Mechanical Behaviour of Materials - 1

Prof. Shashank Shekhar
Department of Materials Science and Engineering
Indian Institute of Technology, Kanpur

Module - 1
Lecture - 3
Stress as a Tensor

Good morning, students. So, today we will be talking about elasticity. And in this particular lecture, we will be talking more about stress and strain, their tensor characteristics and related concepts. So, last week we looked at how the, some of the elastic characteristics are related to the atomistic structure of the material. And based on that, we were able to calculate and predict what should be the elastic modulus of the prop material and also what would be another property that is directly related, the thermal coefficient or thermal expansion.

And we also saw that these two are related to the atomic bond energy and the stiffness of the bond. So, moving on, today we will be talking about elasticity, stress and strain.

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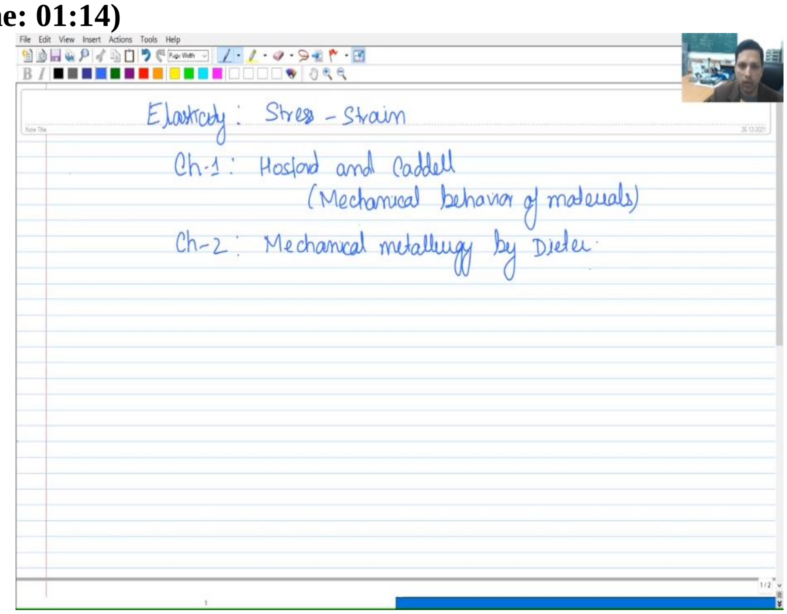

So, content wise, what we will be discussing today is, you can find in chapter 1 of Hosford and Caddell. This is the book that I have already informed you, is my go to book. Hosford and Caddell, the book name is Mechanical Behaviour of Materials. And another book that would be useful in understanding these concepts is the Mechanical Metallurgy by Dieter. So, what we need to understand is the basics of stress-strain elasticity and plasticity at the macroscopic level.

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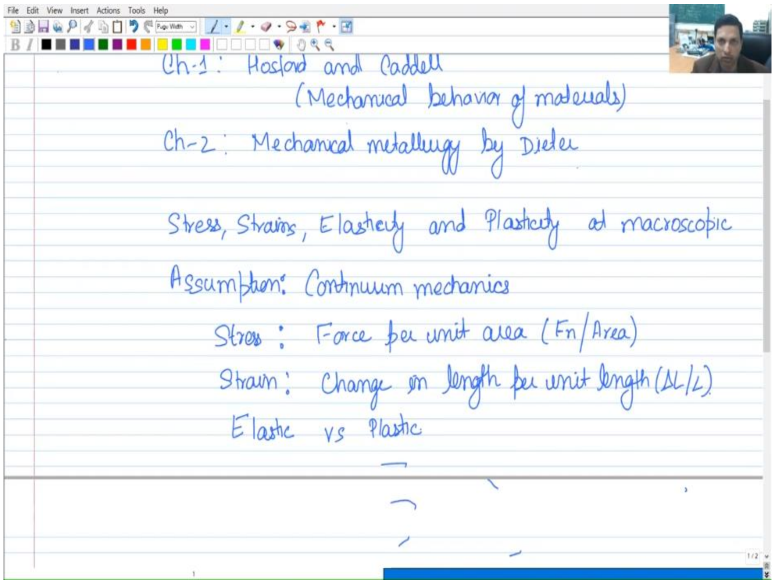

So, in this context, what we need to understand are the stress, strains; macroscopic meaning, I have gone level higher than what we have already looked at, which was the atomistic origin. So, we will now come out at a little higher or in that scale it is larger, much larger than the atomistic scale. When we are dealing with the overall bulk characteristics, you can say, and trying to understand these behaviour, elastic and plastic behaviour at the bulk level, then we need to keep few things in mind.

