Indian Institute of Science

Variational Methods in Mechanics and Design

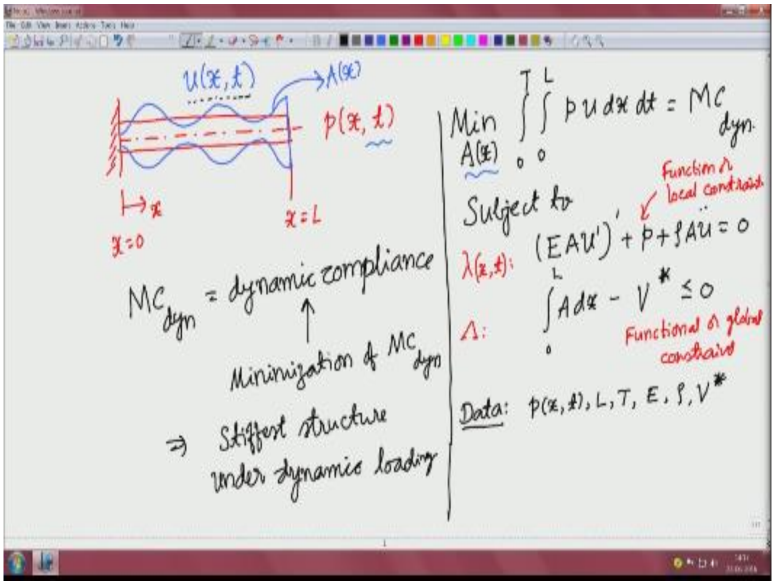
Prof. G. K. Ananthasuresh
Department of Mechanical Engineering
Indian Institute of Science, Bangalore

NPTEL Online Certification Course

Hello we have discussed several aspects of calculus of variations and its applications in mechanics and some in structural optimization. Today to cap it all we are going to discuss variational framework for structural optimization of structures that involve other domains such as electrical, electrostatic magnetic and other things.

We will just take two examples one from dynamics which will not have other domains but other one will have domains from electrical and thermal involving the structural mechanics just to show that variational methods are very general and we can apply to a number of situations okay. So let us start with what how a structure can be done under the dynamic conditions.

(Refer Slide Time: 01:15)



So let us take the example of what we call dynamic compliance that we want to minimize for a structure what we mean by that is that if you have a structure let us say a bar we would like to minimize its strain energy not under static conditions which you have already done, but we will do it under dynamic condition. So if I have a bar okay which will not be what we want to design is actually its cross-section.

So if this is the axis of the bar let us say that there is a load P which earlier was just a function of X, X being this coordinate with X = 0 here and X =L here p is a function of X it varies from point to point along the axis of the bar, but it is also a function of time. What can a function we do not know just like we do not know what kind of function it is in terms of X this pxt is given to us. Now we would like to make it stiff under dynamic conditions because the fact that this is a function of time tells us that when the load change the function of time the structural deformation that you that we have axial deformation that will also be a function of time okay.

So what we would like to minimize now to make it stiff under dynamic condition is to minimize integral over L will say pudx okay, but since p and u are both varying with X and time we will do an integral over certain time also some capital T is given to us and put dt okay. So this one we can call MC_{dyn} meaning that MC_{dyn} okay, let us define here what would let minimize is MC_{dyn} is what we call dynamic compliance okay.

You want to minimize this if you minimize this, this dynamic compliance minimization of MCdyn employs stiffest because you are doing an optimum stiffest structure under dynamic loading under that is what we will have okay. We like to do this subject to and what is our variable it is A(x) we will write the constraint first let us discuss what this A(x) is that we have been using that will be the cross section profile of the beam okay.

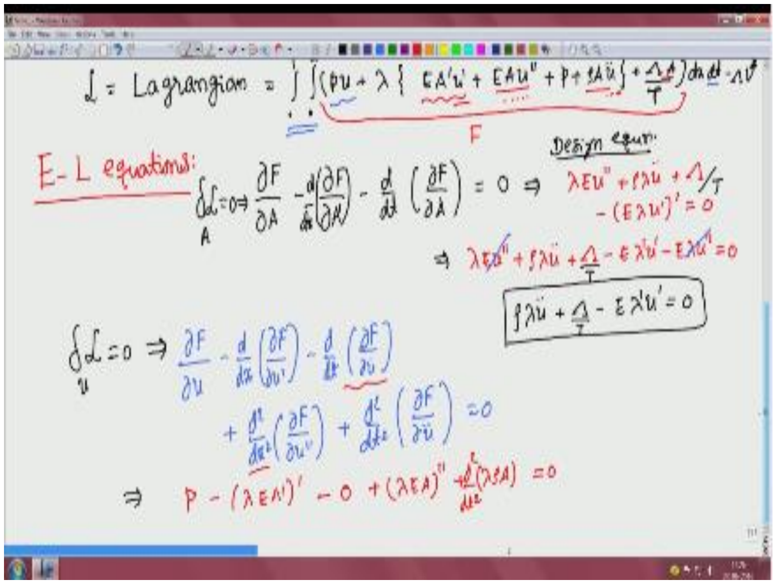
The change is A(x) changing area that is our design variable or optimization variable okay. Now we have to read the constraints for the constraints we have now dynamic situation, so we need to have a function that controls this YX(t) okay that is a state variable now with a function of X and time right. So we will write this as (EAU')'+P that we have okay, that was equation that we had for statics.

