Sound and Structural Vibration
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Module No # 02
Lecture No # 08
Introduction to the Coupled Problem

Good morning and welcome to this lecture on sound and structural vibration.

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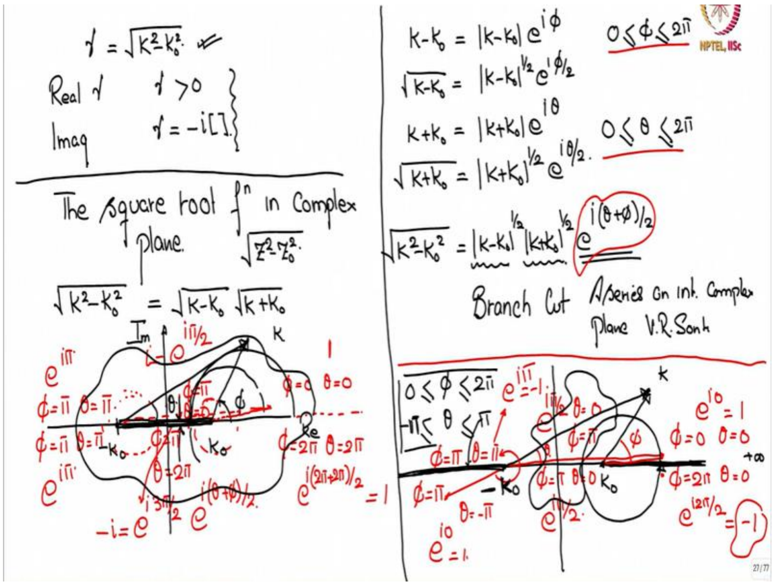

We were discussing the γ issue which is square root of $\sqrt{k^2 - {k_0}^2}$ and real values of γ. Then we said γ must be a positive real number if γ tends out to be imaginary then γ should be of the form negative imaginary time some number. So, we have look at how to make a continuous branch for γ and how to find such a branch that this is also satisfied.

But let us just first look at this square root function in the complex plane so let us consider a $\sqrt{k^2 - {k_0}^2}$. I am deliberately using case in a textbook you will probably find it as $\sqrt{z^2 - {z_0}^2}$. But I would like you to get used to k now being the complex number, so I am using this. So, what is this? This is

$$\sqrt{k^2 - {k_0}^2} = \sqrt{k - k_0}\,\sqrt{k + k_0}.$$

So let us draw the complex plane k_0 is the real number so I have k_0 here and I have $-k_0$ here. And so, this is the real axis, and this is the imaginary axis. So let us say my k is somewhere

here a complex number k is here. So, $k - k_0$ is this vector and $k + k_0$ is this vector now let us describe those vectors in terms of phasor notation amplitude and angle. So let me say that my

$$k - k_0 = |k - k_0|\, e^{i\phi}.$$

So, this angle is my ϕ that means $\sqrt{k - k_0} = |k - k_0|^{\frac{1}{2}}\, e^{i\frac{\phi}{2}}$. Now I will define the limits of ϕ so let us say that $0 \leq \phi \leq 2\pi.$ ϕ ranges between 0 and 2π similarly $k + k_0 = |k + k_0|\, e^{i\theta}$ so this angle is θ again $0 \leq \theta \leq 2\pi.$

Now so similarly $\sqrt{k + k_0} = |k + k_0|^{\frac{1}{2}}\, e^{i\frac{\theta}{2}}$. So, this square root function

$$\sqrt{k^2 - {k_0}^2} = |k - k_0|^{\frac{1}{2}}\, |k + k_0|^{\frac{1}{2}}\, e^{i\frac{\theta+\phi}{2}}.$$

So, these are real positive number, so they do not have a significant effect here they are just amplitudes the actual continuity is decided by this function $\frac{i(\theta+\phi)}{2}$. And how is that we will see?

So let us see how so here at this location if I take my k very close to the real axis then this is the one vector, and the other is this vector one from k_0 one from $-k_0$. So, I can say my ϕ is 0 θ is 0. If I reach this point, I come here then ϕ has come in a half circle so ϕ will be π and θ is still 0. If I come here even θ has come half circle so ϕ is π and θ is also $\pi.$

And if I go just below the negative real axis here also ϕ is π and θ is π. If I come here between then this θ has gone full circle whereas ϕ is still π so I get $\phi = \pi$ and $\theta = 2\pi$. And if I just go below the real axis positive real axis here, I get ϕ has gone full circle θ as gone full circle. So, ϕ is $2\pi,$ θ is also 2π now we want to look at this term $\frac{\theta+\phi}{2}$.

