Sound and Structure Vibration
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Module No # 05
Lecture No # 23
Physics of the Coupled Waves

Good morning and welcome to this next lecture on Sound and Structural Vibration.

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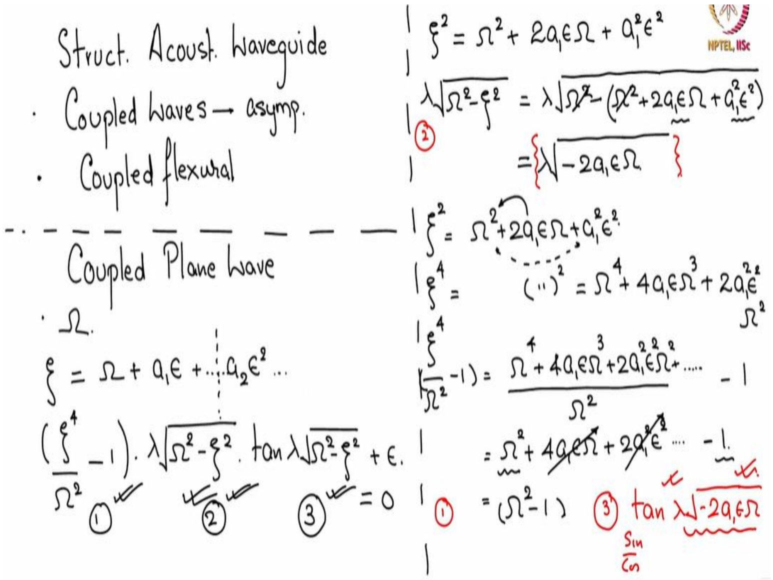

The current topic we are looking at is the structural acoustic waveguide, and we are trying to derive the coupled waves using asymptotic method. So, we just last time looked at the flexural wave we looked at the coupled flexural wave and we derived a correction we will get back to that. The next couple wave we are interested is the coupled plane wave**.** So, the original plane wave had the wavenumber Ω so it is a straight line in the wavenumber plane.

So, now to derive the coupled wavenumber I use this expansion $\xi = \Omega + a_1\epsilon$. I can use higher order corrections like $a_2\epsilon^2$ etc, but for demonstration purpose this is adequate**.** So, now to remind ourselves we have this dispersion equation which we should not forget this will remain with us for this whole topic $\left(\frac{\xi^4}{\Omega^2} - 1\right)\ \lambda\sqrt{\Omega^2 - \xi^2}\ \tan\lambda\sqrt{\Omega^2 - \xi^2} + \epsilon = 0$.

So, we have to substitute that approximation in each of these terms here. So, let us look at we have to look at the plane wave term, so this is one. And so permanently let us remember this is $\left(\frac{\xi^4}{\Omega^2} - 1\right)$ the flexural wave this is $(\lambda\sqrt{\Omega^2 - \xi^2})$ the plane wave these are $(\tan\lambda\sqrt{\Omega^2 - \xi^2})$ the cut-ons. So, let us look at substitution in the plane wave so we need $\xi^2 = \Omega^2 + 2a_1\epsilon\Omega + a_1{}^2\epsilon^2$.

Therefore, $\lambda\sqrt{\Omega^2 - \xi^2}$ looks like $\lambda\sqrt{\Omega^2 - (\Omega^2 + 2a_1\epsilon\Omega + {a_1}^2\epsilon^2)}$. So, this cancels out and now I will shorten my derivation we are looking at the biggest term here ϵ^2 is a very small term compared to ϵ so I will take it as $-2a_1\epsilon\Omega$**.**

$$\lambda\sqrt{\Omega^2 - \xi^2} = \lambda\sqrt{-2a_1\epsilon\Omega}\,.$$

I will ignore the square term so that is the plane wave second terms what about the first? Has ξ^4. So, let us see $\xi^2 = \Omega^2 + 2a_1\epsilon\Omega + {a_1}^2\epsilon^2$**,**

$$\xi^4 = (\Omega^2 + 2a_1\epsilon\Omega + {a_1}^2\epsilon^2)^2.$$

So that turns out to be to the relevant terms that I need $\Omega^4 + 4a_1\epsilon\Omega^3 + 2{a_1}^2\epsilon^2\Omega^2$ all others are higher order in ϵ**.** So

$$\left(\frac{\xi^4}{\Omega^2} - 1\right) = \frac{\Omega^4 + 4a_1\epsilon\Omega^3 + 2{a_1}^2\epsilon^2\Omega^2 + \cdots}{\Omega^2} - 1,$$

