Sound and Structural Vibration
Prof. Dr. Venkata Sonti
Department of Mechanical Engineering
Indian Institute of Science - Bengaluru

Lecture – 15
The Coupled Vibration Field

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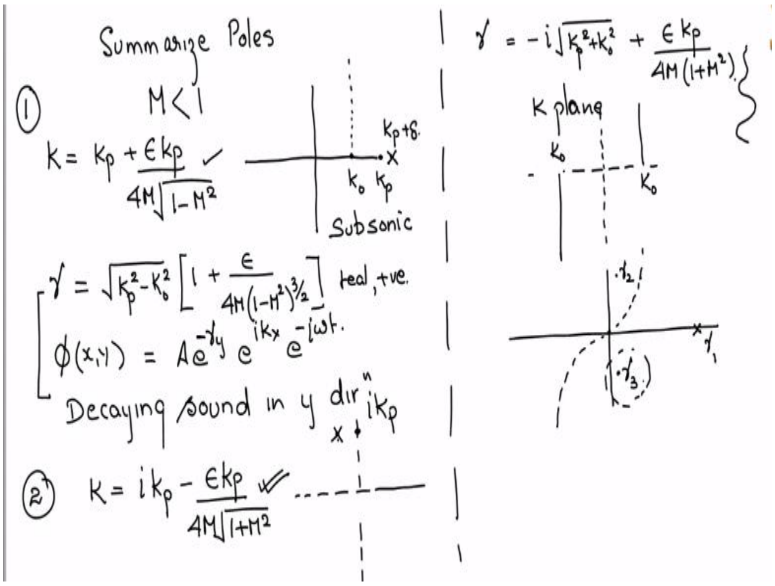

Good morning and welcome to this next lecture on sound and structural vibration. Last time I said we will summarize the poles or singularities that we found. So, first for M less than 1,

$$k = k_p + \frac{\epsilon k_p}{4M\sqrt{1 - M^2}},$$

and this pole is the subsonic pole so there is k_0 here, I will stick with this cuts, so there is k_0 here and k_p was here, now k_p gets modified to $k_p + \delta$ that is this pole.

So, it is subsonic meaning k_p is greater than k_0. The corresponding γ is

$$\sqrt{{k_p}^2 - {k_0}^2}\left[1 + \frac{\varepsilon}{4M(1 - M^2)^{3/2}}\right].$$

It is a real positive number if you remember the $\phi(x, y)$ had this form $A\, e^{-\gamma y} e^{ikx} e^{-i\omega t}$. So, we are talking γ, so γ is real and positive. It is a decaying field.

So this is a decaying field, decaying sound field in y direction. So, this is first. Then below M = 1 the second one is the one near ik_p. So, k then becomes equal to

$$ik_p - \frac{\epsilon k_p}{4M\sqrt{1+M^2}},$$

and the corresponding γ this pole is here ik_p for our original pole it is corrected over here. The corresponding γ is equal to

$$-i\sqrt{{k_p}^2 + {k_0}^2} + \frac{\epsilon k_p}{4M\sqrt{1+M^2}}.$$

So this is propagating in the y direction but decaying. Now I would like to say that corresponding to this cut in decay plane k_0, k_0, k plane the corresponding cut in the γ plane was this. And for M less than 1 we have a γ_1 here, a γ_2 here and a γ_3 here. So, corresponding to this γ_3 is this root k this being consistent.

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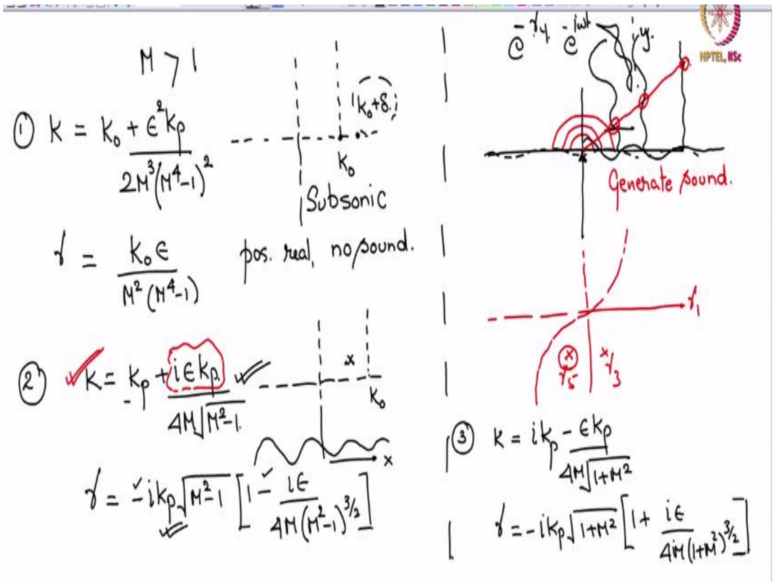

