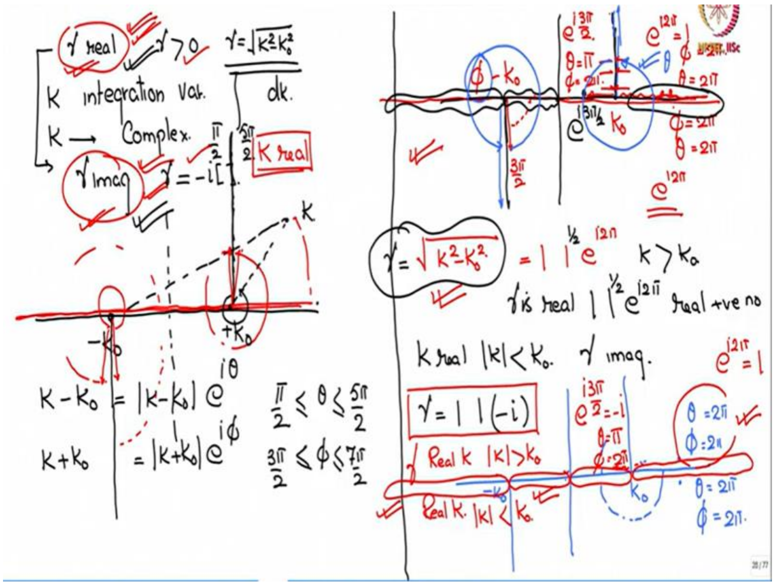
Sound and Structural Vibration
Prof. Venkata Sonti
Department of Mechanical Engineering
Indian Institute of Science, Bangalore

Module No # 02
Lecture No # 09
The Coupled Roots

Good morning and welcome to this next lecture on sound and structural vibration.

(Refer Slide Time: 00:31)



Last time we ended up here we were looking at contours in the complex plane for our integral. And one of the ways for defining a continuous branch for this γ functions was using this particular cut over here. So, there is a vertical cut from k_0 that goes off to ∞ this way and one from $-k_0$ that goes off to ∞ downward and the angular definition here is θ and here is ϕ.

And θ ranges from $\frac{\pi}{2}$ to $\frac{5\pi}{2}$, ϕ ranges from $\frac{3\pi}{2}$ to $\frac{7\pi}{2}$. So, I will just repeat a few points so here this is a vertical cut here. So, if I look at θ, ϕ values so ϕ value here is a $\frac{\pi}{2}, \frac{2\pi}{2}, \frac{3\pi}{2}, \frac{4\pi}{2}$ that is 2π and θ here $\frac{3\pi}{2}$, 4 pi twice pi both twice pi here just on the top. And just below we have same values $\theta = 2\pi, \phi = 2\pi$.

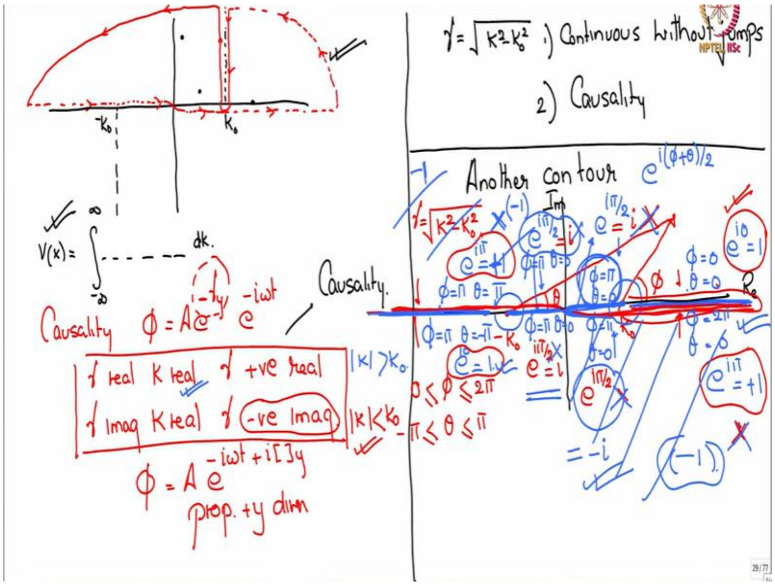
Now if we just cross this line now here θ is $\frac{2\pi}{2}$ which is π and ϕ is again 2π so I get the summation 3π and I have a square root, so I get $e^{i\frac{3\pi}{2}}$ which is negative i. And what did I say last time I said γ given by this entity for real values of k when magnitude of k is greater than

k_0 I should get a real positive number. So, this is the region is where magnitude of k is greater than k_0 and k is real.

So, I get $e^{i(2\pi+2\pi)/2}$ which is $e^{i(2\pi)}$ which is $+1$. So, this satisfies the criteria similarly when I am below k_0 for real values of k_0 I mean real values of k such that magnitude k is less than k_0 I should get negative imaginary. γ should be imaginary with a negative value so you can see I am getting a $-i$ in this region.

