Sound and Structural Vibration
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Lecture – 2
Harmonically Excited Systems

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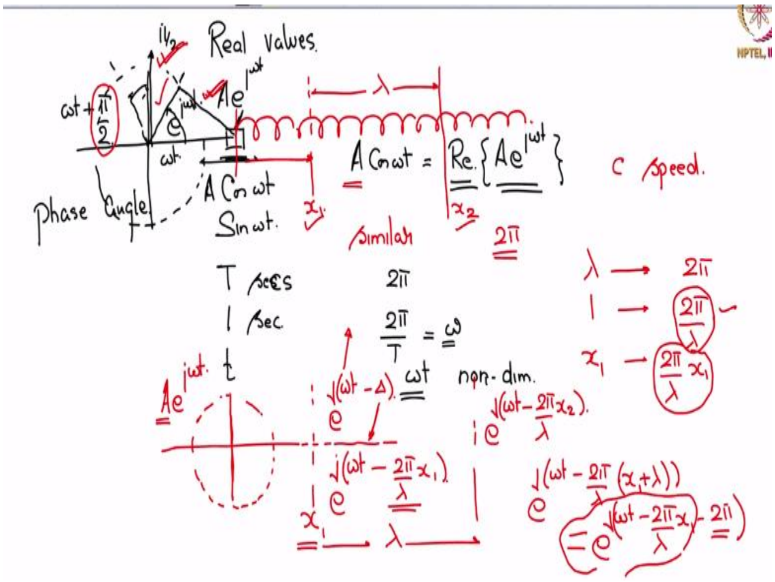

Welcome to you all. Now, we said that we are interested only in real values, right, only in real values although we are bringing in the complex phasor notation we are interested in real values. That means as this slider rotates or as this crank rotates and this mass does oscillatory movement, this mass is doing oscillatory movement back and forth, so either a $\cos(\omega t)$ type movement or a $\sin(\omega t)$ type movement.

So, then how do we get it back? We get it back by saying that $A\cos(\omega t)$ which could be the movement of this mass. This mass could be moving like $A\cos(\omega t)$, this is equal to the real part of $Ae^{j\omega t}$.

$$A\cos(\omega t) = \mathrm{Re}\{Ae^{j\omega t}\}.$$

So, this crank is rotating at $e^{j\omega t}$. It is rotating as $e^{j\omega t}$ and therefore there is some amplification due to this slider and so let us say that slider has an amplification and becomes $Ae^{j\omega t}$.

And out of that I am trying to pull a real value movement for my mass, so it is the real part of $Ae^{j\omega t}$ or the imaginary part whichever I want and this angle that is covered ωt is the phase angle. So, we said that we will consider the starting to be vertical at $\frac{\pi}{2}$. So, the phase at any instant for this first mass is going to be measured from here, so it is going to be $\omega t + \frac{\pi}{2}$.

Now, how does it work out? So, in T seconds, the phase covered by the phasor is 2π and so in one second the phase covered is $\frac{2\pi}{T}$ which is ω. So, in small t seconds the phase covered is ωt, this is actually a non-dimensional number that is why it is an angle, non-dimensional number. So that is what is covered by the phasor and to that you add the reference value your starting value.

So, that happens to be the phase of the first mass point and beyond that is our spring. Now, let us see over here the information as it travels along the spring, it takes time right we said c is the speed, so it takes time to travel. So if I assign a phasor to the starting point, let us say that phasor we will align with the original phasor here. So, it moves its own descriptor moves like this and we assume there is is an amplification A.

So, let this phasor be $Ae^{j\omega t}$. This A is unimportant, but let us just keep it because if we say this is 1, there might be some amplification of A over here, so well let us keep it. Now, as the pulse moves along the spring, if I look at a certain location $x_1$it is delayed. So, it will see the information at a later time. So, the phasor should be representable here as $e^{j(\omega t - \Delta)}$.

But ωt being the phase or being the angle, this should also have the same unit, a non-dimensional angular unit. So, now what should that be? Now, we say that wavelength happens to be the distance between two points, let us say these are two points, wavelength the distance between two points undergoing similar motion or same motion that means the phase is 2π, right, because the phasor went round once and these two points were generated.