Then for M greater than 1, first we have

$$k = k_0 + \frac{\epsilon^2 k_p}{2M^3(M^4-1)^2},$$

M greater than 1 we are in the supersonic beyond coincidence, so k_0 is high. Now this correction is further higher. So let me take k_0 here. Again the correction is $k_0 + \delta$, so this is also subsonic, so, whether you are above $M = 1$ or below there is always a subsonic wave. Below $M = 1$ it is close to k_p, above $M = 1$ it is close to k_0.

The corresponding γ is $\frac{k_0\epsilon}{M^2(M^4-1)}$, this is a positive real quantity and therefore no sound or decaying sound in the y direction. The sound field does not take away energy, the energy propagates along the plate with this wave number. So let us call it 1. Now second the root near k_p that is

$$k = k_p + \frac{i\varepsilon k_p}{4M\sqrt{M^2 - 1}}.$$

This positive is chosen with reference to appropriate damping of a pole for positive x direction.

Then γ, so this let us see where it is. This is my k_0 cut, it is here, it is here. So this is a propagating root on the plate, but it also decays slowly and the decay depends on the amount of damping you have. This is positive x direction. Along with that the γ is given by

$$-\mathrm{i}k_p\sqrt{M^2 - 1}\left[1 - \frac{i\epsilon}{4M(M^2 - 1)^{3/2}}\right].$$

So, this root the γ for this how does it behave?

So, we have $e^{-\gamma y}$, so this minus, this minus cancel and this i this i gives a negative real value, negative real value placed here this is increasing. This is increasing in the y direction from the plate. Then, of course we have this propagating, the i imaginary part tells me it is propagating because the time is $-i\omega t$. So, this is propagating and increasing. So it seems a little counterintuitive, but if you look at it another way that my source is here.

The line sources here and this is my plate and we seem to see that the sound field is growing however the x wave is decreasing. The x direction wave number is decreasing. So it will decrease as we move further and further **(12:33)** it will decrease So, here we go a little to the right and look at the sound field, it will again grow but not so much and you go further it will grow but not so much.

So, given that the causality or cause lies here at the origin if you draw a line, a radial line along any direction, you will see that the sound field is decreasing. So, it is alright. So, here this supersonic pole will generate some sound, and this root comes from again the γ plane how we find that this is the valid γ plant. So there is a root here, there is a root here and there is a γ_1 here. So, this is γ_3, is γ_5, this is γ_1.

So, this root generates this root. Lastly, the imaginary root that we have close to ik_p does not change that

$$k = ik_p - \frac{\epsilon k_p}{4M\sqrt{1+M^2}}\,,$$

and the corresponding

$$\gamma = -\mathrm{i}k_p\sqrt{1+M^2}\left[1+\frac{i\epsilon}{4M(1+M^2)^{3/2}}\right].$$

This is propagating and decaying in the y direction. So, I have maintained consistency with Crighton's cut and his statements.

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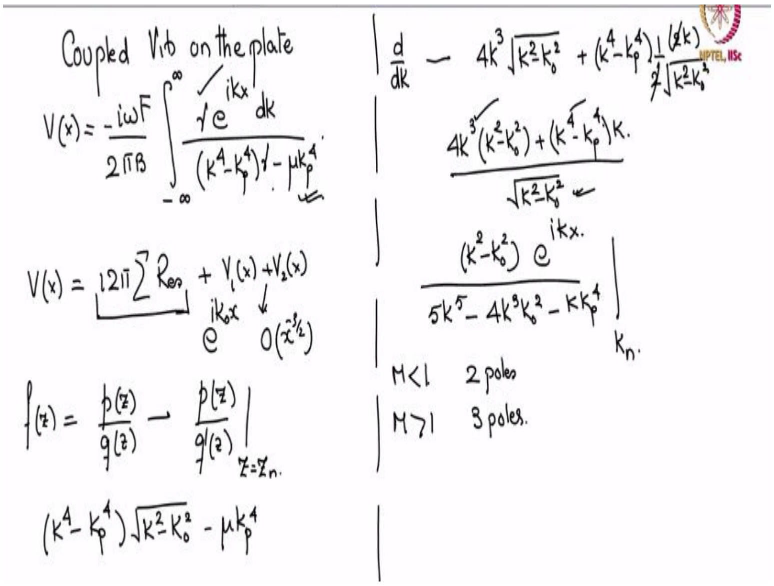

