Sound and Structural Vibration
Prof. Venkata Sonti
Department of Mechanical Engineering
Indian Institute of Science, Bangalore

Lecture - 28
Derivation of Acoustic and Vibration Response

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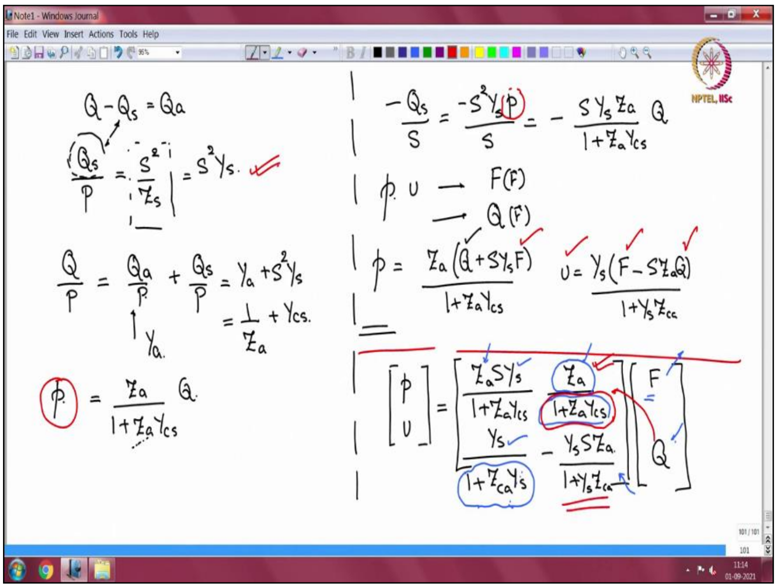

Good morning and welcome to this next lecture on sound and structural vibration. So, we ended up in the last class over here where we found pressure due to an input volume velocity and due to a force applied to the structure. Similarly, we found the velocity at the common point when a force is applied, and a volume velocity is given into the cavity. So, we will just make some physical comments.

So, if I put this together in a matrix form, I get

$$\begin{bmatrix} p \\ U \end{bmatrix} = \begin{bmatrix} \dfrac{Z_a s Y_s}{1 + Z_a Y_{cs}} & \dfrac{Z_a}{1 + Z_a Y_{cs}} \\ \dfrac{Y_s}{1 + Z_{ca} Y_s} & -\dfrac{Y_s s Z_a}{1 + Z_{ca} Y_s} \end{bmatrix} \begin{bmatrix} F \\ Q \end{bmatrix}.$$

I have to make the comments here only or I take a new page and write again. So, let me make the comments here. So, if there is no flow into the system that means my Q is 0.

Then if a force is applied a pressure and a velocity are generated and if the system is in vacuum suppose then Z_a is actually 0 then pressure is actually 0 you can see the pressure is 0 if the system is in vacuum. Then the particle velocity is just F over Z_s. So, this term goes to 0 particle velocity is just F over Z_s now if the system is not in vacuum a pressure gets generated which opposes the applied force and thus reduces the particle velocity to U which is given by this entity with the denominator.

And if in addition to that a flow Q is given a pressure increment of QZ_a should have happened provided Z_s was infinity or Y_{cs} was 0 but as Z_s finite, the pressure is reduced since the structure deforms and therefore you have this additional denominator you do not get that much pressure. Similarly, if F happens to be 0 then Q is the inflow when Z_s is infinite Y_{cs} is 0 thus the velocity is going to be 0 velocity the interface is going to be 0.

But a pressure is going to be generated so, related to the pressure there is no velocity since the structure is infinitely rigid if the structure has a finite impedance Z_s firstly the pressure generated is lowered Q into Z_a divided by this because the structure now takes up some deformation and the corresponding velocity is a plane my multiplying with s over Z_s it is a negative. So, this matrix makes physical sense.