One is that the assumption; there is an inherent assumption when we talk about these characteristics and that is of continuum mechanics. You may have heard this term. What do we mean by continuum mechanics? What we mean is that there is no discontinuity in character. So, if we are talking about stress from one point to another point to another point to another point, then we are assuming that there will be a continuity.

It is not that at one point it will be very high, other point it will be very low. There is no discontinuous function of stresses and strains, which in turn means that when we are looking at the macroscopic model, we are assuming that there are no atoms or basically the material is distributed uniformly throughout and hence the characteristics that it represents are also uniformly distributed.

Uniformly meaning, not that the overall characteristics become homogenous, but that atoms are, or the material, the atom or the mass is not concentrated in atoms but it is distributed. And then hence you get a continuous property of the material. So, this is one very important thing to keep in mind. Now, we have already looked at the concept of stress and we said that stress although it may be very familiar concept today, but it was not so easy to understand at some stage.

But now we understand it to a good extent and for now, we will consider it as force per unit area, where Fn, I am representing to represent force acting normal to the given area, but there are shear stresses which are basically forces acting in the plane. Strain: This is change in length per unit length. Now, the strain can be defined in 1-dimension, 2-dimension or 3-dimension.

So, it is not that here for simplification we have kept delta L by L, which means it is a single dimensional, but you can also have strain in 2-dimension and 3-dimension. And another important concept to remember while we are moving ahead right now is that whenever we are applying stresses on a material, then there are 2 distinct regimes; one is the elastic regime and the other is the plastic regime.

In the elastic regime, the material regains its original shape, while in the plastic regime, there is a permanent shape change in the material. So, right now, we are talking about the elastic properties of the material. So, always keep in mind whether we are in the elastic or the plastic regime. And as you would see, as we go on that elastic regime, you have much more exact and accurate equations to define relation between stress and strains, but when we move on to the plastic regime, that equations become more vague, more complicated and they are not very accurate to put it simply.

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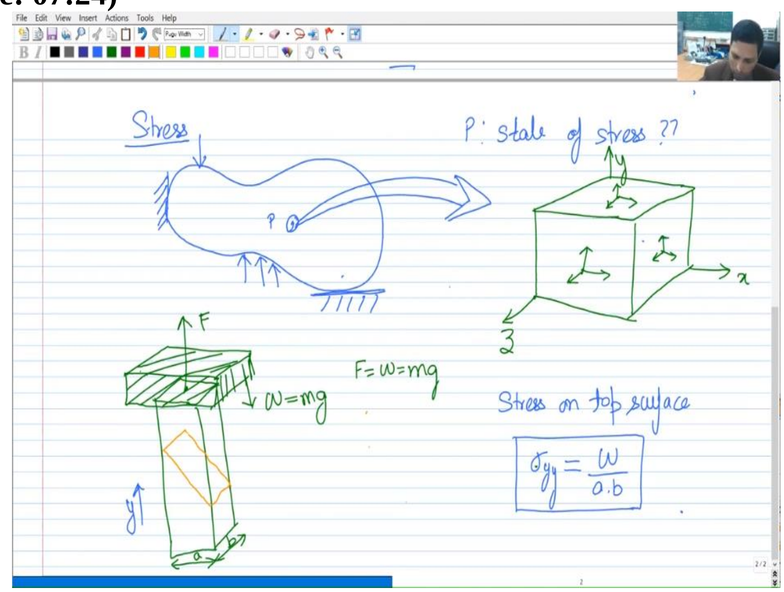

So, coming to the concept of stress. Let us say we have a free body, something like this, which is constrained at one region and then it is also constrained at another region like this, and there are some loading that has taken place over there; so, complex loading. It means, in a real life application, we know that the loading would be very complicated in nature and not a simple uniaxial loading.