Now, we started off looking at coupled vibration field on the plate, just to recall the

$$v(x) = \frac{-i\omega F}{2\pi B}\int_{-\infty}^{\infty}\frac{\gamma e^{ikx}dk}{(k^4-{k_p}^4)\gamma-\mu {k_p}^4}.$$

And what did we say this velocity field total is going to be now

$$v(x) = i2\pi\sum Res + v_1(x) + v_2(x).$$

This was dominantly a pole at ik_0 and decays as $O(x^{-3/2})$ and the residues were left here.

So, the residuals have to be computed at the singularities which we just listed summarized. So, if you look at this denominator, now there is a theorem that if we have a function that is a ratio of two polynomials then the residue can be computed. So, if you have a function, I am not being very rigorous here $p(z)$ by $q(z)$ and the residue of this can be $p(z)$ over $q'(z)$ provided $q'(z)$ does not go to 0 and whatever pole.

So, if we use that we need the derivative of the denominator. So, we have $\left(k^4 - {k_p}^4\right)\sqrt{k^2 - {k_0}^2} - \mu {k_p}^4$. So, we need to take the derivative of this thing. So, if we do that, so derivative with respect to k , so

$$\frac{d}{dk} \rightarrow \; 4k^3\sqrt{k^2 - {k_0}^2} + (k^4 - {k_p}^4)\frac{2k}{2\sqrt{k^2 - {k_0}^2}} ,$$

$$\frac{4k^3\left(k^2 - {k_0}^2\right) + (k^4 - {k_p}^4)k}{\sqrt{k^2 - {k_0}^2}} ,$$

$$\left.\frac{\left(k^2 - {k_0}^2\right)e^{ikx}}{5k^5 - 4k^3{k_0}^2 - k{k_p}^4}\right|_{k_n} .$$

So this has to be evaluated at any of the roots, of course including e^{ikx} .So, if it is for this cut we have taken if M is less than 1 you get 2 roots, if M is greater than 1 you get 3 roots. **(21:20)**.

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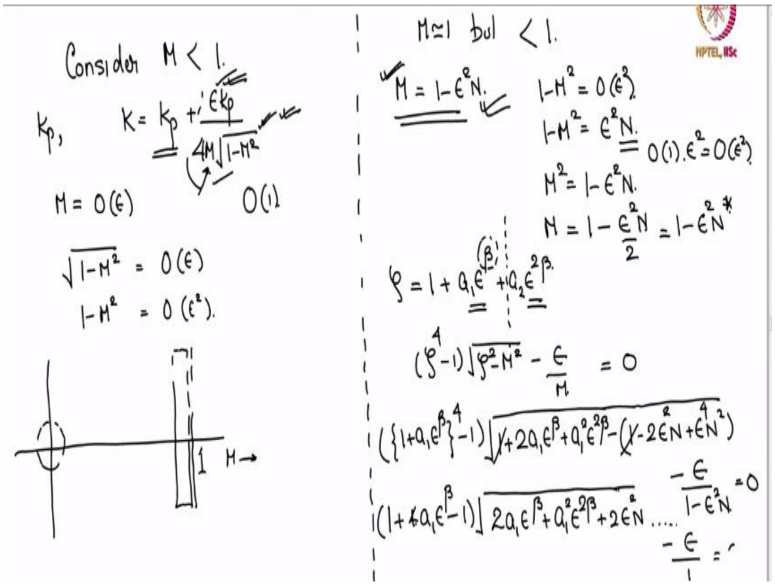

Now let us look at M less than 1. Consider M less than 1, so the root near k_p. The correction happens to be

$$k = k_p + \frac{\epsilon k_p}{4M\sqrt{1 - M^2}} ,$$

Now, if M becomes small that means I go very close to 0 frequency, M becomes order of ε, this term becomes order of ε and this ε and that order ε will cancel and this will become an order one term which is not allowed in asymptotics.

