Sound and Structural Vibration

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Module No # 02
Lecture No # 06
Uncoupled Solution to the Classical Problem

Welcome to this lecture on sound and structural vibration. Last time we had started looking at the model problem. We had posed it and we decided to look at the uncoupled part of the problem and the uncoupled part looked like this.

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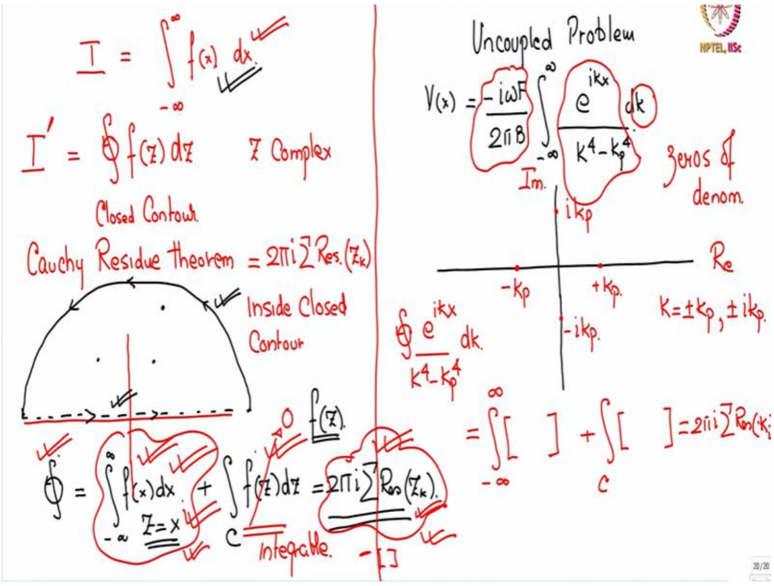

We were desiring

$$v(x) = \frac{-i\omega F}{2\pi B}\int_{-\infty}^{\infty}\frac{e^{ikx}}{k^4 - {k_p}^4}\,dk.$$

Here we decided to use principles of complex variables and I told you about the contour integration. So now we are going to integrate this using principles of contour integration so let me show you the contour and the singularities of this function.

This now k becomes complex k enters the complex domain so far k was real but now k enters the complex domain. So, what are we trying to do so now what we have for the integral is we have a contour integral $\oint \frac{e^{ikx}}{k^4 - {k_p}^4}\,dk$, k is now complex. This is now what we are going to do but I have excluded the constants in front we will add them later. So, this is going to be equal to the integral we want from minus infinity to infinity on the real axis plus there will be portion from a contour.

And it is going to equal to $2\pi i$ times the sum of the residues at whatever singularities we have.

$$\oint \frac{e^{ikx}}{k^4 - {k_p}^4}\, dk \; = \; \int_{-\infty}^{\infty} [\quad] + \int_C [\;\;] \; = \; 2\pi i \sum Res(k_i).$$

So let us look at these singularities of this functions once k has become complex that means where does it blow up that means zeros of the denominator. It is not very difficult to see that k = $\pm\, k_p$ are singularities and $\pm$ i k_p are also singularities.

So, if I plot those I have $-k_p$ I have $+k_p$ I have $+ik_p$ and $-ik_p$ so these are the singularities.

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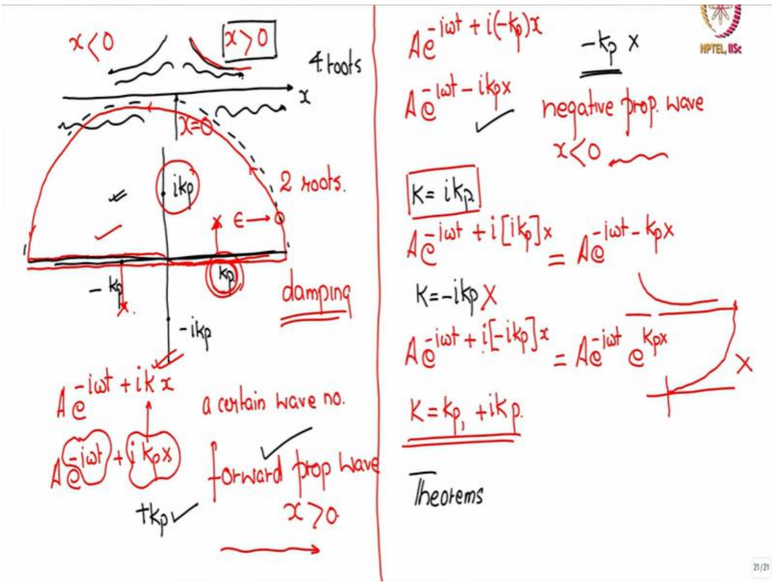

