Sound and Structural Vibration
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Module No # 05
Lecture No # 21
Derivation of Coupled Waves Using Asymptotic Method

Good morning and welcome to this next lecture on sound and structural vibration as you can see, we ended in the last class by deriving the non-dimensional form of the coupled dispersion equation.

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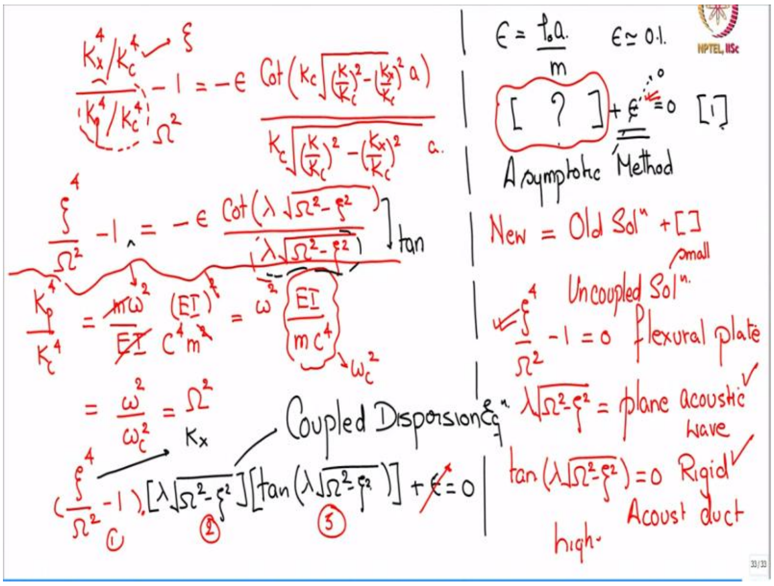

This is called the let me call this the coupled dispersion equation today we will see what it means and how to solve it? Now ϵ I said was $\frac{\rho_0 a}{m}$. Now typically this is a small quantity in the sense of ϵ is in the range of 0.1 and so on and so forth. Now ϵ is a small quantity and I have a certain equation plus a small quantity = 0.

$$[\ ?\ ] + \epsilon = 0. \qquad [1]$$

And I have to solve for some unknown in here we have seen this in the first problem we saw the classical problem from Crighton. That when you have a small parameter, we use asymptotic methods we use the asymptotic method. So again, to repeat the idea is that if we know the solutions to this particular equation let me again call it some number will call it one. If you know the solution to this equation 1 when ϵ is 0 and this ϵ is 0 if we know the solutions.

Then if ϵ is added as the small quantity the intuitive idea is that my solutions here should not have changed that much by the presence of the small quantity that is the idea. So, if I know these solutions when ϵ was 0 and my query is what are they when ϵ is not 0 but a small quantity. Then they should be the new solution must be a perturbation that means old solution plus something small.

It cannot be hugely different which makes physical sense so first of all let us see when ϵ is 0 what are we getting. When ϵ is 0 we get these 3 we get $\frac{\xi^4}{\Omega^2} - 1$ and we set it 0. We put ϵ to 0 and the equation is set to 0 then there are 3 products $1, 2, 3$ there are 3 terms in the product. And so, anyone of them can be 0 so this can be 0 for example.

And this represents the flexural wave solution wave in the 1D plate will see that again then we have $\lambda\sqrt{\Omega^2 - \xi^2}$ which represents the plane acoustic wave in the duct. And then lastly the $\tan\left(\lambda\sqrt{\Omega^2 - \xi^2}\right)$ equal to 0 this is the rigid acoustic duct higher order waves.

So, there are 3 solutions so these are what I call uncoupled solutions. When the fluid is bounded by a rigid wall it behaves on its own that is the plane acoustic wave and rigid duct higher order wavenumbers. When the flexural plate is not in contact with the fluid that is in vacuum, so this is the solution. So, these are the uncoupled solutions now when ϵ is brought in the solutions will become coupled.