The correction M should always be smaller, of a smaller order than the first term. So this correction is not valid anymore, we go to low frequencies or if my $\sqrt{1-M^2}$ becomes order ε which means $1-M^2$ becomes order ϵ^2, even then this order ε and that order ε will cancel and this whole term become order one and is not valid. So, let we have this frequency regime now.

We have this frequency regime which goes from 0 to 1 let us say this is 1, M. So, we will look at these very close to 0 and we will look at this regime close to 1. So, let us now first look at very close to 1 but less than 1. So, we are going to look at M close to 1 but less than 1. So how do we take a correction? We take a correction of this form $M = 1 - \epsilon^2 N$. Where do we get this idea from?

We said when $1-M^2$ is equal to order of ϵ^2 the expansion is invalid and therefore I make $1-M^2$ is $\epsilon^2 N$,this (N) is an order one term, but multiplied by ϵ^2 it is becoming an order ϵ^2, so that is where I get this. We can do further so $M^2 = 1 - \epsilon^2 N$ then M is root of that, so it will become $1 - \frac{\epsilon^2 N}{2}$, still ϵ^2 that is where I get this.

So, we can replace N by some N^* and then $M = 1 - \epsilon^2 N^*$.

$$M = 1 - \frac{\epsilon^2 N}{2} = 1 - \epsilon^2 N^*.$$

So, we will look at this frequency, but we will look at this particular root which is $\zeta = 1$. So, we will extend this root now using

$$\zeta = 1 + a_1\epsilon^{\beta} + a_1\epsilon^{2\beta},$$

and so forth because I do not know what balancing power will come in here, I do not know here. So, I am expanding in an unknown, but I will take only one term because simple to derive.

So, what do we do is this we have to get back to our famous dispersion equation $(\zeta^4-1)\sqrt{\zeta^2-M^2} - \frac{\epsilon}{M}$ this has to be balanced again. So, if we do that and correct till the first term,

$$\left(\{1+a_1\epsilon^{\beta}\}^4-1\right)\sqrt{1+2a_1\epsilon^{\beta}+{a_1}^2\epsilon^{2\beta}-(1-2\epsilon^2N+\epsilon^4N^2)}-\frac{\epsilon}{1-\epsilon^2N}=0.$$

So, now if we expand this I get

$$(1+4a_1\epsilon^{\beta}-1)\sqrt{2a_1\epsilon^{\beta}+{a_1}^2\epsilon^{2\beta}+2\epsilon^2N}-\frac{\epsilon}{1}=0.$$

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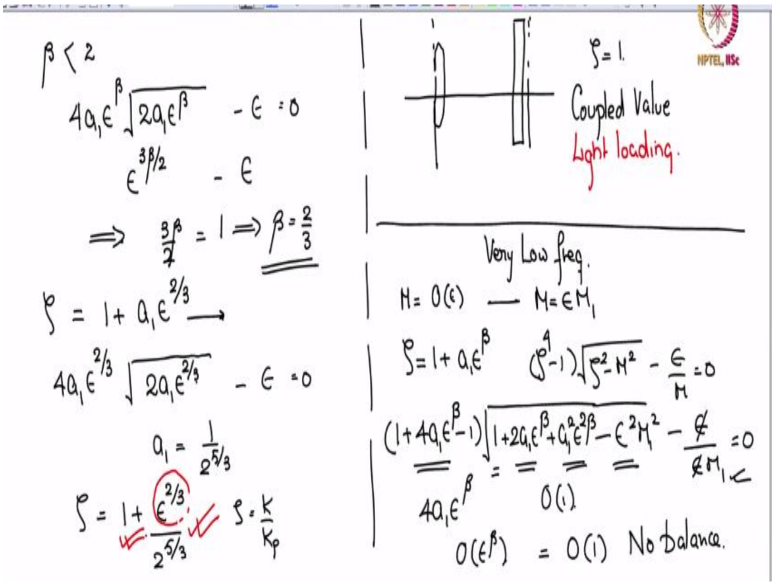