Now we would also have PAU..=0 okay. So we are now the constraint that we have governing equation is for dynamics because we have included the acceleration PAU.. is there and then we will also have the volume constraint as usual 0 to L Adx that volume should be less than V* that is specified. Again remember that whenever we have a function type constraint the corresponding Lagrange multiplier is going to be a function of X.

And here the time also just like this equation, this equation is valid for all values of x all values of time so this will be a function of λ will function of X and T this λ will just be a scalar because is a functional type constraint okay. Again let us remind ourselves that this is functional or global constraint and this is a function or local constraint okay.

And we should also complete the problem by indicating what is given what is data that here loading is given we assume the length of the bar is given time or which we want to consider optimization T is given E modulus is given ρ is given and V* is given okay these are all the things that are known. Now how do we solve the problem the first step of doing it as always is to write the Lagrangian okay.

(Refer Slide Time: 07:43)



So you would do that looking at the problem let me write the Lagrangian so we will have objective function 0 to L, 0 to T pu and then we have all of this λ times I will expand this $EA'U'+EAU''+P+\rho AU..$ and $+\lambda k$ all of this $dxdt-\lambda V^*$ okay that is just a constant last one we do not have to worry about it okay this is lambda V^* okay.

Now if you see we have put dx dt whereas the integral here for this constraint is only for up to dx there is no dt, so it is safe to divide by so that when you do integral over time that T gets canceled although it does not really matter a whole lot. Once we have that we notice what our integrand is that is our integrand F for which we write our are rather Euler-Lagrange equations.

So here we have F1t so when you take a variation we have to take into account that things are varying with space and time. So when you have two independent variables how do we do it we write the equations as first there will be if I take first of all variation of the Lagrangian with respect to area of cross section okay.

And that would apply $\partial f/\partial A$-$\partial f/$ well it will be $d/dx(\partial F/\partial A')+d^2/$ we do not have that actually A'' is not there so this term is not needed but instead we have the time so we have $d/dt(\partial F/\partial A.)$ okay that is also not there. So it does not matter so A has only A and A' so that is all we would have and we equate to 0 that gives us that what we call design equation and that is in this case where our A is in okay let me just underline A is here, A is here those done in A is also here.

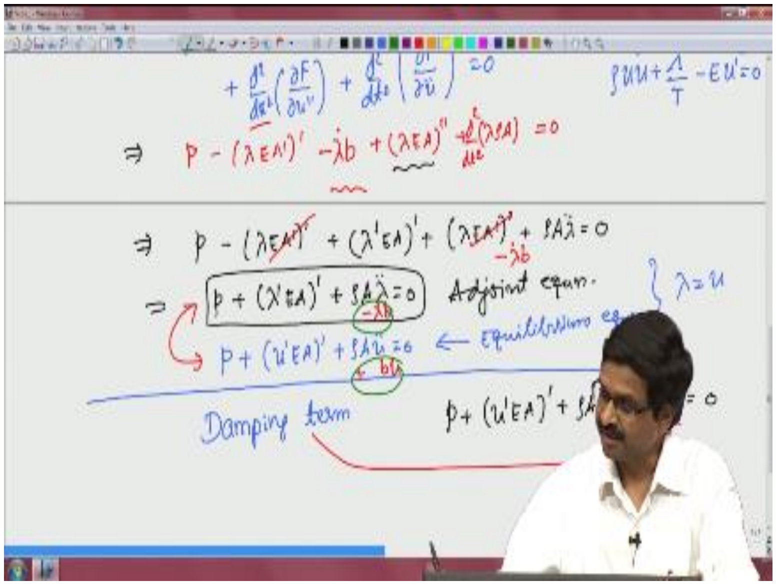
So I can write this equation $\partial F/\partial A$ as λEU'' and we have PλU..+λ all this is for A-s $\partial F/\partial A'$ that so we have that here so that will become –E A will not be there because we have taken respect to A' (EAλU')' =0 so we can simplify a little bit. So we have λEU''+ρλU..+λ and this if I expand the last term that will be Eλ'E'-EλU''=0. So now this term and this term cancelled.