So, let us say this is how the loading condition is on the material which is fixed at these two regions; and then there is a load here; and then there is a load here and it is still very simple, but even in this simple condition, we will not be able to go and uniformly define the overall stresses. So, there will be a continuously varying stresses over the whole body. So, what we do is, we take or define the stress at a particular point or what is called a state of stress at a point.

So, let us say this is the particular point P. So, this is the point P and we want to understand the state of stress. What is the state of stress at this particular point? So, in order to understand this, what we do is that, we will assume; one, we have already said, continuum mechanics; and now we will assume that this, because the material is very, you can say continuously varying characteristics, so, we can assume that we can take one cube type of structure from there.

And when we talk about the state of stress, we will be defining the state of stress for that one particular cube. So, let us say this is the point we are talking about. It is a very small cube that we have taken from this particular point and infinite number of such cubes which will fill this whole bulk to make up this component. And we can define the state of stress for each and every of these cubes. So, it is a cube, although it has not come out very accurately.

And then we can say, first that there will be these axes, so, let us say this is x-axis, this is y-axis and this is z-axis. In a general condition, there will be state of stress; so, there will be shear stress. These are the shear stress; this is normal stress. Similarly, there will be a normal stress here and there will be shear stress along these. These are the shear stresses and this is the normal stress.

So, overall, this is how the stresses in a general condition would look like. And we will come back to this, understand more about it, but for now, this explains how the state of stress is defined at different regions. So, this is one point; there can be another point and the values would keep changing at each and every of these points. In the most simplest of the cases, it may remain constant; but given the loading condition, we can say that it will keep varying.

Now, to understand it in a better way, let us look at a simple example. So, let us say you have a bar kind of shape like this, which is standing vertically and which has dimensions a and b. And let us say we have put a brick over here, whose weight is W equal to mg. Now, since this body is not deforming, so, an equal and opposite force must be acting in this direction, which is equal to F.

Therefore, F = W = mg. So, mg is the force acting onto the rectangular bar and F is the force which is equal to mg, is the force acting as a resistance from the bar. So, what is the stress, state of stress on the surface here? So, that can be given as follows. So, stress on the top surface can be given by sigma. Assume that this is y direction; so, σ_{yy}. And we will come to why it is called σ_{yy}.

σ_{yy} equal to force per unit area; force acting is

$$\sigma_{yy} = \frac{w}{ab}$$

So, this will become stress acting on material or acting on this, on the surface. And this is acting normal to the surface over here. In other cases, as you would see, it may even act on the plane. So, under what condition? We will see. So, for example, let us say, now we want to talk about, on a different plane inside the material.

So, let us say we are interested in this particular plane. So, because we are talking about state of stress and we looked at those very small cubes, and those cubes fill up the whole material, and we can define stress of state in each of these cubes, so, we can also look at one particular plane. So, over there also there are several of those cubes and we are looking at the state of stress for that particular cube in that particular plane. So, here, we will take the section which is given here, which will look; so, if we take away this section, cut this bar at this particular plane, what we will have is a section like this.

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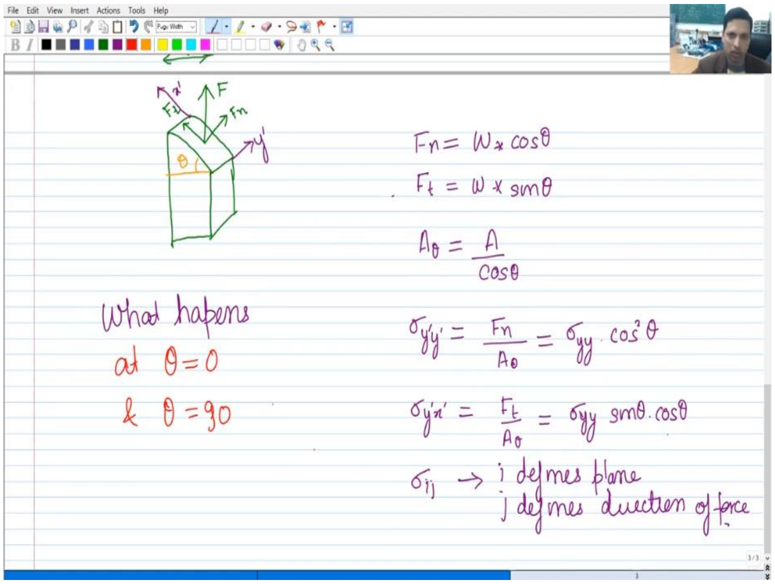

So, here you will have some force; this is the force which was acting in this particular direction. And now there will be components which will be acting; one, normal to this particular plane; so, we can call it F_n. And then, one will be in the plane, which we can call shear or tangential. So, for now, I will call it tangential. So, this is the force acting over here and this is our angle theta. So, the plane is inclined at an angle theta.