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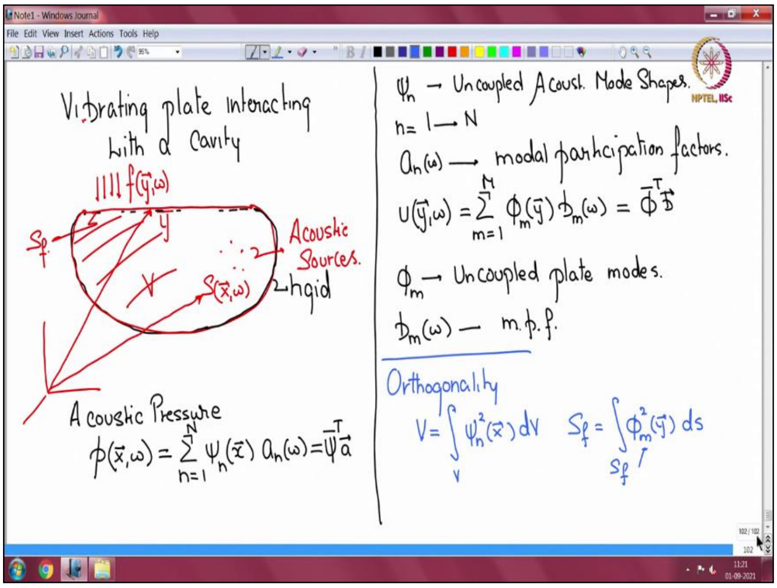

Now let us look at the actual system which is this. So, there is a structure. So, we will call this what do we call this topic we will call this vibrating plate interacting with the cavity vibrating plate backed by a cavity interacting with the cavity. So, the diagram I will use is very similar to what is shown in Kim Brennan's paper. So, this is a vibrating structure, and it is backed by some arbitrarily shaped cavity.

That cavity wall is rigid small as rigid. Then there is this volume V inside there are forces applied to this structure that is described by y the structural surface is described by $\vec{y}$ bar and it has a surface given by S_f and then within there are acoustical sources acoustical sources that source distribution is described by $S(\vec{x}, \omega)$. So, the 3D geometry of the cavity is described by the vector $S(\vec{x}, \omega)$ and the plate surface which is a plane is described by $f(\vec{y}, \omega)$.

So, now what do we do? We describe now the acoustic pressure we describe the acoustic pressure using the modal sum pressure is described by where we assume that the mode shapes are the rigid body mode shape that means the panel was rigid. So, that is equal

$$p(\vec{x}, \omega) = \sum_{n=1}^{N} \psi_n(\vec{x}) a_n(\omega).$$

ψ_n are uncoupled acoustic mode shapes and we take n of them.

So, that is why the index n goes from 1 to n. So, we take n acoustic mod shapes and then $a_n(\omega)$ are the amplitudes or also called modal participation factors and we can write this in a matrix in a vector notation. So, we will do it $\vec{\psi}^T \vec{a}$ the bar indicates vector similarly the plate velocity described by

$$U(\vec{y}, \omega) = \sum_{m=1}^{M} \phi_m(\vec{y}) b_m(\omega).$$

Here what are Φ_m they are the uncoupled plate modes that means plate vibrating in vacuum not backed by a cavity then b_m are the amplitudes or model participation factors and in a matrix notation or a vector notation this $\vec{\phi}^T \vec{b}$. Now these uncoupled modes they satisfy orthogonality that is the advantage. So, how is. So, let us write orthogonality what is that orthogonality?

$$V = \int {\psi_n}^2(\vec{x}) dV.$$

$$S_f = \int {\phi_m}^2 (\vec{y})\, dS.$$

That means if this index is different ϕ_m into ϕ_r integrated will be 0 that is the orthogonality.

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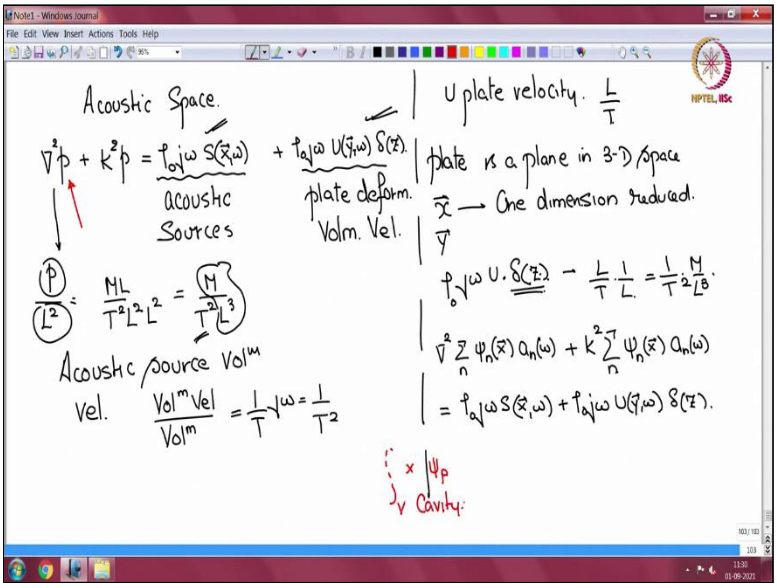