So, the phase difference between them is 2π. So, over λ distance the phase covered is 2π. So, over unit distance the phase covered is $\frac{2\pi}{\lambda}$. So, over a distance x_1, the phase covered is $\frac{2\pi}{\lambda}x_1$. So, at $x_1$location the phasor will be denoted by $e^{j\left(\omega t - \frac{2\pi}{\lambda}x_1\right)}$ that is the amount of phase delay. This is a phase delay over unit distance, this is the phase delay over x_1.

So, that is the amount of phase delay, so that is what the location at $x_1$will be seen. Another location $x_2$let us say will be seeing $\mathrm{e}^{\mathrm{j}\left(\omega t \,-\frac{2\pi}{\lambda}x_2\right)}$and similarly if there was a point λ away from x_1, let us call it $x_1 + \lambda$, let us do this here.

So, let us say $\frac{2\pi}{\lambda}$ and it is λ upon more than x_1. So, this is going to be equal to $\mathrm{e}^{\mathrm{j}\left(\omega t - \frac{2\pi}{\lambda}x_1 - 2\pi\right)}$.

$$\mathrm{e}^{\mathrm{j}\left(\omega t - \frac{2\pi}{\lambda}(x_1 + \lambda)\right)} = \mathrm{e}^{\mathrm{j}\left(\omega t - \frac{2\pi}{\lambda}x_1 - 2\pi\right)}.$$

So, 2π gives me just one. So, a point λ away from x_1 behaves like x_1, so that is the answer I get.

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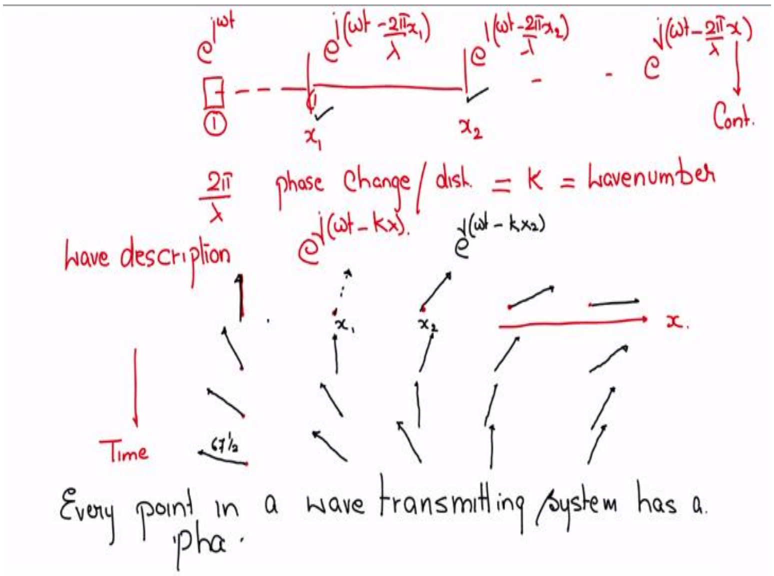

So, now if we look at my spring, so this first mass is moving like $\mathrm{e}^{\mathrm{j}\omega t}$ at a distance x_1, I have a phasor that moves like $\left(\omega t \;-\; \frac{2\pi}{\lambda}x_1\right)$, at a distance x_2 I have $\mathrm{e}^{\mathrm{j}\left(\omega t \,-\frac{2\pi}{\lambda}x_2\right)}$ and so forth. And if my x is a continuous variable, not discrete points, I will say the description along the spring for a continuous variable x is $\frac{2\pi}{\lambda}x$, x is now continuous along the spring.

It is a continuous variable along the spring, not discrete points. This entity $\frac{2\pi}{\lambda}$ which is actually phase change per unit distance is given the symbol k and the name wave number. Wave number is a very important entity in acoustics and sound structure interaction. The symbol is dedicatedly used for wave number. So that means what? The wave description has become $\mathrm{e}^{\mathrm{j}(\omega t - kx)}$.

So, now just to drive home the point a little bit more I will take stations, I will take this station right here the 1, the mass the origin. I take another station a little to the right, one more little to the right, on more little to the right and may be the last one and this is my spatial direction and this below vertical will be my temporal direction. This way is my temporal direction, increasing time.