So, this now gives me

$$= \Omega^2 + 4a_1\epsilon\Omega + 2{a_1}^2\epsilon^2 + \cdots - 1.$$

Now again we take the dominant term so these are small compared to this and this order one terms so these can be ignored so this effectively is

$$\left(\frac{\xi^4}{\Omega^2} - 1\right) = (\Omega^2 - 1).$$

So, that is one now the tan term has the same thing inside as this so let me write it here term 3 the tan $\lambda\sqrt{-2a_1\epsilon\Omega}$**.** So, tan for small values is small value because numerator is sin right tan is sin by cos.

So, for small values of this sign is the argument itself and cos is one so we know this is going to be just the argument.

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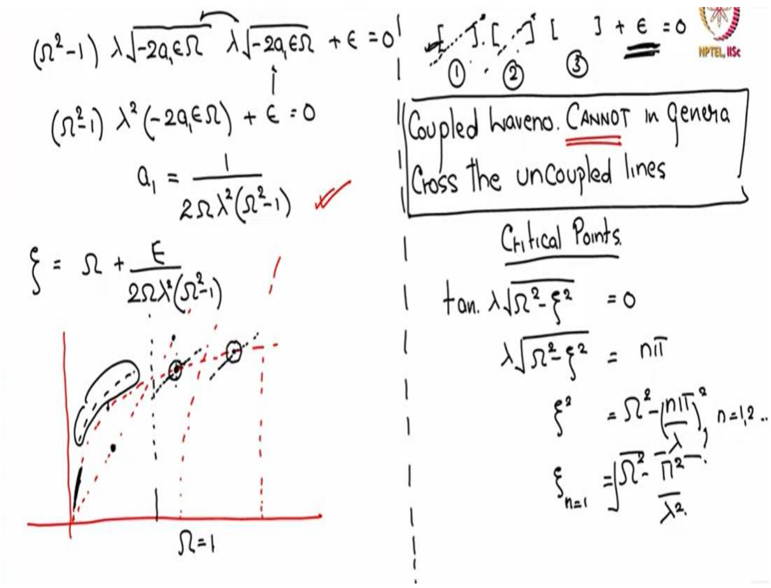

So, let us put this together so if I put it together, I have

$$(\Omega^2 - 1)\lambda\sqrt{-2a_1\epsilon\Omega}\,\lambda\sqrt{-2a_1\epsilon\Omega} + \epsilon \ = 0,$$

$$(\Omega^2 - 1)\lambda^2(-2a_1\epsilon\Omega) + \epsilon \ = 0.$$

So now this can be balanced there are 2 terms with ϵ value so if we balance, I will get

$$a_1 = \frac{1}{2\Omega\lambda^2(\Omega^2 - 1)}.$$

So now what is the total coupled wave coupled plane wave is given by original plus the correction.

$$\xi = \Omega + \frac{\varepsilon}{2\Omega\lambda^2(\Omega^2 - 1)}.$$

So, just to show you what is happening?

I will do the uncoupled waves in a in a very light dotted manner. So, this is the uncoupled acoustic plane wave, then this is the uncoupled flexural wave, then these are the rigid duct cut-ons, this is the second rigid duct cut-on. So now we have done the flexural wave and the plane wave right. So, the flexural wave also had a two-way description plus and minus. So, this is my coincidence you should know that by now this is my coincidence frequency ω_c or in the non-dimensional terms $\Omega = 1$.

So, the flexural way first below coincidence it looks like this and above coincidence it looks like this it has there is a point where it crosses there is a point which where it crosses the uncoupled wave it is below before that point it is above beyond that point**.** Similarly, here also again there is another point it crosses and below it is lower above its higher**.** Now a point that should be mentioned here is that again we have 3 terms the flexural term which is 1 multiplied by the plane wave term which is 2 multiplied by the cut-ons which is 3 + ε = 0.

Now the coupled wave I will write it here the coupled wavenumbers cannot in general cross the uncoupled lines is a rule they cannot cross the uncoupled lines. What am I trying to say is that? This coupled line this coupled solution that I am trying to find this coupled solution for example the flexural wave cannot go and touch this red dotted line or cross this red dotted line.