So, now let us look at the acoustic space acoustic space. So, the equation is $\nabla^2 p + k^2 p$ is equal to there will be a component coming from acoustic sources there will be a component coming from plate deformation and therefore a volume velocity from there volume velocity from there.

$$\nabla^2 p + k^2 p = \rho_0 j\omega\, S(\vec{x}, \omega) + \rho_0 j\omega\, U(\vec{y}, \omega)\delta(z).$$

So, let us look at the units now the units of this are pressure over length square pressure divided by length squared.

So, let us take the force first mass into acceleration divided by L square now I have pressure units divided by one more L square this one. So, I have M over T square length cube. So, this is density. So, I will have acoustical density over here that means these terms should have the same units I am trying to generate those terms. So, density is there and then I need something over T square and the acoustic source has to be a volume velocity acoustic source is what a volume velocity source with some spatial distribution.

So, this is going to be the remaining is what I just one over T square. So, how do I do that I will say it is volume velocity that gives me one over volume over T per unit volume velocity per unit volume this will give me 1 over T because volume cancels out and then I will put $j\omega$ to make it one over T square. So, I have $j\omega$ and an S which is volume velocity per unit volume.

So, this is S has volume velocity per unit volume. So, that is how I get. Now what about the plate? The plate as its own velocity you plate velocity locally there is a plate velocity. So, that has length over time units. So, now what I do is that the plate is a plane the plane in 3D space.

So, out of the $\vec{x}$ space which is 3d it occupies. So, one dimension is reduced one dimension gets reduced that means we remain in a plane.

So, one dimension is dropped and that becomes my y. So, I put. So, what do I do I do take the velocity and then I multiply by a certain delta function which has dimension in it whatever dimension let me put z here temporarily? So, what this gives me is length over time divided by length. So, I have one over time now I have 1 over time. So, what do I do I multiply by $j\omega$ that gives me 1 over time square and I multiply by a density the fluid density which gives me mass over L cube.

Suppose you follow so then what do I do I have $\rho_0 j\omega$ velocity locally multiplied by a certain delta that delta takes me from x space to y space the x dimension 3D to y dimension to 2D that is what that $\delta(z)$ does okay. So, now given this situation I am going to substitute for my pressure the modal sum. So, what does that do I get

$$\nabla^2 \sum_n \psi_n(\vec{x})\, a_n(\omega) + k^2 \sum_n \psi_n(\vec{x})\, a_n(\omega) = \rho_0 j\omega\, S(\vec{x}, \omega) + \rho_0 j\omega\, U(\vec{y}, \omega)\delta(z).$$

So, what are we going to do to this we are going to use the orthogonality. So, we are going to multiply by another mod shape of the same basis and integrate over the volume of the cavity that is what we are going to do.

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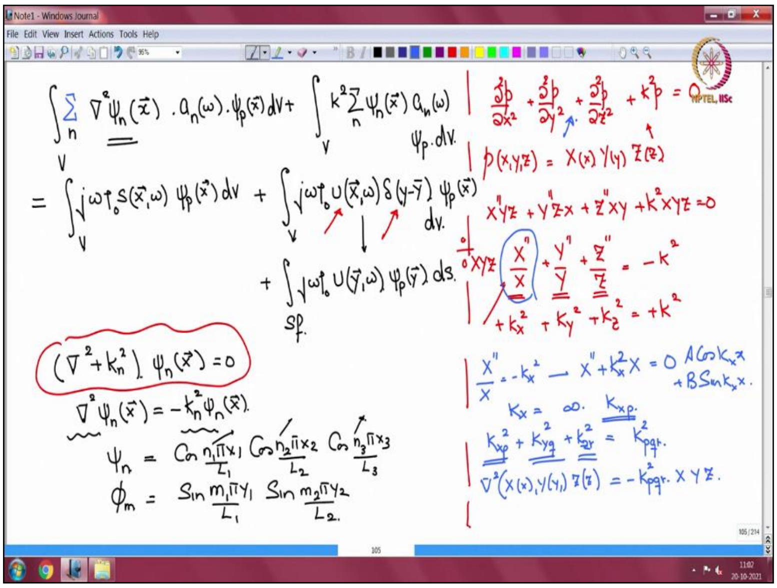