Now the problem is deliberately is very symmetric so if I am going to just describe the problem, we have an infinite plate and we have a line force acting at x= 0. So, it is very symmetric so there will be a propagating wave that way and a decaying wave that way a propagating wave that way and decaying wave that way. That is why we have 4 groups 4 singularities that are why we have 4 roots.

And therefore, which are the once to include we have ik_p here we have k_p here we have $-k_p$ here $-ik_p$ here. So, there is an x greater than 0 solution there is a x less than 0 solution that are going to be symmetric. So, we are going to attempt x = 0 solution and therefore we have to choose 2 of the roots you have to choose 2 roots. So let us see my displacement let us say some amplitude A it has $-i\omega t$ time dependence.

Then let us say it has $+ikx$ now k here is anything a certain wave number so I am going to substitute each of this.

$$Ae^{-i\omega t+ikx}.$$

I am going to substitute first k_p this means a forward propagating wave in the positive x direction one should know this sign flip is the indicator.

$$Ae^{-i\omega t+ik_px}.$$

Once you have the negative here and a positive here that sign flip is an indicator for forward propagation.

Now so k_p we have looked at now suppose we look at $-k_p$ so I have

$$Ae^{-i\omega t+i(-k_p)x}.$$

so, which means what this is

$$Ae^{-i\omega t-ik_px},$$

this is a negative propagating wave that means in the negative x direction. This wave is going leftward this wave is going rightward starting with x = 0 my source or line force is at x = 0.

So, this one represents a wave moving positive x and this one represents a wave in the negative moving x. So $-k_p$ is not to be chosen $+k_p$ to be chosen should be part of my contour even within the mathematics some physical aspects coming our judgement comes in blindly we cannot do the mathematics. We have to choose a contour and choose the roots such that meaningful results come.

So, you can see that having $-k_p$ does not give a meaningful reason for meaningful results for positive x. Similarly, if I choose ik_p if I put ik_p for k what do I get? I get

$$Ae^{-i\omega t+i[ik_p]x},$$

this gives me $Ae^{-i\omega t-k_px}$ what is this? This is a wave decaying in the positive axis as I move in the positive x direction it decays away.

So that is what I have drawn here it decays away which is possible so $k = ik_p$ should be included. The last choice is what $k = -ik_p$ so that gives me

$$Ae^{-i\omega t+i[-ik_p]x}.$$

So, I get

$$Ae^{-i\omega t}e^{k_px}.$$

So, this starting at x = 0 blows up to ∞ at x = ∞ this is not as admissible.

Such results are not physical so it is not as admissible so what is admissible? $k = k_p$ is admissible $k = ik_p$ is admissible. So, any contours I choose these are the 2 routes that should be included. So now the contour I choose is I come from negative infinity I comes slightly above. I cross over here I include $+k_p$ in my contour then I go around I go off to infinity at infinity I take a semicircle I come back and meet.

Now this is one way of looking at it by taking contour that is slightly above the real axis here and slightly below the real axis here. The other way typically done in engineering is that every route as a little bit of damping. We are doing undamped analysis in my plate equation or acoustic wave equation there is no damping. But all systems have some damping that means the routes are damped that means they will have imaginary portions.