But it is doing so here that that reason I will give you later just one point it is doing I will give that reason later. But otherwise, it cannot the reason is that when this uncoupled line for example is these solutions so this exactly goes to 0 on the coupled uncoupled line this term. If I land on this uncoupled flexural curve first number one goes to 0. If number one goes to 0 you can there is a balance ε sitting here which cannot be balanced.

Similarly, number 2 is an uncoupled plane wave so if my solution exactly lands on the plane wave here or here let us say then number 2 is exactly 0. Then there is this ε which cannot be balanced so, the new solutions in general cannot cross the old solutions**.** But there are critical points where they do so we will look at the critical points there are critical points on the diagram where crossing occurs, we will look at those**.**

Now the third solution is where $\tan \lambda\sqrt{\Omega^2 - \xi^2}$ goes to 0. Which means $\lambda\sqrt{\Omega^2 - \xi^2}$ is equal to some $n\pi$**,** or my $\xi^2 = \Omega^2 - \left(\frac{n\pi}{\lambda}\right)^2$ for some values of n, going from $1, 2$ etc. That means the first fellow first coupled rigid duct cut-on let me call it $n_1$is given by

$$\xi_{n=1}{}^2 = \Omega^2 - \frac{\pi^2}{\lambda^2}.$$

Now I will show you see these are cumbersome to do by hand. This calculation that I am showing here still they are probably manageable but in order to do them by hand is cumber some and the geometry is simple, but geometry gets complicated these terms are all complicated, so we actually use a software called Maple**.** So, I will show you calculations on Maple how to do this**?**

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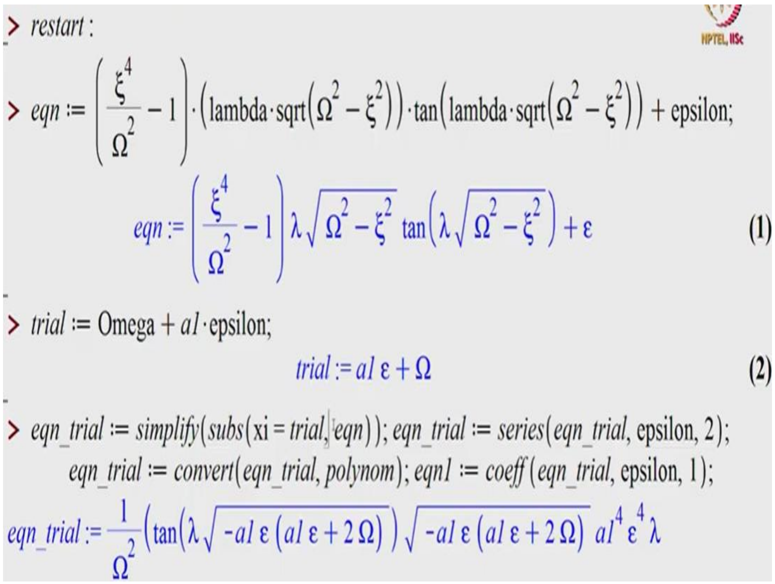

So let me show you those so let us see this is a Maple sheet I am showing you the formulas are already there you can see**.** So, starting is using a restart command that means everything is reset to 0, so let me run it restart. Then I have set up the equation which is $\left(\frac{\xi^4}{\Omega^2} - 1\right) \lambda\sqrt{\Omega^2 - \xi^2}\ \tan \lambda\sqrt{\Omega^2 - \xi^2} + \epsilon$ so if I type return, I get that equation**.**

Now I am trying to look at the plane wave coupled plane wave, so I have a trial solution $\Omega + a_1\epsilon$. Now thus trial solution has to be substituted, so as I show you can learn Maple commands also so one is I am going to substitute which is the subs command ξ is equal to trial in equation so equation is given above, and I am saying simplify it because there are fourth power and so forth simplify the equation**.**

Then what I do these are Maple system things that you have to do otherwise it does not work. So now I am going to expand in a Fourier series the resulting equation I am going to expand in a Fourier series**,** about what about ϵ equal to 0. So that this equation trial is going to be expanded in a Fourier Taylor series not Fourier series my apologies Taylor series around epsilon equal to 0 up till 2 terms that is what this says**.**