So, let us do that we have

$$\int \sum_n \nabla^2 \psi_n(\vec{x})\, a_n(\omega)\, \psi_p(\vec{x})\, dV$$
$$+ \int k^2 \sum_n \psi_n(\vec{x}) a_n(\omega)\;\; \psi_p\, dV = \int j\omega\rho_0 S(\vec{x},\omega)\, \psi_p(\vec{x})\, dV$$
$$+ \int j\omega\rho_0 U(\vec{x},\omega)\, \delta(y-\vec{y})\psi_p(\vec{x})\, dV + \int j\omega\rho_0 U(\vec{y},\omega)\psi_p(\vec{y})\, dS.$$

Now we have to use a certain bit of information that is relevant to the mode shapes. So, the mode shape if you realize from your acoustics of interior spaces the mode shape satisfies the Helmholtz equation.

$$\left(\nabla^2 + {k_n}^2\right)\psi_n(\vec{x}) = 0.$$

This you should know this that means I am going to use the fact that

$$\nabla^2 \psi_n(\vec{x}) = -{k_n}^2 \psi_n(\vec{x}),$$

this is very important because the derivative suddenly vanishes you see.

The left side is a derivative the right side is an algebraic meanwhile let me tell you that the mode shapes are as follows the acoustic mode shape is for rigid walls

$$\psi_n = \cos\frac{n_1 \pi x_1}{L_1}\,\cos\frac{n_2 \pi x_2}{L_2}\,\cos\frac{n_3 \pi x_3}{L_3}\;,$$

where n_1, n_2 and n_3 are individual mode numbers in the $x_1,\, x_2\; and\; x_3$directions this is for a hard wall enclosure rectangular enclosure then the plate modes are simply supported modes.

So, they are

$$\phi_m = \sin\frac{m_1 \pi y_1}{L_1}\,\sin\frac{m_2 \pi y_2}{L_2}\;.$$

Now just in case you have forgotten how this aspect works out I will just give a sketchy proof or a sketchy derivation of this. So, if we have

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} + k^2 p = 0,$$

that is your Helmholtz equation and then we do this $p(x,y,z) = X(x)Y(y)Z(z)$we are doing separation of variables and we substitute it in here.

So, what we will now get

$$X''YZ + Y''ZX + Z''XY + k^2 XYZ = 0.$$

So, if I divide by if I divide the whole thing by XYZ. So, I get

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -k^2.$$

So, this is exclusively a function of X this is exclusively a function of Y and a function of Z which is equal to a constant.

So, each of these must be a constant. So, I write it as

$$-{k_x}^2 - {k_y}^2 - {k_z}^2 = -k^2.$$

So, if I remove the minus, I get this now

$${k_x}^2 + {k_y}^2 + {k_z}^2 = k^2,$$

if I try to solve each one of these. So, that means $\frac{X''}{X} = -{k_x}^2$ if I try to solve that that ends up as $X'' + {k_x}^2 x = 0$ is like a spring and a mass system. So, this has you know $A\cos k_x x + B\sin k_x x$ type of solutions.

Then we apply the boundary conditions applying the boundary conditions we get values for k_x which are infinite not just one. So, it will acquire in index kind of k_x let us say you know p whatever requires an index p equal to 1, 2, 3, 4 and so forth. So, now what happens is I will have

$${k_{xp}}^2 + {k_{yq}}^2 + {k_{zr}}^2 = {k_{pqr}}^2.$$

So, what that means now is that I have from here $\nabla^2\big(X(x), Y(y), Z(z)\big) = -{k_{pqr}}^2 XYZ$. So, I will stop here we will continue from the next class.