Let us assume and see, let us assume that β is less than 2. Then what do I get? I get

$$4a_1\epsilon^{\beta}\sqrt{2a_1\epsilon^{\beta}}-\varepsilon=0,$$

$$\epsilon^{3\beta/2}-\epsilon=0,$$

$$\Rightarrow\frac{3\beta}{2}=1\Rightarrow\beta=\frac{2}{3}.$$

So, if we now use this, so that means now I have found the correction ε order for the right extension, so that means ζ should be expanded as $1+a_1\epsilon^{2/3}$, earlier we wrote β as we did not know, so now β has been found to be $\frac{2}{3}$, so this is the right expansion. So, if we again substitute this in the dispersion equation I get

$$4a_1\epsilon^{2/3}\sqrt{2a_1\epsilon^{2/3}}-\varepsilon=0.$$

I will just compute this $a_1=\frac{1}{2^{5/3}}$. So, ζ is given by $1+\frac{\epsilon^{2/3}}{2^{5/3}}$. ζ is what? ζ was $\frac{k}{k_p}$. So, now that is the correction when we are very close to $M=1$ and we are in this range of frequency that is the correction for the ζ equal to one root. So, it is still a perturbation, the coupled value that we found here is still a small change over the real value.

So, we will still call this light loading. Now, let us try very low frequency. That means what? M will be taken as order ϵ. And how do we represent it? We represent it as M is equal to ϵM_1. So, at this frequency range that means we are very close to the zero frequency somewhere here. So, let us again perturb this root 1. So, we will do $\zeta = 1 + a_1\epsilon^\beta$.

So, we have again, we should not forget the dispersion equation $(\zeta^4 - 1)\sqrt{\zeta^2 - M^2} - \frac{\epsilon}{M}$ that is to be balanced. So, I get

$$(1 + 4a_1\epsilon^\beta - 1)\sqrt{1 + 2a_1\epsilon^\beta + {a_1}^2\epsilon^{2\beta} - \epsilon^2{M_1}^2} - \frac{\epsilon}{\epsilon M_1} = 0\,,$$

$$(1 + 4a_1\epsilon^\beta - 1)\sqrt{1 + 2a_1\epsilon^\beta + {a_1}^2\epsilon^{2\beta} - \epsilon^2{M_1}^2} - \frac{1}{M_1} = 0.$$

So, if we try to find out the dominant term here all others have ϵ to the power some value, so they are small terms, so this is order one term, this term is an order one term and here we have $4a_1\epsilon^\beta$ and therefore this is an $O(\epsilon^\beta)$ term. So, this total term on the left is $O(\epsilon^\beta)$ and the right hand side term is order one, so no balance. There is no balance possible.

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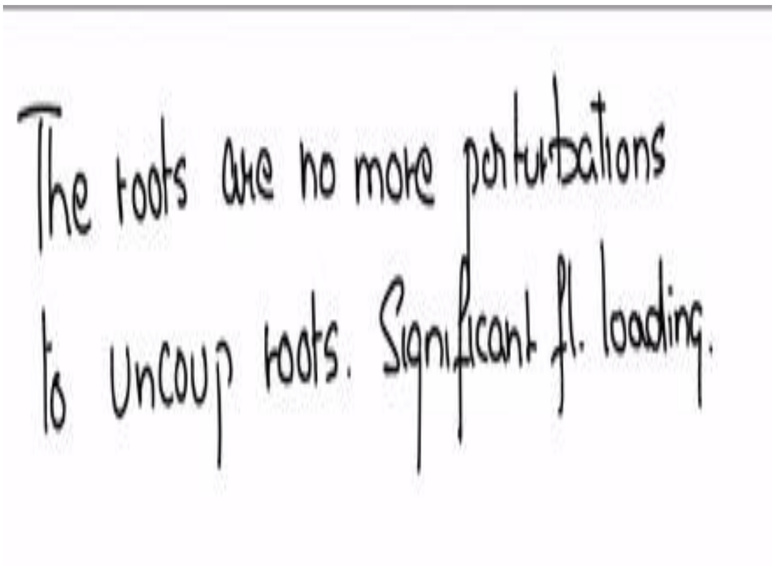

That means what? The roots are no more perturbation to uncoupled roots. So, fluid loading is not light, this significant fluid loading. I will end the topic here. We will continue in the next class. Thanks.