Remember that F is a vector. So, at any given angle theta, we can find what will be the force F_n and F_t. So, let us write down those. F_n which is equal to mostly W or mg is at an angle theta. So, this is also equal to angle theta. Therefore, F_n can be, since it is a vector, it can also be given as

$$F_n = W \times \cos\theta$$
$$F_t = W \times \sin\theta$$

. So, we have now resolved the force components in normal and tangential direction, but now we also need to find out the area over which it is acting.

So, what is the area over which it is acting? So, this area is the same; so, we will not define it as A_n and A_t; So,

$$A_\theta = \frac{A}{\cos\theta}$$

Now, once we have the forces and the area values, then we can calculate what will be the sigma along this normal direction, which is the stress in the normal direction and stress in the tangential direction, which is called as shear stress.

So, this is not the normal x and y; so, we will define it as y prime and this one will be defined as x prime. So, what it comes to, $\sigma_{y'y'}$; and again we will explain in few minutes, why it is called sigma y prime y prime and not just σ_y . So, this is $\sigma_{y'y'}$ and this is normal. So, the force acting is F_n and the area is A_θ. So, it becomes;

$$F_n = W \times \cos\theta$$

and

$$A_\theta = \frac{A}{\cos\theta}$$

And hence

$$\sigma_{y'y'} = \frac{F_n}{A_\theta} = \sigma_{yy}\,(\cos\theta)^2$$

$$\sigma_{y'x'} = \frac{F_t}{A_\theta} = \sigma_{yy}\sin\theta\cos\theta$$

Now, why do we define the stresses as 2 subscripts? So, when we say sigma ij, there are 2 subscripts i and j. What is their significance? i defines the plane and j defines the direction. The direction of what? So, the two terms have been defined as i defines the plane, j defines the direction. So, here, when we are seeing y prime y prime, it means we are talking about the y prime plane.

So, you can see, if this is the normal y prime direction, so, this becomes the y prime plane. And when we are talking about normal force, so, it is force is acting in the normal direction, so, it is acting perpendicular to this y prime plane. This is the y prime plane and the force is acting perpendicular to this. On the other hand, y prime x prime, is still acting on this y prime plane, but the force is in the x prime direction; so, it becomes sigma y prime x prime and that is how it is defined.

That is why we have 2 subscripts. And in fact, as you will see in the next slide you will have the possibility of 9 different components. So, now that we have an equation here, let us think about what will happen at theta equal to 0 and theta equal to 90. So, what we see is that, if you put in the equation for this

$$\sigma_{y'y'} = \sigma_{yy} \qquad at\ \theta = 0$$

What does that mean?

That means that we are looking at the original stress where the force was this original condition, where the force was acting on the surface. So, θ equal to 0 is very simple, but then, let us also put the θ equal to 0 in this equation, which is y prime x prime. So, here, sin theta term is there; so, when you put theta equal to 0, then the whole thing becomes 0, which means the tangential or the shear stress is 0, which means that in this particular condition, we had no shear stress, which also is true.

So, θ equal to 0 is very easy to understand. Now, let us put θ equal to 90. Now, when we put θ equal to 90, what happens to this first equation? cos theta A at theta equal to 90; let me just write this square term properly. So, when you put theta equal to 90 in cos theta, it becomes 0; so, this term is 0. Here also you have one cos theta term; so, here also this term will go to 0. So, what happens at theta equal to 90?

Your normal stresses as well as shear stresses are 0. So, how is that possible? Both normal and shear stresses are going to 0, what is the meaning of this? So, think about it. We will come back to it; we will answer that question in a moment. So, with this basic understanding of stress, let us define the stresses as tensile quantity. So, we will look at stress as tensors.