The same thing will happen if you calculate here and here so here you will get a real value positive here you will get a negative imaginary value this is valid cut. So how will the integral look like I will just show you.

(Refer Slide Time: 04:51)



So, we have line over here and I have a cut going from k_0 have a cut going negative from $-k_0$. So, I want the real line integral my $v(x)$ is a real line integral of the form $-\infty$ to ∞. Some function dk it is a real line integral so I will come from $-\infty$ above the real line cross over here go below. So, this entire real line I have and that integral we do not touch using principles of complex variables.

We compute using residues and other indirect contours so now I go to ∞ and take an arch at ∞ I come to this line which is a branch cut I cannot cross it. So, I come down till the real axis go around this k_0 again go up and go off at ∞ as an arc and join at $-\infty$ that is my contour. So, this is my close contour and for causality again I repeat for causality because my ϕ is $A\,e^{-\gamma y}$ with the time dependence of $-i\omega t$ if γ is real for k values on the real axis then γ must be positive real.

Then this term decays with y else it will blow up with y similarly if γ is imaginary when k is on the real axis then γ must be negative imaginary. Then I will have ϕ as an $A\, e^{-i\omega t+i[\;]y}$ which is a propagating wave in y direction. It goes upwards from the plate it goes to ∞. So, these 2 are must be respected so when we do this contour integrals 2 things becomes important. one is that the γ that you define must be continuous function without jumps continuous branch without jumps.

The other is this is one the other is causality both are separately satisfied or separately important. So, this is the causality condition so we will use this contour later to compute our $v(x)$. And then of course there will be singularities there will be singularities in the inside the contour and we have to compute residues. So, the singularities is another big topic we will see.

Now just so that I churned your mind I will use another contour means another cut. So let us see if we can understand that this is my complex plane and I have γ which is $\sqrt{k^2 - {k_0}^2}$. So, this is my k_0 this is my – k_0 and here let us say is my k so this is my $k - k_0$ and this is my $k + k_0$ vector or phasor. This again is my θ and this I am sorry this is my ϕ and this is my θ.

Now ϕ I will keep it again going from 0 to 2π and theta going from $-\pi$ to π. So, if we now look at values here ϕ is 0, θ is 0 here ϕ is π θ is 0. Similarly, here ϕ is π θ is 0 here ϕ is π θ is π. Here ϕ is $\pi,$ θ is $-\pi$ because θ moves from $-\pi$ to π this way and ϕ goes from 0 to 2π this way. Again, here my ϕ = $\pi,$ θ = 0 here also ϕ = $\pi,$ θ = 0 here ϕ equal to 2π, θ equal to 0.

Now if I look at this $e^{i(\theta+\phi)/2}$ factor I get e^{i0} which is 1 I get $e^{i\pi}$ which is -1. Now we come here let us see here I get $e^{i\frac{\pi}{2}}$ here also I get $e^{i\frac{\pi}{2}}$ here $e^{i\pi}$ here e^{i0} to the -1 and a 1. Now what did we say here when k is on the real axis and γ is going to be real that means magnitude of k is greater than k_0.

So here basically in this region I should have γ positive that means e^{i0} must be positive number. So, on the top it satisfies but, on the bottom, it is negative, so this is not correct in this region the bottom is not correct and this is discontinuous here which is not good. Because our main concern is always this real line integral mind you, we are always interested in this real line integral which we have to compute for inverting the transform everything else that comes is incidental.

So, when I am here, I cannot be discontinuous on the top I have one value bottom one value no so this is not ok first of all. Now when I move down when I move below k_0 when my k goes below k_0 and γ is going to be imaginary it should be negative imaginary. And if you see here, it is $e^{i\frac{\pi}{2}}$ which is i which is positive imaginary, so it is also not good.

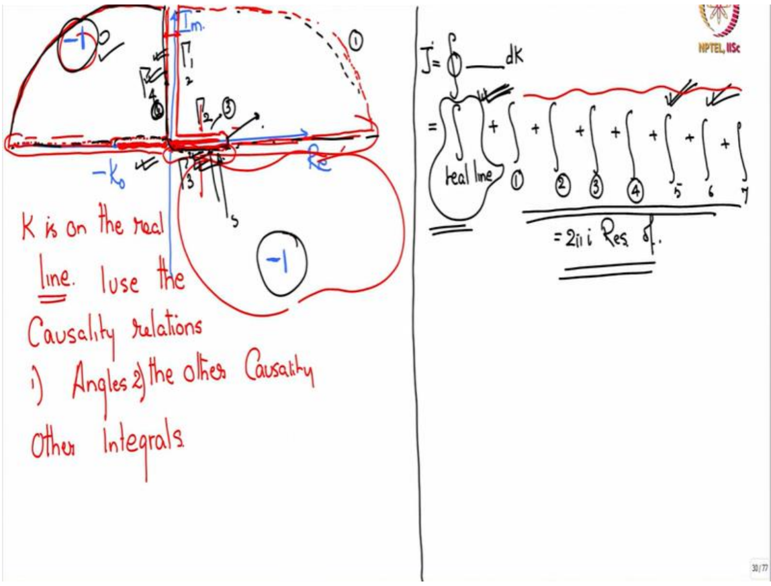
And below again $e^{i\frac{\pi}{2}}$ that is also not good so what should we do? Similarly, here on the left side here on the top I get $i\frac{\pi}{2}$ which is i here lower part I get $e^{i\frac{\pi}{2}}$ which is again i here also k magnitude is less than k_0 that is one thing. Then on the left over here on the top I have -1 and the lower side +1 again I have a discontinuity across this line which is my main line that I want to move through.