So what we get is PλU..+λ-Eλ'U'=0 okay that turns out to be our design equation. So now if we write the variation with respect to U and equated to 0 we get the adjoint equation okay, now we got the design equation going to write the adjoint equation and I see that I forgot this t that will be λ/t does not change much in just that the t that we introduced here because the constraint was only this constraint was only having dx when we brought it since we have both of them I just put t so that when you do integrals respect to t that T will cancel and we will get the constraint as it is, okay.

The volume constraint small change now for u where we have you there u′ u″ and ü, right so we have to write this as δf/δu-d/dx or δ/δx same thing δf/δu′-d/dt of δf δu̇ which we do not have plus d^2/dx^2/δf/ δu″ +d^2/dt^2 not dx^2 dt^2(δf /δü)=0 okay. Let us work it out this implies u is there that is you so that is only case u is alone that will be p δf/δu and then we have δf/δu′ that is over here and that is it.

So that gives us λEA′ since (u′)′ d/dx and ü is not 0 it is not there that that this will be 0 then we have the next one that is u″ is over here so that will give us λEA″ because we have d^2/dx^2 + we have the double dot term that will be λρA yeah that is it but this will be double dot meaning d^2/dt^2okay, that is equal to 0.

(Refer Slide Time: 16:14)



Now let us simplify it this is a joint equation so we want to see that it will be similar to the governing equation so what we will do is we will not expand this because this is going to get cancelled with something that is if I expand this, okay that is going to give($\lambda'EA)' + (\lambda EA')' + \rho$ an A do not vary with a row and A do not vary with time so double dot will only be secondary only by $\ddot{\lambda}$ equal to 0 now we see that this and this get cancelled so what I get here is P + $\lambda'EA'$ +$\rho a \ddot{\lambda}$ equal to 0.

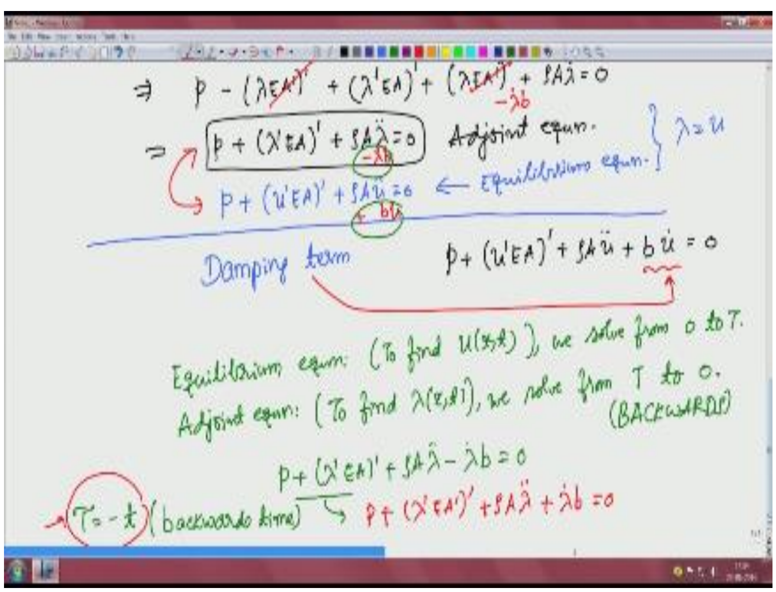
This is our adjoint equation and if you notice the governing equation that we had which is EAu′ I just dashed line this our governing equation $(EAu')'+P+\rho\ddot{u}$ equal to 0, is how we got it okay, so that was also just like this wherever λ was it was you so this was u equilibrium equation. So we can conclude from these two that λ is equal to u, λxt is equal to uxt we substitute this back in the design equation we get $\rho u\ddot{u}+\lambda/t - Eu'^2$ equal to 0, that is what we get.

So once you know u you can compute you will u′ and they will say what λ is then we can use a plurality method to solve this problem, okay. So adjoint equation can be easily solved alright so the same setup that we had for static problem also came from dynamic problem but there will be a small change if we consider damping term okay, damping term in the governing equation let us say we add that so I will write the equation that we have $p(u'EA')'+\rho\ddot{a}$ now let us add that there is a $\dot{u}$ term with some damping now let us say we will call it $b\dot{u}$ equal to 0.