So, if I do that here I get e^{i0} which is 1 here $e^{i(2\pi+2\pi)/2}$ which is $e^{i(2\pi)}$ which is also 1 no problem. But here I get $e^{i\pi/2}$ here I get $e^{i\frac{3\pi}{2}}$. So, this is actually equal to i this is equal to $-i$ so if I move across here that line, I get a jump in the function value so that region is not allowed I cannot cross.

Now we go back to here ϕ and θ are both π across so I am continuous, so I get $e^{i\pi}$ which means what? If I have some contour, it cannot cross the region here because the function takes a jump. I can do this I can take a contour which can goes around I can do this, but I cannot do this I cannot take a contour it goes and cuts this line. So, this line is called a branch cut and it happens to be between the values $+k_0$ and $-k_0$.

I have given a series of lectures on integration on the complex plane a short series of about 24 half hour lectures where this branch cut has been explained in great detail. So, and these are there on NPTEL or YouTube so you can look at it. It is something like a series on integration in the complex plain and if you type my name, you will get this on YouTube also. So, you can see as discussed in branch cuts in great detail.

So, I won't repeat the entire deal here so that is the branch cut not this branch cut is unique to this choice of θ, ϕ values. So now let us see here if I choose $k_0, -k_0$ let say my k is here this is $k - k_0$, $k + k_0$. Now I will choose for ϕ I will choose the same limits $0 \leq \phi \leq 2\pi$ but theta I will choose like this $-\pi \leq \theta \leq \pi$ I have chosen a different range for theta. So, what does that do?

So, I will quickly look at it here so ϕ is still 0 here θ if I take a point close by this is one vector and that is the other vector ϕ is 0, θ is also 0. So, if I come over here this is ϕ of course this θ. Now ϕ has come to π now θ is still 0 I am $-\pi$ here and I am so 0 over here θ is still 0. If I come here my ϕ is π and θ is π because theta has gone full circle it is π.

And just below I have ϕ still equal to π but theta jumps –pi and here my ϕ is still π and θ is 0. And here ϕ is 2 π just below and θ = 0 so if I look at this expression now, I get e^{i0} which is 1 I get $e^{i2\pi/2}$ which is $e^{i\pi}$ which is -1. So, it is different here $\theta\ and\ \phi$ are same I get $e^{i\pi/2}$ on top and bottom no problem here.

But here I get $e^{i\pi}$ above which is a -1 and I get e^{i0} which is a 1. So, you can see now that with this choice of θ, ϕ I have discontinuity here. So, the branch cut happens from k_0 to $+\infty$ and from $+\infty$ it comes $-k_0$ that means what? I can have a contour integral which goes and does this, but I cannot have a contour integral which goes and does this I cannot cross this line same here.

I cannot have contour which crosses that line so that is a branch cut now there are several ways to do cuts I said. So let us take one more and these will be used in the problem they will come of use in the thought process for this problem.

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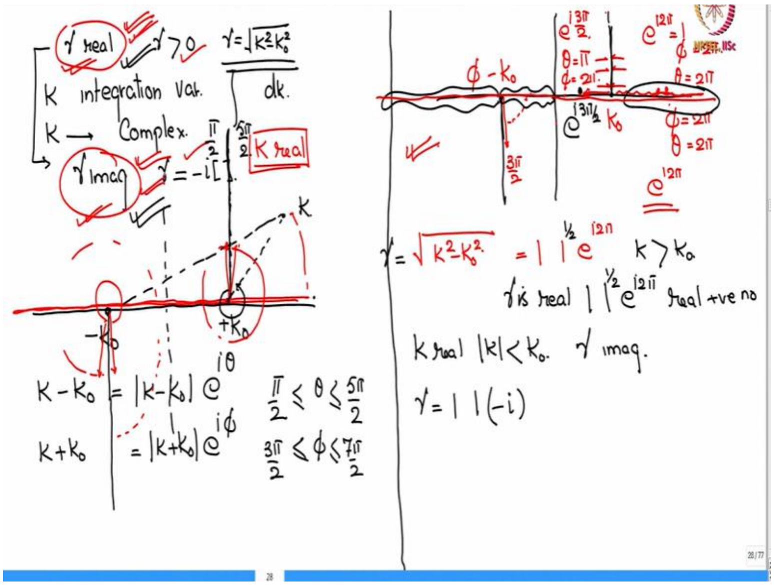

So now we said γ has you know issues for bounds γ real then γ should be positive, and γ is what $\sqrt{k^2 - {k_0}^2}$. And mind you if you remember k is the integration variable that means the integrals are with respect to dk and k is going to be made complex k becomes complex in the plane. And to carry on this idea when γ becomes imaginary γ must be negative imaginary $-i$, times something.