So here also I am not okay sign wise so several places I am not ok so what we do is we multiply this region by -1 and we multiply this region by -1. So, what happens when I multiply by -1 that means fourth quadrant is multiplied by -1 I get a + 1 here therefore I am continuous with this part so this part beyond k_0 has become continuous. And let us see here I had $e^{i\frac{\pi}{2}}$ i become this is + i it will become $-i$.

So, imaginary should be negative I am also ok here and, in this part, it does not matter what I do whatever is the result is the result we will leave it like that. Because we are going to move on this contour so that means what now there is a discontinuity from the top it is $e^{i\frac{\pi}{2}}$ from the lower side $-\frac{i\pi}{2}$. So, we create a discontinuity here this line is discontinuous.

Now on the left similarly on the top will multiply by -1 second quadrant so the second quadrant is multiplied then what happens $e^{i\pi}$ which is -1 becomes $+1$ and I have $+1$ here anyway. So, I am continuous here but here what happen $e^{i\frac{\pi}{2}}$ which is i now becomes –i so imaginary negative I am okay here but from the bottom side it is +i still so I have a discontinuity here.

(Refer Slide Time: 19:58)



So, what we now have now essentially is we come so let us see so we come this $k_0, -k_0$ real imaginary. Here we multiply by -1 here we multiply by -1 so I begin my contour, and this is a discontinuity, and this is a discontinuity what happen this is the discontinuity, so we come from $-\infty$ we get on to this cut over here we cross over below we get below the cut okay then we jump of the cut we go off to ∞.

And we come all the way on an ∞ contour and there is a jump now between these 2 lines over here. So, we cannot cross this line, so we come down you come straight down go to the right go around this come back cross over go up go over all the way on an arc of ∞ radius back here and join up. So essentially as long as k is on the real line, I use the casualty relations.

So that one I respect the angles and the other I respect causality so when I am in this quadrant, I will multiply whatever integrations comes with -1 then I move on to the first quadrant which I do not touch I cannot cross this because it is a discontinuity, I come down I go again here. I am not worried because the main parts I want is below here that is what maintains the cut over here, I go around then I am below here again I have to respect the causality.

And I go over here again I have to respect the causality because there is a -1 and I go off to ∞ come down and I join up. We will not be using this cut so this is all we are going to see essentially in contour integration we do not touch the real line at all. It is indirectly computed and therefore in the indirect computations when you get the other portions of the curves other integrals the extra integrals that you get that is where you have to enforce your causality impositions.

So, by implication I am coming from the left I am getting on to this cut coming down here getting off the cut and going up. But I do not touch that integral anyway so how are these causalities imposed? These causalities are imposed when I am here and here when I get back right, I have portions of the cut where I get back here, I cross over here. So, in these parts I implement the minus in these parts I implement the negative.

So, I have a γ_1 for example I have γ_2 for example I have γ_3 portion for example I have γ_4 portion for example let me write it out. Let me write this out so I have let us say the closed from integral for velocity whatever it has dk, k has become complex that has what the real line integral real line which I do not touch. So plus let us call this the one 1 let us call this 2 let us call this 3 there is a small circle call it 4 that is the bottom line call it 5 then this is vertical call it 6 I go up so that is 6.

Then I go of this arc with ∞ radius call it 7 those are equal to the $2\pi i$ times the residue of the function at the singularities inside. So, there are some singularities again, so I have to compute the residues. So now in this portion whatever was the number I forget 1, 2, 3, 4, 5 in 5 and in 6 I have to implement the negative. So, when I implement the negative in the 5 and the 6 portions then what happens in the main integral causality and the cut definitions are implemented.

So, we do not do anything to this the cut definitions are implemented in the remaining portion and of course the singularities and residues that is how both angles and causality are maintained. The main idea I am trying to convey is that we do not touch this integral so how do we respect the angles and the causality by implementing that in the other portions by implementing that in the other portions of the contour integral that is how it is done time is running out. So, I will close the lecture here we will continue from the next class thank you.