So earlier we only had this but now we have this extra term this is our damping term right, if we do that and go back and then see what happens okay, so somewhere here let us add okay, let us add the damping term okay, so that will be our $b\dot{u}$+$b\dot{u}$ that means that our integrand we wrote yeah this is our f has that okay that means that in the governing equation itself we will add damping term over there okay over there we add +$p\dot{u}$, so what changes we had said that this thing does not exist where that becomes 0.

But now it will not be 0, okay focus here that is not0 that will be without multiplying λ b will be there -λ b be the -sign so that kind of continues here okay, so that -λb will be there and then here also - λ P is there sorry in this case – λ will be there whereas here give you +λ that is what we added we said that damping term so not λ ok that one second this will be $b\dot{u}$ in the governing equation it will be $\lambda\dot{b}$ because when i take this there is d/dt is where you become $\dot{\lambda}$ so there will be $\dot{\lambda}$ b the $\dot{\lambda}$ P now if you compare these two equations they are not exactly the same because over here over here it's a +$b\dot{u}$ whereas here it is - $b\dot{u}$ so adjoint equation and equilibrium equation are not exactly the same because the damping term which depends on $\dot{u}$ has a +sign in the governing equation whereas -sign in the adjoint equation so we cannot just solve.

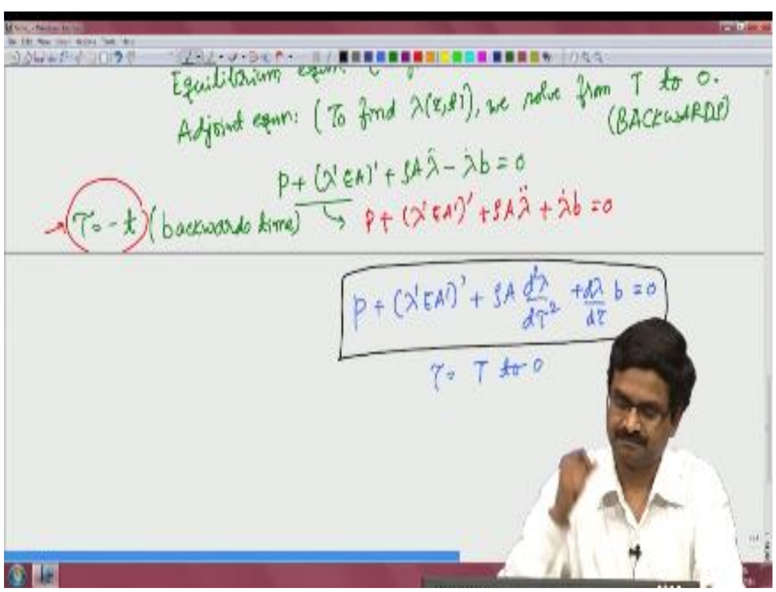
(Refer Slide Time: 22:09)



But we could solve it if we reverse the sequence meaning that for solving equilibrium equation that is to find that is to find you xt we solve from 0 to T okay when there is -sign I can switch the time you know for solving adjoint equation Oh since we have -sign I cannot do the structure of the equilibrium equation so adjoint equation I say I want to solve to find λx,t we solve from t to 0 that is backwards why is that important.

Because when you look at the equation adjoint equation P+$\lambda'EA'$+ $\rho a \ddot{\lambda}$-λ dot P equal to 0 now let us say that instead of the time T let us actually have minus time with another variable defined τ, τ is – time okay backwards time backwards time if you do that how does this equation change it changes p does not change and $\lambda'EA'$ has not no dependence on the time so that also is just fine okay, but here $\ddot{\lambda}$ you have to take derivative twice now we are doing with respect to τ which has --T so when you take two derivatives of τ then it will become plus again so that remains plus there will be $\rho a \ddot{\lambda}$ now with respect to τ this - $\dot{\lambda}$ will become +$\dot{\lambda}$ b equal to 0 because of what we have assumed that T-τ.

(Refer Slide Time: 24:49)



So over here what we have is $P(\lambda' EA')' + \rho a d^2\lambda/d\tau^2$ and this one will be $d\lambda/d\tau$ times be equal to 0 τ goes from final time to 0, now if you look at this equation this as exactly the same form as the governing equation so for us to solve we can use the same parent element routine to solve it rather going backwards in time okay, this is what happens when there is a sequential time, time equals 0 happens one second two seconds whatever 2.5 seconds when it goes like this we go forward for equilibrium equation.

And then come backwards in negative time for a joint this is a characteristic of the adjoint method when there are these first order terms there this can also happen sequent in sequential coupled multi physics problems which you will see after this into the next half of the lecture, thank you.