And there is for further a little bit of adjustment you have you have to do I use a command convert whatever is resultant as a Polynomial**.** Otherwise, it adds the higher order terms as order epsilon square and cube, which cannot be dealt with so polynomial this command will remove the extra terms**.** Then what I do is? Out of that resulted equation I collect that particular equation which is coefficient of ϵ to the power 1 because that is the one, I have to balance so let me run this command**.**

(Refer Slide Time: 21:41)

$$+4\tan\left(\lambda\sqrt{-a1\,\varepsilon\,(a1\,\varepsilon+2\,\Omega)}\right)\sqrt{-a1\,\varepsilon\,(a1\,\varepsilon+2\,\Omega)}\;\Omega^{3}\,a1\,\varepsilon\,\lambda$$
$$+\tan\left(\lambda\sqrt{-a1\,\varepsilon\,(a1\,\varepsilon+2\,\Omega)}\right)\sqrt{-a1\,\varepsilon\,(a1\,\varepsilon+2\,\Omega)}\;\Omega^{4}\,\lambda$$
$$-\tan\left(\lambda\sqrt{-a1\,\varepsilon\,(a1\,\varepsilon+2\,\Omega)}\right)\sqrt{-a1\,\varepsilon\,(a1\,\varepsilon+2\,\Omega)}\;\Omega^{2}\,\lambda+\varepsilon\,\Omega^{2}\Big)$$

$$eqn\_trial := \frac{-2\,\Omega^{5}\,a1\,\lambda^{2}+2\,\Omega^{3}\,a1\,\lambda^{2}+\Omega^{2}}{\Omega^{2}}\,\varepsilon+O\left(\varepsilon^{2}\right)$$

$$eqn\_trial := \frac{\left(-2\,\Omega^{5}\,a1\,\lambda^{2}+2\,\Omega^{3}\,a1\,\lambda^{2}+\Omega^{2}\right)\varepsilon}{\Omega^{2}}$$

$$eqn1 := \frac{-2\,\Omega^{5}\,a1\,\lambda^{2}+2\,\Omega^{3}\,a1\,\lambda^{2}+\Omega^{2}}{\Omega^{2}} \qquad (3)$$

So, I have already run it so you can see, I will take you the so this is the first substitution it looks horrendous then what I do I make a series of it till to just 2 terms. So, I get this term and you can see it is telling me there are higher order terms, I need to remove this higher order term so the next one removes that, so it does the polynomial command it removes that.

Now within there is only one term fortunately here because I have expanded only 2 terms and I remove the higher order term. Then I collect the equation which is coefficient of just ϵ to the power 1 which is this.

(Refer Slide Time: 22:43)

$> soln := solve(eqn1, a1);$

$$soln := \frac{1}{2\,\Omega\,\lambda^{2}\left(\Omega^{2}-1\right)} \qquad (4)$$

$> soln\_tot\_ac := \mathrm{Omega} + soln\cdot \mathrm{epsilon};$

$$soln\_tot\_ac := \frac{1}{2}\,\frac{\varepsilon}{\Omega\,\lambda^{2}\left(\Omega^{2}-1\right)}+\Omega \qquad (5)$$

$> trial2 := \Omega^{\left(\frac{1}{2}\right)} + a1\cdot \mathrm{epsilon};$

$$trial2 := \sqrt{\Omega}+a1\,\varepsilon \qquad (6)$$

$> eqn\_trial2 := simplify(subs(\mathrm{xi} = trial2, eqn)); eqn\_trial2 := series(eqn\_trial2, \mathrm{epsilon},$

Which is this now I have to solve it for a_1 because that is the correction term, so now I solve it and I get this is the correction term. So, what is the total new coupled wave it is given by the total and ac means acoustics acoustic wave number so this original Ω plus this correction term. So, you can this is what I derived on the tablet also**.** So that is the acoustic wave correction now we will do the correction for the flexural wave.

So here I am taking the flexural wave correction which is $\Omega^{1/2} + a_1\epsilon$. So, I get this, and I go through the same set here substitute this into the original equation.

