I distinguish between ψ1 and ψ2 definitely in case where my measurement is E_1 or E_2 now what happens when my measurement is E_3 in that case no complement can be made the conclude on thereof is that using theory I am able to distinguish that two states not all ways because if I get a result $E_{3\ I}$ I do not do avoid that state wise that when I do distinguish them I never make a mistake.

In other words if I get E_1 I know the state is ψ2, if I get E_2 I know the state is ψ1 is I whenever I am able to distinguish I distinguish definitely occasionally I cannot compute conclusion. So this is the positive operator value pressure which is non projective measurement and we have seen it has some interesting aspectual.

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