So let us say here I begin as before as my phasor is vertical. Now this point a little bit later is delayed, so that phasor is let us say some $22\frac{1}{2}^{\circ}$ behind. Another point to the right is let us say 45° behind. Why is that because it is $e^{j\omega t}$ and delayed by k times let us say x_1, x_2, k times x_2. Similarly this one is say delayed by $67\frac{1}{2}^{\circ}$ and this is delayed by 90°.

Now here what do we have? This is advancement in time, so the phasor let us say moves forward $22\frac{1}{2}^{\circ}$ and here it is $22\frac{1}{2}^{\circ}$ behind, so this is going to be vertical. Here it is further $22\frac{1}{2}^{\circ}$ behind, it is going to be here. Here it is going to be 45° behind and here it is going to be $67\frac{1}{2}^{\circ}$ behind and so on. Here this will be 45° ahead.

Here it will be $22\frac{1}{2}^{\circ}$ ahead. Here it will be vertical. Here it will be $22\frac{1}{2}^{\circ}$ behind. Here will be 45° behind. Here this will be 67, my angles are a bit off but here this is a $67\frac{1}{2}^{\circ}$ ahead. Here it is 45°. Here it is $22\frac{1}{2}^{\circ}$. Here it is vertical. Here it is $22\frac{1}{2}^{\circ}$ behind and so forth and all points in between, x is a continuous variable time in a continuous way.

So, every point in a wave transmitting system or a wave bearing system has a phasor description. I am having some trouble here, so let us see.

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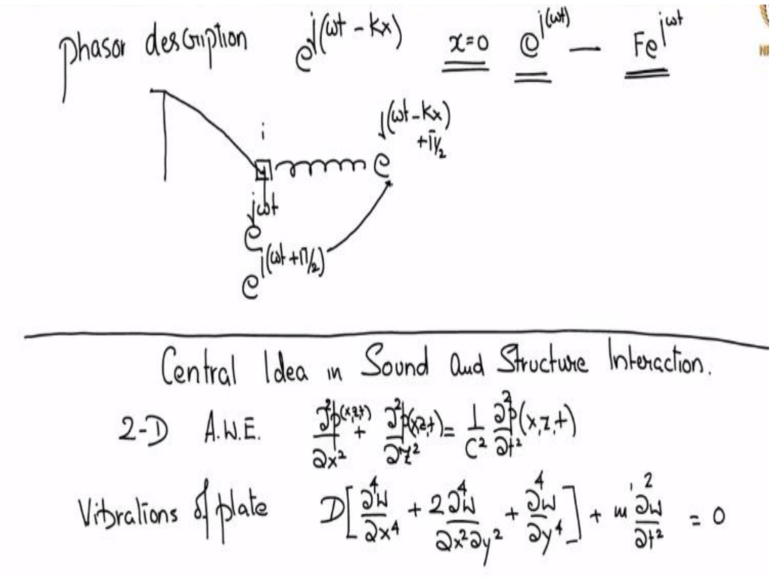

Has a phasor description of this form $e^{j(\omega t - kx)}$. At some reference $x = 0$ whatever it may be you choose it is going to be $e^{j\omega t}$. So, you can consider that to be your reference for phase, many times this happens to be the forcing agency, so your forcing agency is $Fe^{j\omega t}$.

So, the rest of the response is phased with respect to the forcing agency and therefore the absolute phase is not important, it is with respect to any reference that you choose. Here we said our phasor moves starting from vertical, so we tied that to the mass, we said let us say that is the phase of your mass, then everything else starts to move relative to that phase and so you have this description $e^{j(\omega t - kx)}$ where this was described as $e^{j\omega t}$.

If this is described by $e^{j\left(\omega t + \frac{\pi}{2}\right)}$ it does not matter, this $\frac{\pi}{2}$ will get added everywhere so that the relative difference between the two is the same. Now, let us see. We go now to the very first important idea in sound structure interaction, main or central idea in sound and structure interaction. In order to do this, we will need the two dimensional acoustic wave equation.