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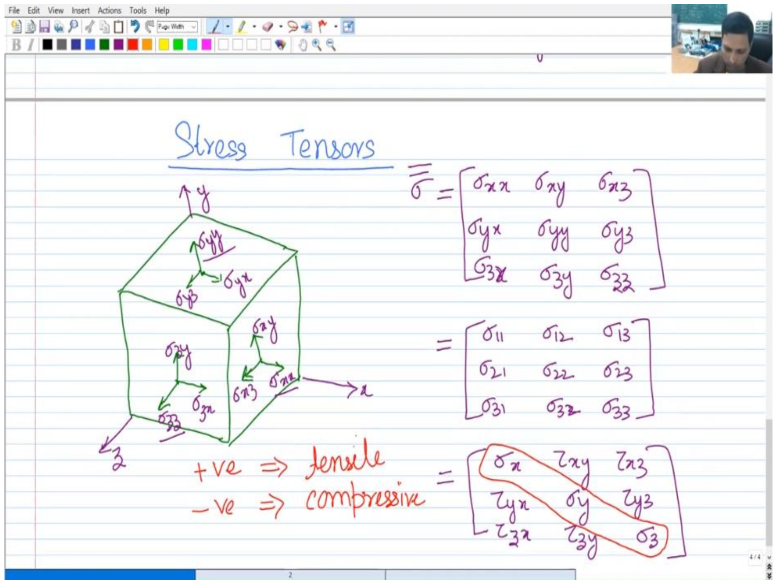

So, we will again go back to our cube. So, let us say this is the cube on which the stresses are acting. So, this is used to define the stresses in all the 3 different directions, x, y and z, and the normal and the shear stresses. So, let us say, over here our x-axis is, let us say like this, y-axis is like this and z-axis is like this. Over here, we will have the stresses, let us say the stress is acting over here, so, there will be 2 shear stress components and 1 normal stress component.

Similarly, here we will have 2 shear stress component and 1 normal stress component. And also here we will have 2 shear stress component and 1 normal stress component. And now, let us see how we define it. You remember, it has 2 subscripts i and j, where i defines the plane and j defines the direction. So, if we are talking about this one, so, the plane on which it is acting is this particular plane and this plane is x.

In fact, all the 3 stresses here are acting on the x plane, but this is acting in x direction, this is acting in y direction, this is acting in z direction. So, this becomesσ_{xx}, σ_{xy} and this becomes , σ_{xz}. Now, here it is the y plane; so, the first term would be y for all of these. And then, second term will be this one along x, this one along y and this one along z. So, this is , σ_{yx}, this is , σ_{yy}, , σ_{yz}.

And here, this is a z plane and the 3 stresses are acting in the x y z; so, this becomes, σ_{zx}, , σ_{zy}, , σ_{zz}. So, what we see here? , σ_{xx}, , σ_{yy}, σ_{zz}; these are the normal quantities. And therefore, normal quantities meaning stress, normal stresses acting perpendicular to the plane. These are the only 3 ones out of these total 9 quantities. So, therefore, this has the form ii; so, σ_{xx}, , σ_{yy}, σ_{zz} and so on.

On the other hand, the shear stress elements have two different indexes. So, when i is not equal to j, it is a shear stress, and when i is equal to j, it is a normal stress. And the way it is usually defined or put it in the tensor form is like this; sigma, this defines the tensor quantity and sometimes it will be written as x y z, sometimes it will be written as 1 2 3. So, let me write all the different versions here for sake of completeness, so that, in case you encounter any of these, you know that it is representing one and the same thing.

$$\bar{\bar{\sigma}} = \begin{matrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{matrix}$$

Or sometimes people write it in terms of 1 2 3

$$\bar{\bar{\sigma}} = \begin{matrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{matrix}$$

And still one more form of representing this is like here. So, here, in the other one, shear tresses are presented as tau and the normal stresses are represented by only ones index, 1 subscript.

$$\bar{\bar{\sigma}} = \begin{matrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{matrix}$$

So, what are the few things that we notice here? That the normal stress elements are along the diagonal for all of these. I mean, it is one and the same thing. So, normal stress elements lie along the diagonal and the shear stress elements are the non-diagonal elements.

And another important thing to keep in mind is that when we are talking about stresses, positive implies tensile and negative implies; so, if the value is given in negative, it means it is a compressive force, it is trying to compress; and if the value is positive, it means it is trying to pull out. So, now, let us look at one example problem.