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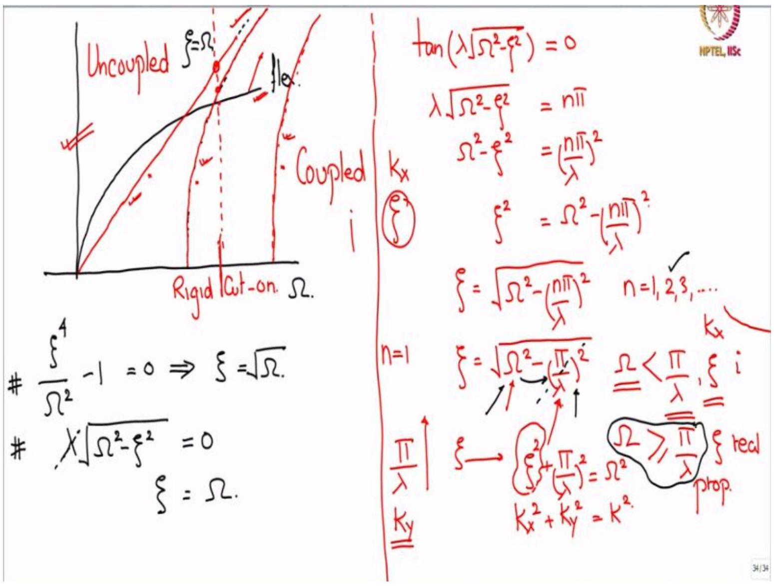

But further to this let us see on a diagram what these are so let me take an axis this is the non-dimensional frequency axis. So let us look at $\frac{\xi^4}{\Omega^2} - 1 = 0$ which implies that ξ is equal to $\sqrt{\omega}$. So, this is the non-dimensional flexural wave in the plate. So, it will look like this, so this is the flexure next we have $\lambda\sqrt{\Omega^2 - \xi^2}$ when that is equal to 0.

λ is dimensional so it is not 0 then we get ξ is equal to of course $\pm\Omega$ so we will take Ω. So, ξ is equal to Ω is this plane wave 1D sound wave and lastly, we have $\tan\left(\lambda\sqrt{\Omega^2 - \xi^2}\right) = 0$ and this is the tan. So, what I will say is $\lambda\sqrt{\Omega^2 - \xi^2}$ must be some $n\pi$. Or $\Omega^2 - \xi^2 = \left(\frac{n\pi}{\lambda}\right)^2$ or $\xi^2 = \Omega^2 - \left(\frac{n\pi}{\lambda}\right)^2$ or ξ is equal to $\sqrt{\Omega^2 - \left(\frac{n\pi}{\lambda}\right)^2}$.

We will take n going from $1, 2, 3$ etc., now for example if we take $n = 1$ then my ξ is equal to $\sqrt{\Omega^2 - \left(\frac{\pi}{\lambda}\right)^2}$. So, π is a fixed number λ is a fixed number. Ω is the only variant in here Ω varies so if Ω is less than $\frac{\pi}{\lambda}$ then ξ is imaginary. We have seen that if you have something imaginary it amounts to a decaying wave.

So below for the values of the non-dimensional frequency below $\frac{\pi}{\lambda}$, ξ will decay away. ξ is an actual wave number dimensional form is k_x so in the axial direction it decays away. So, it is not of interest if it is decays away a little distance away you do not pick it up. So, it is not of interest whereas if Ω just is greater than or equal to $\frac{\pi}{\lambda}$ then ξ starts to emerge as a real quantity.

That means propagation that means this wave cuts-on so the non-dimensional wave number $\frac{\pi}{\lambda}$ is in the y direction. Whereas ξ happens to be in the axial direction from here you can see that $\xi^2 + \left(\frac{\pi}{\lambda}\right)^2 = \Omega^2$ is very similar to ${k_x}^2 + {k_y}^2 = k^2$. So now $\frac{\pi}{\lambda}$ is non-dimensional wavenumber in the k_y direction it is related to a shape of the wave in the y direction.