So, this k_p can be you know given an imaginary portion such that it moves into this. This $-k_p$ can be given an imaginary portion such that it moves downward. So, then I move straight away along the real axis there automatically my $+k_p$ is within the contour $-k_p$ is outside the contour. And they have been given a very small ϵ damping then after the answer is computed I send ϵ to 0 do it that way also.

So, in anyway what is included is $+ik_p$ and $+k_p$ for computing answers in the positive x direction and this is the shape of the contour. So, what has happened is that the portion I wanted minus infinity to plus infinity is included however I have ended with a contour on a semicircular arc. So that has to be evaluated otherwise you would not get the full answer. So, we need some theorems now from complex variables so let us see.

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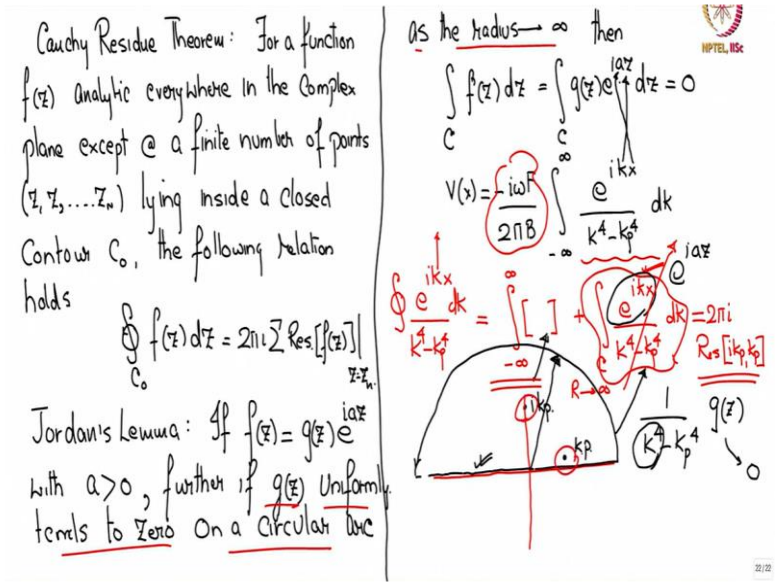

So, one theorem is the Cauchy residue theorem so I will just write it here for a function $f(z)$ analytic everywhere in the complex plane except at a finite number of points z_1, z_2 through z_n lying inside a closed contour C_0 the following relation holds. What is that?

$$\oint_{\mathrm{C}_0} f(z)\,\mathrm{dz} \;=\; 2\pi\mathrm{i} \sum Res\,[f(z)]|_{z=z_n}.$$

That is one result we will need the other result is called Jordan's Lemma it says that if $f(z) = g(z)e^{iaz}$ with $a > 0$, a being a positive number. Then further if $g(z)$ uniformly tends to 0 on a circular arc as the radius tends to ∞. Then

$$\int_C f(z)dz = \int_C g(z)e^{iaz}dz = 0.$$

So, these are the 2 theorems we will need so now let us see we have

$$v(x) = \frac{-i\omega F}{2\pi B} \int_{-\infty}^{\infty} \frac{e^{ikx}}{k^4 - {k_p}^4}\,dk.$$

Here k from if we want to take Jordan's theorem k plays the role of z and x positive plays the role of a. So now we are as I said earlier, we will keep the constants outside for the moment we will think about this integral.

So, we are going to replace it with a contour integral let us keep the same variable, but k has become complex decay. So that is going to involve the real portion of the integral which I want plus an integral over a contour whose radius tends to infinity.

$$\oint \frac{e^{ikx}}{k^4 - {k_p}^4}\,dk \;=\; \int_{-\infty}^{\infty} [\;] + \lim_{R\to\infty} \int_C \frac{e^{ikx}}{k^4 - {k_p}^4}\,dk \;\;=\; 2\pi i\; Res\big[ik_p, k_p\big].$$

And what did we say the contour we are choosing is we now choose to go straight on the real axis you go on the real axis. I am drawing it separately so that you can see I go off to ∞ so my k_p is inside ik_p is inside. And I come I close the value close the contour at $-\infty$. So, I have the portion I want the real integral here I have ended with extra portion over here.