(Refer Slide Time: 23:43)

$$+ 6\tan\left(\lambda\sqrt{-a1^2\varepsilon^2 - 2\sqrt{\Omega}\,a1\,\varepsilon - \Omega + \Omega^2}\right)$$

$$\sqrt{-a1^2\varepsilon^2 - 2\sqrt{\Omega}\,a1\,\varepsilon - \Omega + \Omega^2}\;\Omega\,a1^2\,\varepsilon\,\lambda + \Omega^2\Big)\Big)$$

$$eqn\_trial2 := \frac{4\tan\left(\lambda\sqrt{\Omega^2-\Omega}\right)\sqrt{\Omega^2-\Omega}\;\Omega^{3/2}\,a1\,\lambda + \Omega^2}{\Omega^2}\,\varepsilon + \mathrm{O}\left(\varepsilon^2\right)$$

$$eqn\_trial2 := \frac{\left(4\tan\left(\lambda\sqrt{\Omega^2-\Omega}\right)\sqrt{\Omega^2-\Omega}\;\Omega^{3/2}\,a1\,\lambda + \Omega^2\right)\varepsilon}{\Omega^2}$$

$$eqn2 := \frac{4\tan\left(\lambda\sqrt{\Omega^2-\Omega}\right)\sqrt{\Omega^2-\Omega}\;\Omega^{3/2}\,a1\,\lambda + \Omega^2}{\Omega^2} \qquad (7)$$

Make a Taylor series collect the coefficient of ϵ to the power 1 and I get this**,** see this is a simple geometry yet it looks quite complicated.

(Refer Slide Time: 23:56)

> $soln2 := solve(eqn2, a1);$

$$soln2 := -\frac{1}{4}\,\frac{\sqrt{\Omega}}{\tan\left(\lambda\sqrt{\Omega^2-\Omega}\right)\sqrt{\Omega^2-\Omega}\;\lambda} \qquad (8)$$

> $soln\_tot\_fl := \Omega^{\left(\frac{1}{2}\right)} + soln2\cdot \mathrm{epsilon};$

$$soln\_tot\_fl := \sqrt{\Omega} - \frac{1}{4}\,\frac{\sqrt{\Omega}\,\varepsilon}{\tan\left(\lambda\sqrt{\Omega^2-\Omega}\right)\sqrt{\Omega^2-\Omega}\;\lambda} \qquad (9)$$

> $\mathrm{epsilon} := 0.25 : \mathrm{lambda} := 3.0 : plots[multiple]\Big(plot, [\mathrm{Omega}, \mathrm{Omega} = 0..2.0, color$

So, then I solve for a_1 that is the solution, then I add it to the original that is my total correction I mean total new wavenumber fl implies flexural and this is what it is? Now so I am going to plot there I will select $\epsilon = 0.25$ and $\lambda =$ say 3.0 and there is a plotting command I plot it and that is how it looks let me try to raise the size. So now let us see this, so the blue dotted line and the green dotted or dashed line are uncoupled waves.

Let me say this that uncoupled implies when there is flexure when there is a plate there is no fluid. And when there is a plane wave the plate is rigid so when one medium is there the other medium is not there. So, it is kind of in some way incorrect to draw these two pictures together, because when the plane wave is there the plate is not vibrating in the when the plate is vibrating there is no acoustic fluid those are that is what is meant by uncoupled.

But these lines together give us markers so now on top of that we plot the coupled curves so this black line is the coupled flexure you can see it is gone above. In this region this is the coincidence line below it is gone above and as it approaches coincidence it starts to blow up. So, the actual solution does not blow up our method is inadequate the expansion we have used $\Omega^{1/2} + a_1\epsilon$ that expansion is inadequate and so the result is not correct a_1 value is not correct.

The actual solution does not blow up if we do it numerically it will come out right, so what that means is we have to watch, and it searches regions we have to find a new expansion. So, we will do a patch work of expansions. So, this is one critical point where the solution blows up a_1 value blows up, so we do not take the value till there so the flexural correction is this black line.

Further the next flexural correction as you saw the quad term starts to oscillate so close to this flexural curve is again this black line so there is a point where it crosses is another critical point. So now just before that point the wavenumber is below the flexure just above the value is above the flexural wave and this repeatedly continues. And between every two rigid cut-ons there is one point like that where the crossing will occur, so that is the flexural wave.

Next, we have look at the plane wave the blue dotted line is the uncoupled plane wave, and the correction to it below coincidence is below and then above coincidence it is just above that is the plane wave correction term. We can now let us see if we can do it over here let me try this on the fly.