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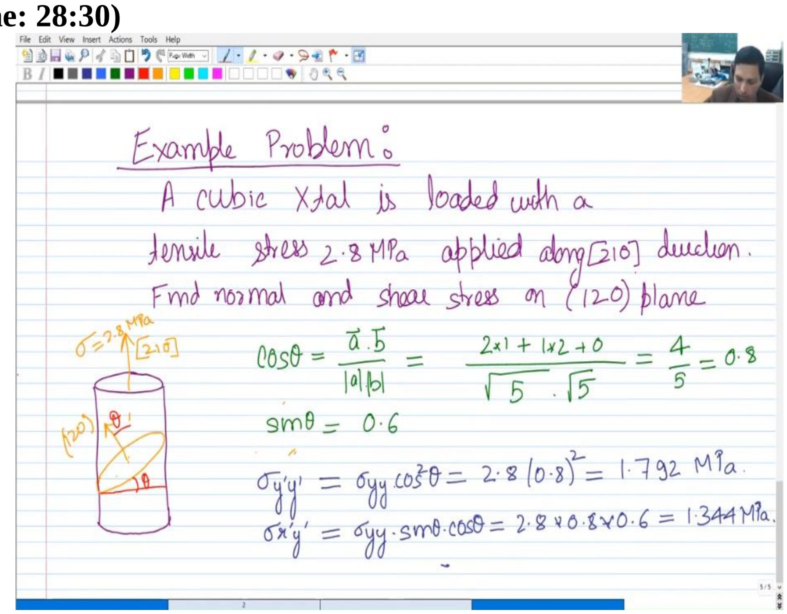

So, it is given that a cubic crystal is loaded with; so, let me quickly write down. A cubic crystal is loaded with a tensile stress of 2.8 mega Pascal, which is applied along [2 1 0] direction. Find the normal stress and shear stress acting on the (1 2 0) plane. So, the stress is acting along [2 1 0] direction, but you have to find the normal and shear stress on the (1 2 0) plane.

So, remember that when we are talking about direction, we write it in square bracket; and when we talk about planes, we write it in this regular brackets. Now, to solve this, what you need is a simple refresher of the vector algebra. So, if you want to know the angle between 2 vectors, then we know that it is given by

$$\cos\theta = \frac{(\vec{a} \, . \vec{b})}{|a||b|}$$

So, here, the 2 directions that we know that we are given are [2 1 0] and [1 2 0].

So, this will give us $\cos\theta$. Once we have the$\cos\theta$, we can find the $\sin\theta$ .And these are, $\cos\theta$ and $\sin\theta$is what comes into, in our equation, if you remember, that we used earlier to find the stresses on a different plane, not a normal to which a stress has been applied. So, with that, we have a as[2 1 0] and b as [1 2 0]. So, a dot b would be 4.

Now
$\cos\theta = 0.8$ and $\sin\theta$=0.6
And now that we have the sin theta and the cos theta values, we can calculate

$$\sigma_{y'y'} = \sigma_{yy}\,(\cos\theta)^2 = 2.8 \times 0.8^2 = 1.792 MPa$$
$$\sigma_{y'x'} = \sigma_{yy}\sin\theta\cos\theta = 2.8 \times 0.8 \times 0.6 = 1.344\ MPa$$

So, you can see that once we have established the relation, you do not even need to know how the 2 angles or directions are oriented with respect to each other.

We have derived the normal stresses and the sheer stresses without actually going into detail of how [2 1 0] and [1 2 0] are oriented. But just for sake of completeness, again I will draw it over here roughly how they would look like and it may not be accurate; I am just drawing it schematically. So, let us say this is the [2 1 0] direction there, along which a stress of 2.8 megapascal was applied.

And let us say this is the (1 2 0) plane. So, this is the (1 2 0) plane; so, the normal too, it is [1 2 0] direction. And what we have found is the theta between these two, which is same as saying that this is the theta. So, this is the plane which is oriented at theta and we calculated that by calculating the angle between these two directions. Now, this is when we are talking about x and y axis; but when you have more than 2 axes involved, then the overall formulation is a little is a bit more complex, and I will just show you the overall equation.