So, the previous cuts I showed branch cuts they are just to show that how to take cuts you for a branch. And what lines are not allowed what are lines are allowed that you can cross. Now here I am going to show you a region in a complex plane where this issue will be satisfied where this point and this point will be satisfied. It is important to emphasize that this should be satisfied only when k is real.

Because the integral I want is going to be on the real line that is the integral I want anything else is extra in the complex integration I have shown you that you get extra portions which I have to be evaluated and used along with Cauchy residue theorem. However, the one integral I want is actually moving on the real axis. So, when k is moving on the real line these should be satisfied that is the important part.

Once k has moved into the complex domain something can happen, but we are not worried about that only when it is moving on the real line, I want these criteria to be satisfied. So now let us see one branch here I have a k_0 and I have a let us see $-k_0$. So now I have complex number k sitting over here with these 2 vector components $k - k_0$ and $k + k_0$. So, $\sqrt{k - k_0}$

or let us say $k - k_0$ to begin with I will define using $|k - k_0|\, e^{i\theta}$ where θ now goes $\frac{\pi}{2} \leq \theta \leq \frac{5\pi}{2}$.

So that results in a cut like this so my θ moves from $\frac{\pi}{2}$ here and it goes around and comes to $\frac{5\pi}{2}$. Similarly, at $-k_0$ I have $k + k_0$ which is given by magnitude $|k + k_0|\, e^{i\phi}$and ϕ now takes values between $\frac{3\pi}{2} \leq \phi \leq \frac{7\pi}{2}$. So, it begins here goes around and comes back here whereas the first fellow begins here goes around and comes back here.

So now let us look at a region over here so if I have to redraw so my θ is going to be $\frac{\pi}{2}, \frac{2\pi}{2}, \frac{3\pi}{2}, \frac{4\pi}{2}, \frac{4\pi}{2}$ which is 2π right here just above the real axis. Now what is ϕ over there? ϕ is right here ϕ is $\frac{3\pi}{2}$, so here it will be $\frac{4\pi}{2}$ so 2π. So, if I take the half of that I get $e^{i2\pi}$ I get a 1 which is equal to 1 all the way it is 1.

Now just below right here ϕ is still equal to 2π, θ is still equal to 2π so I get $e^{i2\pi}$ so no problem here. Now suppose I cross and come here then θ is $\frac{2\pi}{2}$ which is π, θ is π if I consider a point θ is π over here. And then ϕ is $\frac{3\pi}{2}, \frac{4\pi}{2}$, ϕ is 2π and what is the angular function $i(\theta + \phi)/2$ gives me $i\frac{3\pi}{2}$.

So, you can see that there is a discontinuity across this line so now you can also see that I have k_0 over here and – k_0 over here. So, and k is what I am looking at on the real axis right so when k is here I get for this function which is $\sqrt{k^2 - {k_0}^2}$ the combined functions I get a magnitude and $e^{i2\pi}$ here in this region and that region is what?

k is greater than k_0 which means γ is real so when γ is real I get an amplitude into $e^{i2\pi}$. So, it is a real positive number whereas when I come over here, I cross over and come here at this location k is real of course but magnitude of k is less than k_0. So, γ becomes imaginary so what values does it have here I get γ is again some amplitude but in addition I get a negative i.

I have $e^{i\frac{3\pi}{2}}$ so I get a negative i so it is a negative imaginary number. So, the same will be the story here also in this region you will get a negative imaginary number and here again will get a positive real number. So, with this kind of a cut going vertical I not only give a continuous branch for a function γ. But I also satisfy, these 2 criteria when k moves on the real axis for the integral that I want.

So, this is very important so we will be looking at this cut this is the cut will be looking at however there is one more cut which will actually give you the same uniqueness continuous behavior of the branch and these 2 criteria satisfies. I will show that in the next class time is running out thank you very much.