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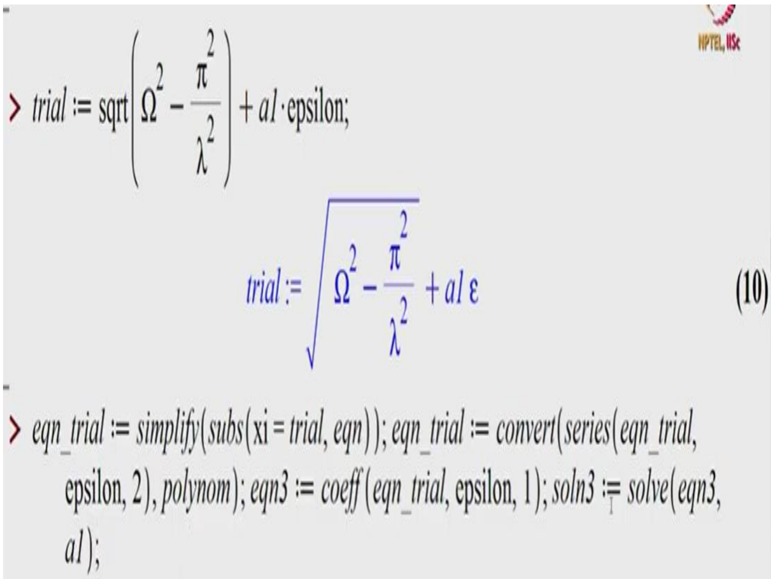
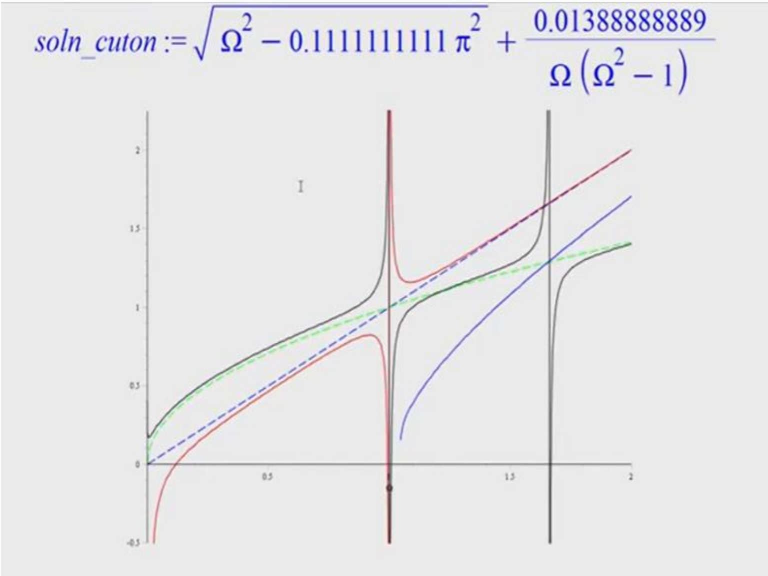


So, I will say that trial is equal to square root of Ω square that is why I have it written down. So, this is now my first rigid cut-on then I do a correction for it + a_1 star ϵ. So, I will have to start from the top because it is taking the λ values so I will just start see the λ, ϵ values were defined and so it is taking the values which at this point I want them to be symbols**.**

So now we go through equation underscore trial equal to simplify substitute ξ equal to trial in equation close this. Then equation trial is equal to series equation trial ϵ two terms and I will say polynomial. So, I am doing it in one shot let us see this works and we will also say that equation let us see equation 3 equal to coefficient of equation trial ϵ, 1. See if it works it seems to work hopefully, so now let us try and plot this well we still have to solve it of course we still have to solve this thing.

Let us let us solve it so we will call it solution 3 is equal to solve equation 3 for a_1 let us see if it works, I do not know if it worked**.** Let us plot and see multiple plots so we have to do here is that solution cut-on is equal to square root of Ω at 2 $-\pi$ at 2 slash. So let us plot so this happens to be the solution cut-on correction let us plot if it works**.** So, I will copy this solution cut-on and we will give it a colour what colour will we give it Let us give it blue.

(Refer Slide Time: 36:37)



Let us do it blue we got the curve right I think time is up and we will explain it in the next class.