Now if you look at it this in the Jordan theorem this portion looks like e^{iaz} and $\frac{1}{k^4 - {k_p}^4}$ looks like $g(z)$. And k is the complex variable taking the role of z so as my semicircular arc goes to ∞ its radius goes to ∞, $g(z)$ uniformly tends to 0 that is the Jordan's theorem. If you recall $g(z)$ uniformly tends to 0 on a circular arc as the radius goes to ∞.

So that is true Jordan's Lemma applies and this portion is equal to e^{iaz}. So therefore, this integral on the circular arc goes to 0 therefore the integral I want is simply the sum of the 2 residues at ik_p and k_p. So let us just compute that and see we have just about enough time.

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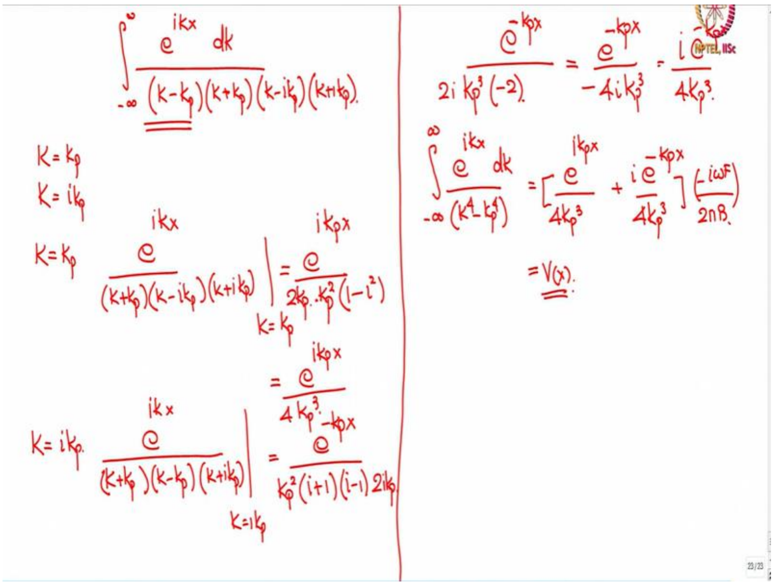

So, I have let me just recall I have this integral

$$\int_{-\infty}^{\infty} \frac{e^{ikx} \quad dk}{(k - k_p)\,(k + k_p)\,(k - ik_p)(k + ik_p)}.$$

So, I need the residues at k = k_p and k = ik_p so you should know how to find residues multiplied by the offending term or remove it. So, I have for k = k_p I have to $\frac{e^{ikx}}{(k + k_p)\,(k - ik_p)(k + ik_p)}$ evaluated at k = k_p.

So that gives me let us see

$$\left.\frac{e^{ikx}}{(k + k_p)\,(k - \mathrm{ik}_p)(k + ik_p)}\right|_{k = k_p} = \frac{e^{ik_px}}{2k_pk_p^2(1 - i^2)} = \frac{e^{ik_px}}{4k_p^3}.$$

Similarly for k = ik_p

$$\left.\frac{e^{ikx}}{(k + k_p)\,(k - \mathrm{k}_p)(k + ik_p)}\right|_{k = i\,k_p} = \frac{e^{-k_px}}{k_p^2(i + 1)(i - 1)2ik_p} = \frac{e^{-k_px}}{2ik_p^3(-2)} = \frac{e^{-k_px}}{-4ik_p^3}$$

$$= \frac{ie^{-k_px}}{4k_p^3}.$$

So, we have 2 residues now so the integral I want

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{k^4 - k_p{}^4}\, dk = 2\pi i \left[\frac{e^{ik_px}}{4k_p^3} + \frac{ie^{-k_px}}{4k_p^3}\right]\left(\frac{-i\omega F}{2\pi B}\right),$$

so this is the inverse so this is equal to our $v(x)$. So, I will close the lecture here in the next class corresponding to this uncoupled plate vibration velocity we will find the acoustic response thank you.