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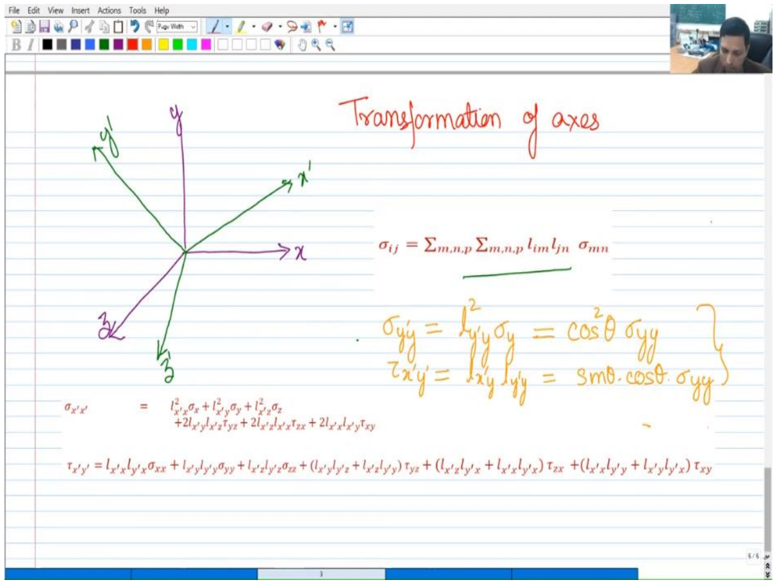

What we need is usually a plane stress transformation, meaning we need transformation along only x and y. So, the third axis mostly remains confined; and therefore, our relations could become much simpler. However, for the sake of completeness again, let me just show what we mean by this transformation of axes and what is the full relation to get the transformation from one access to another.

So, let us say this is x-axis, this is y-axis and this is z-axis. And so, your system was originally in x y z or probably you were doing the calculation in x y z, but now, the different plane that you are looking at is oriented along a completely different axes. So, I will draw this over here. I will call this x prime, y prime and z prime. So, these are the 3 different axes; maybe your new plane along which you want to calculate is defined along this new green axis.

And originally, your stresses that you are given or known are along those maroon x, y and z. So, how do we translate? So, the relation is very simple and I will simply; in a general way it is represented like this. And the more expanded format for this would be; is given over here. Now, this equation is compressed or the general format and this is the expanded format. So, here it is giving you what the normal stress would look like and this is giving you the relation what the shear stress would look like.

Now, what are these l terms that are given here? The l terms are nothing but the cosine values between the 2 axes. So, when it says l x prime x, it means the cosine between, the cosine of the theta between x prime axis and x ; the angle, whatever angle gets calculated between these two, you will have to calculate the cos of that; so, that has to be calculated and squared. So, you will get this term l x prime x square sigma x, l x prime y square sigma y, l x prime z square sigma z and the other term.

So, this way, you would be able to transform from x y z to another x prime, y prime, z prime where you have been, your new plane is defined. So, now that we have this, let us see if we can calculate a simple relation over here. So, the problem that we solved over here; now, if we look over here, so, here what we are given? Let us assume that what we are given is along the y-axis. So, this is the y-axis. And let us say this becomes the y prime axis.

Now, in the context of this problem, all other axes are meaningless; what we are interested is only in y and y prime. So, when we are talking about the sigma in the other one, so, we will be looking at sigma y prime y. So, instead of x prime x, we will replace everything by y or basically rotate x to y, y to z and z to x. And what we will get is basically just this first term with yy, y prime y. So, l y prime y square and sigma y.

And what is l y prime y? This is basically theta between y and y prime, which is what we calculated over there, which was cos is; so, the theta is something we do not know directly, but we know. What we know is the cos theta term; so, we were able to calculate this, and this is the same equation that we actually used. The other one that we wanted to find was the shear stress.

So, $\tau_{x'y'}$ is what we are interested in, but we have only one term which is yy; so, all other terms would go to 0. This will go to 0; all other terms we can see they will go to 0; what remains is only sigma yy. And over here, we want to have 2 terms $l_{x'y}$and $l_{y'y}$ l y prime y. So, y prime y is cos theta. And $l_{x'y}$ where theta, if you look at it, it is basically 90 minus theta. So, cos of that will become sin theta.