Because y is bounded on the top and bottom between 0 and a so ξ can be propagating can be non-propagating. And for Ω greater than $\frac{\pi}{\lambda}$ it becomes propagating so how do we plot this thing? So, at very large values of Ω you can see because $\frac{\pi}{\lambda}$ is a fixed number does not change. So, at very large values of Ω, Ω^2 much greater than $\left(\frac{\pi}{\lambda}\right)^2$ so you can neglect $\left(\frac{\pi}{\lambda}\right)^2$ term and then I get $\sqrt{\Omega^2}$ which is Ω.

So, ξ equal to Ω is my plane wave so this thing will start somewhere over here and as time I mean with frequency it will start approaching and it will approach the plane wave ξ this is ξ equal to Ω the plane wave. But this cut-on wave will approach this fellow at a higher and higher values of Ω let me put it in red ink. So, this is a higher order wavenumber but because it has a cut-on behavior, we call them it is rigid duct cut-ons.

I will use that terminology similarly if I take n higher value, I take $n = 2$ here then I get a higher order cut-on which will start again somewhere here. We will start of vertically and then it will turn, and it will start following this plane wave at higher and higher frequencies and similarly $n = 3$ and so forth. That means if you decide the frequency is here excitation frequency is here then you take it vertically then these are the possible waves that can exist in your acoustic system that's what it means.

Just a word of caution what we have plotted is uncoupled use this language commonly, so these are uncoupled that means actually when there is a plate vibrating there is no fluid. So, these waves should not be there its uncoupled similarly when these plane wave and higher order cut-ons are there, there is no flexible wall, wall is rigid so this curve should not be there. But we use this language we plot both of them on the same plot and look at their relationship.

But in actuality what happens? We are going to find the coupled wavenumbers that is the objective we are going to find the coupled wavenumber dimensional wavenumber k_x or non-dimensional ξ we are going to do that. So, then we will plot ξ on the same graph to see how they change. So, these are the 3 types of waves that exist the red ones are the fluid waves the black one is the plate wave.

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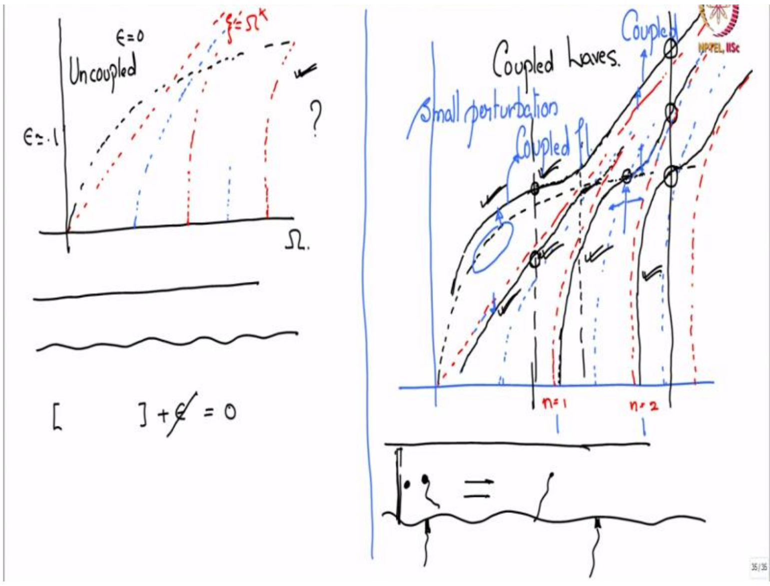

Now in the next picture I will give you an idea here of what is going to happen again this is my Ω these are I am going to uncoupled. I am going to plot them a little light this is the uncoupled plane wave I will show you what I mean by a plane wave ξ equal to Ω plane wave. Then the next cut-on at $n = 1$ it takes off vertically and starts to curve and asymptotically meet the plane wave then the $n = 2$ rigid duct cut-on it takes out vertically too and will start turning and it higher and higher it will approach this one.

And then we have the plate uncoupled wavenumber that also I will plot in a dotted manner. Now what happens the when the coupling has occurred that means now I have a structural acoustic wave guide fluid couple the plate, plate couples the fluid. This picture is going to change this is uncoupled picture but now because they are coupled this picture will change.

So, the question is how it will change that the question we are asking from the coupled dispersion equation we have with this ϵ quantity the question we ask is that how does this picture change. By setting ϵ to 0 this is the picture we got this is the ϵ = 0 picture uncoupled. But when ϵ is not equal to 0 a small quantity what is this picture look like?