We will write it like this A. W. E acoustic wave equation in rectangular coordinates, it is given by

$$\frac{\partial^2 p(x,z,t)}{\partial x^2} + \frac{\partial^2 p(x,z,t)}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 p(x,z,t)}{\partial t^2}.$$

So, we will need this equation, 2-D acoustic wave equation. We will also need the plate equation governing the vibrations of a plate the rectangular plate.

What is that given by? That is given by

$$\mathrm{D}\left[\frac{\partial^4 W}{\partial x^4}+2\frac{\partial^4 W}{\partial x^2\,\partial y^2}+\frac{\partial^4 W}{\partial y^4}\right]+m'\frac{\partial^2 W}{\partial t^2}=0.$$

This is the equation of motion for a vibrating plate, vibrating rectangular plate. So, we will need these two equations.

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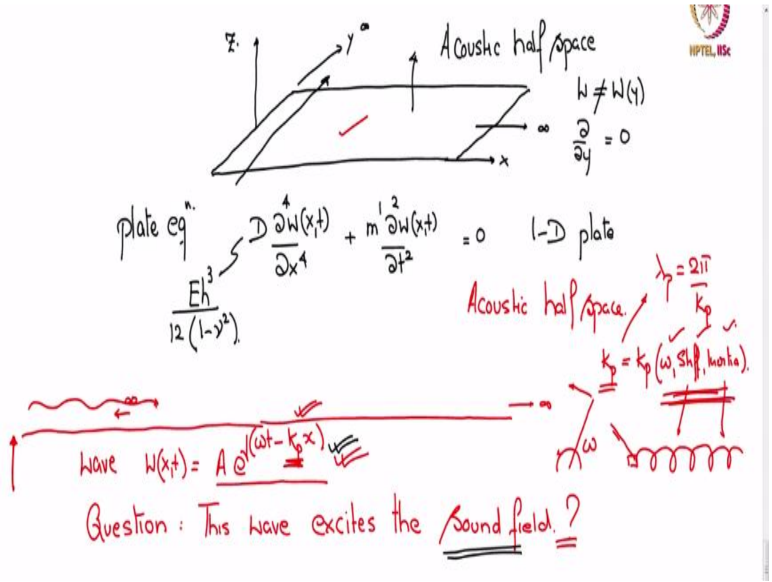

Now. So, what is the system we are looking at? We are looking at a system, let me draw it small first. So, we have a rectangular plate. So, this is my x, that is my y, that is my z. So, we have a rectangular plate that is vibrating and it goes off to ∞ in this direction, it goes off to ∞ in that direction. So, above the plate we have a region we call it the acoustic half space.

So as if the region has been divided into two halves, the world has been divided into two halves, one above the plate, one below the plate. Now further as this plate vibrates, we will assume that the dependence on y is not there. So, the displacement W earlier is not a function of y, so all y derivatives will go to 0, all orders of y derivatives go to 0. So, the plate equation kind of looks like the beam equation.

So, I have

$$\mathrm{D}\frac{\partial^4 W(x,t)}{\partial x^4}+m'\frac{\partial^2 W(x,t)}{\partial t^2}=0.$$

So, this actually looks like a beam equation, however, this is $\frac{Eh^3}{12(1-\nu^2)}$ and therefore so we call this a 1D plate, one dimensional plate and therefore, what happens to this picture now? This picture gets modified, so there is a one dimensional plate. it goes off to ∞ in both directions. So, it goes off to ∞ in both directions and that is an acoustic half space. Now, on this one dimensional plate there is a wave moving. It has been initiated there is a wave moving and what is the description $W(x,t) = Ae^{j(\omega t - k_p x)}$. So by now you should be used to this notation, right. This is the description of a wave moving on a wave bearing system, right. So, ω is in my control.

So, at some distance very far away there is a disturbance given, very far away there is a disturbance, maybe a point force is applied and so that starts off a wave and that wave by the time it arrives at where we are it looks like this and then k_p happens to be the wave number in the plate. I should have mentioned earlier that this wave number in the structure k_p depends on frequency and it depends on the stiffness and the inertia properties.