And therefore, it becomes $\sin\theta \times \cos\theta \times \sigma_{yy}$. And therefore, what we are getting is that, we are coming back to the same equation; but then, what you realise is that, this is a more general format and you can use it for any transformation. So, this is transformation of axes. So, we will again now look at another example problem.

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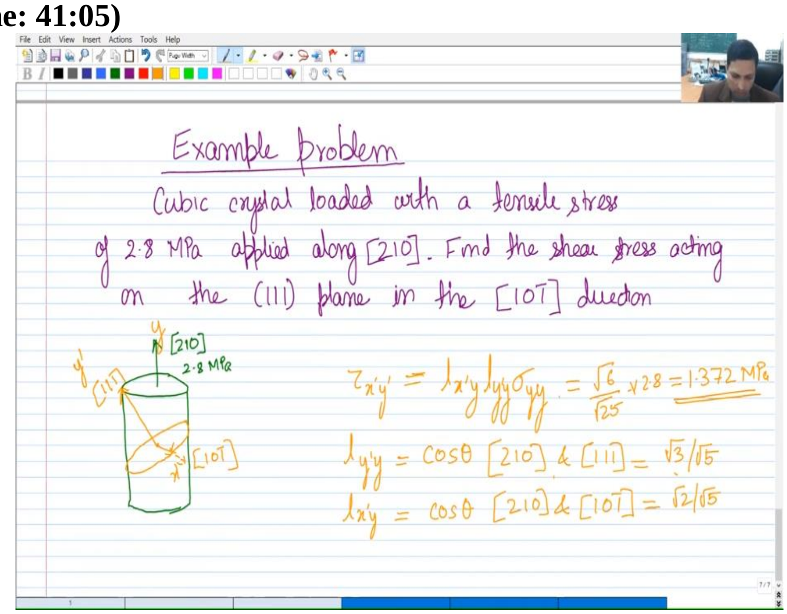

Here, again we are given the same cubic crystal which is loaded with a tensile stress of 2.8 megapascal. So, cubic crystal loaded with a tensile stress of 2.8 megapascal applied along; so far it is same. Now, you are given that you have to find not on the (1 2 0) plane but on the (1 1 1) plane and in the [1 0 $\bar{1}$] direction. So, find the shear stress acting on the (1 1 1) plane in the [1 0 $\bar{1}$] direction.

And this time we will use the equation that we have derived here; so, just for sake of completeness, again I will draw a schematic. So, let us say this is a single cubic crystal and this is loaded along this which is [2 1 0] direction. And the total stress here is 2.8 megapascal; and let us say this is your [1 1 1] direction; and this becomes (1 1 1) plane. And over here, you have to calculate the stress along a particular direction.

So, it is a shear stress but not along any particular direction, not along any general direction but on a particular direction. And what is that particular direction given? That particular direction is [1 0 $\bar{1}$] . So, now, let us go back to this question. Before that, again let me define; this is the y direction and this is the y prime direction and let us say this is the x prime direction.

So, from over here, what we will get, σ_{yy} is the only quantity that exists, which is given; all other terms will go down to 0. So, we are again left with this same equation, $l_{x'y}$and $l_{y'y}$ which is basically cos theta between [2 1 0] and [1 0 $\bar{1}$]and cos theta between [2 1 0] and (1 1 1) , which comes out to be $\sqrt{2/5}$ and $\sqrt{3/5}$ .So, tau xy; now, we have these 2 terms; and sigma yy is given as 2.8. So, $\tau_{x'y'} = \sqrt{6/25} \times 2.8 = 1.372\ MPa$. So, what do we learn here?

We see that using that equation we were able to get the shear stress in a very, you can say in, we can get shear stress in any direction. And the normal direction is of course always easy and shear stresses are, have various orientations along which you can get. And using this equation, you can get that along any direction. So, once you have the direction, you find a cosine value between the two and put it inside those, cosine values into that general equation and you will have the stress value for that particular direction on that particular plane along any particular direction.

So, we come to a close for this particular topic, stress and strain. Actually, we will keep talking about stress and strain. Stress and strain are a lot more; there are a lot more things to discuss in this. Next, we will be talking about principal stresses. So, for now, we will close this chapter and move on.