So, I will give you the story ahead of time so let me see if I can so there are these extra lines in the middle, I will talk about them little later there are these extra lines. So, I might need other pictures so let us see so let me take it here, so I have these red colored acoustic wave the first cut-on let me say $n = 1$ cut-on. So, I will use the language cut-on $n = 1$ and I have the $n = 2$ cut-on then I have the flexural wave let me exaggerate.

Let me even take $n = 3$ here as I said these are uncoupled and they will approach this plane wave at higher frequency. Now I will give you the picture of the coupled wave just the little ahead of the story. So, because ϵ is small every wave gets modified slightly is not a strong modifier to the original wave. So, this wave looks a little like this over here let me plot it looks like this.

In the coupled case so let me I said there are these extra invisible lines let me plot them in a light manner. These extra lines here for now let them be just lines for some convenience I will give the physical picture meaning later. So now this flexural wave how does it change so there will be an acoustic plane wave will change get modified so let me use black get modified to this.

And then as we approach to the flexural wave it becomes the flexural wave and it crosses at this blue line interrupt section. And it follows the next rigid duct cut-on so I will show it more clearly here so this rigid duct cut-on will become this black line over here and as we approach this flexural wave line it turns and then it crosses at a certain point. So, it is below this frequency it is below the flexural wave above this frequency above the flexural wave.

And then it starts to follow the next higher rigid duct cut on asymptotically so this red fellow will become this black line. As we approach this flexural wave returns crosses at the blue line interaction and follows the next rigid duct cut-on. So, this is the story so now what happens to this it will have this behaviour and add this junction which is the coincidence this is a coincidence frequency where the plane wave cuts the flexural wave.

There it will turn and starts following the acoustic plane wave just above, so these black solid lines are the coupled waves. So that means what now in this system over here as they are coupled if I put a pressure sensor in the fluid or I put an accelerometer on the plate somewhere at a certain frequency at a certain non-dimensional frequency. If I take a vertical cut, I will see this wave is dark black line and this dark black line I will see these two waves.

Or these two waves are possible the dispersion equation gives possibility of waves in actuality what will be has to be solved for using a force problem. If I force this system in some manner using some piston or something, then we will see what the propagating waves are that is the actual solution? What dispersion equation does is it gives you this entire set of waves that are only possible whether they will be there or not will depend on the type of forcing you are going to do.

Then you will solve a forced problem which is a more difficult problem so at this frequency if I draw a vertical line this is a possible wave that is a possible wave. But now these are coupled so whether you measure it in this sound field or you measure it on the plate they will pick it up both of them are carrying the same way they will pick it up your sensor will pick it up. Or let us say if you shift to a higher frequency somewhere here.

And I draw now a vertical line then these crossings this and this are the possible waves that could exist in your system. And again, if you put a pressure sensor somewhere or an accelerometer somewhere you will pick up these 3 waves in either of the mediums you will pick up these waves. Now mathematically I would like to arrive at these answers I have given you a schematic picture mathematically we would like to arrive at these answers.

So now how do we do that another thing while we are at this picture the let me use may be blue now. So, there is a physical reason why this flexural wave moved upward there is a physical reason why this red plane wave moved downwards. There is a physical reason why at this frequency below you have the wavenumber below and above the wavenumber is above the flexural wavenumber there is a reason why this black line asymptotically now starts to approach the next rigid cut-on.

So, the physical reason why here the plane wave went up whereas here the plane wave went down. Now because the coupled waves are actually small perturbation to the original waves the language, I will use is that I will use the same name but put coupled in front of it. So, to me this is a plane wave but now it is a coupled plane wave to me this is a flexural wave. But now it is a coupled flexural wave, and I will refer to the region based on the frequency.

So here I will call it coupled flexural wave below coincidence and this I will call coupled flexural wave above coincidence between the first and second rigid duct cut-ons. So, I will use this language we are very close to the end of this half an hour. So instead of starting next section I will close here. So, we will start finding the actual mathematical solutions in the next class thanks.