Earlier we were talking about the spring, same thing. When the phasor is rotating and it is connected to the mass, the mass is connected to the spring, what is the wave number in the spring it is dependent on the frequency ω and the stiffness and the inertia properties on the spring. So, these three independent variables decide what the wave number is going to be and therefore what the wavelength is going to be because why wave length is what?

Wave length is $\frac{2\pi}{k}$. So, wavelength and the inverse wave length wave number are decided by these three properties. So, k_p in the 1D plate got decided by ω and its properties. So, this is my descriptor of the wave in the 1D plate. Now, my question is this wave excites the sound field above the plate, so what is that sound field? This displacement on the 1D plate results in what sound field in the acoustic half space? So, let us see that.

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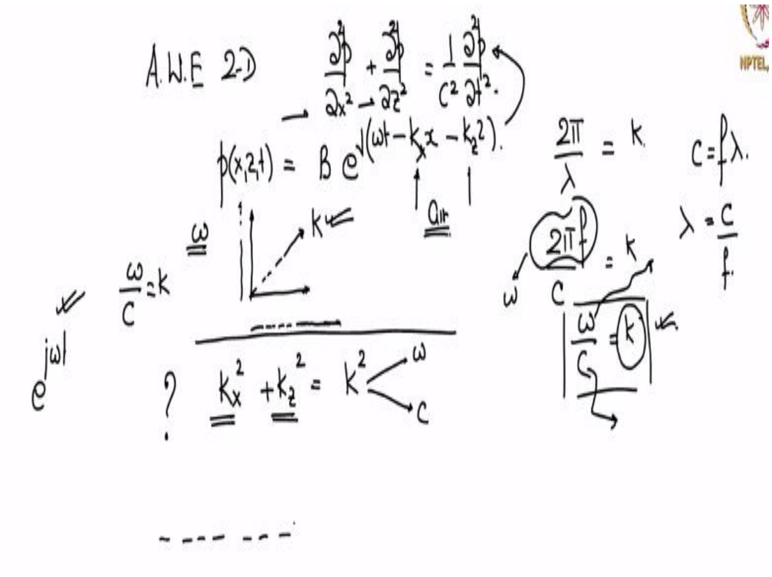

So, we gave the acoustic wave equation earlier for 2D rectangle region. It was given by

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}.$$

You should know now that this has a wave solution given by some constant in front $e^{j(\omega t - k_x x - k_z z)}$

$$p(x, z, t) = Be^{j(\omega t - k_x x - k_z z)}.$$

There is an x component of the wave number, there is a z component of the wave number.

So, there is this 1D plate acoustic half space. So, in the acoustics half space there is a wave number in the x direction, there is a wave number in the z direction and therefore a resultant wave number in some direction which is the acoustic wave number k. Now, the wave number we said was $\frac{2\pi}{\lambda}$, $\frac{2\pi}{\lambda} = k$ the wave number right. And we also said that $c = f\lambda$ or $\lambda = \frac{c}{f}$.

So I have $\frac{2\pi f}{c} = k$ wave number, $2\pi f$ is ω. So, therefore $\frac{\omega}{c} = k$, this is very important. The moment you know the speed of the sound wave in the medium, the moment you know the frequency, wave number in the medium can be found. So, we are actually exciting the beam at ω frequency right and so the sound field being a linear system it will respond at ω.

So sound field will respond at ω and let us say the medium is air, we know the speed of sound in air. So, $\frac{\omega}{c}$ is known to us, this k is known us. The moment frequency is given k is known to us, k in air. Let us say this medium is air for now. So, now if I substitute this into this, substitute this supposed solution into my original partial differential equation you can see I have two space derivatives, I have two z derivatives, I have double time derivatives.

So, that will actually give me, without full derivation I will tell you that $k_x^2 + k_z^2 = k^2$, you can try it out, you will get this. Now, I said this case known to me why because ω is known to me and the speed of sound in air is known to me, but k_x and k_z are not known to me. They are not known to me, so they have to be fixed. How do we fix that? The k_x is the x direction wave number.

That means the wave number in the x direction and k_z is the wave number variations in the z direction, describes the variation and direction. So, let me close the lecture here for today. We will continue with this in the next class. Thank you.