Indian Institute of Science

Variational Methods in Mechanics and Design

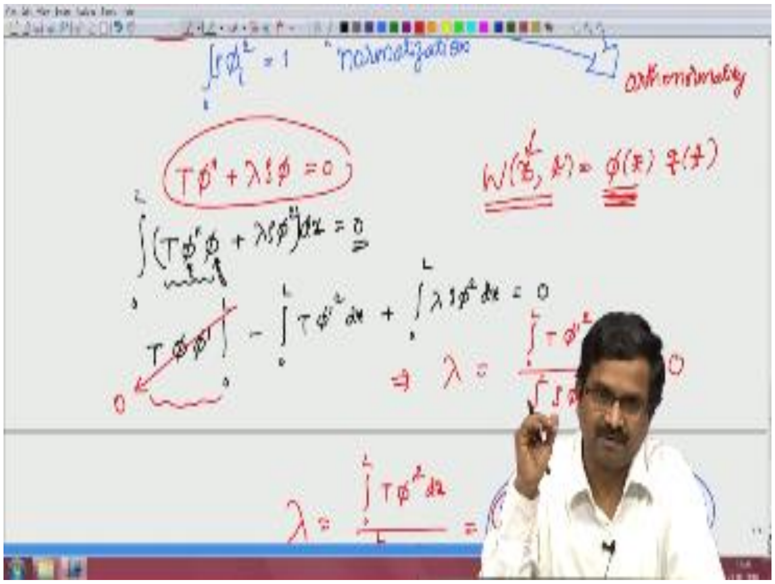
Prof. G. K. Ananthasuresh
Department of Mechanical Engineering
Indian Institute of Science, Bangalore

NPTEL Online Certification Course

So we were talking about the vibrations free vibrations of a string and we wrote the equation of motion of free vibration of a string using Hamilton's principle or extermination of the action integral and wrote a Lagrange equations and got the equation and we use separation of variables technique where W X and T that is a function of space variable X and time T we wrote it as V of X times Q of T and then proceeded in the process we define the Eigen value and Eigen function okay.

We said that Eigen values are always that λ that we have here has to be non-negative we also said that Eigen functions are orthogonal right if you recall the previous part of the lecture.

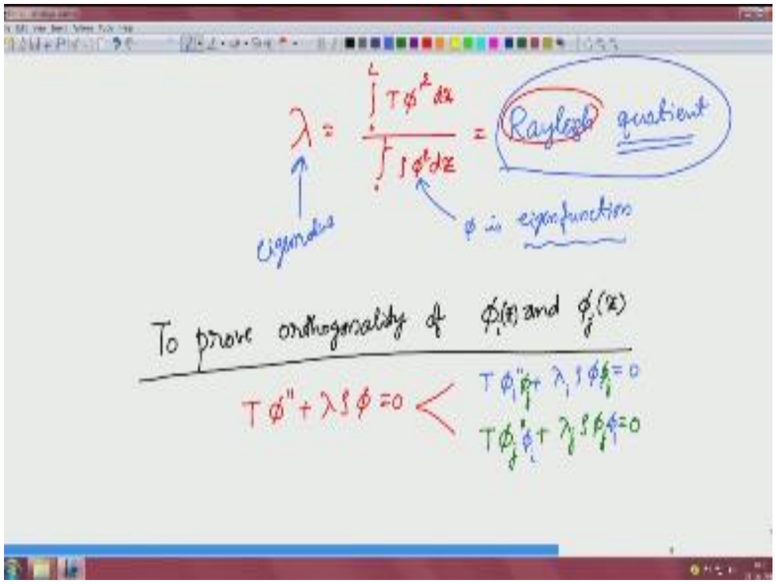
(Refer Slide Time: 01:10)



We said there are terminal we also define what that orthogonality means if I take a pair of them product of those when I integrate from 0 to l to be 0 that is a definition of functions being orthogonal as opposed to vectors being perpendicular to each other okay, now let us prove how this orthogonality comes our starting point again is this equation and our track trick is almost the same the trick is what we did to prove that λ are always non-negative and to define the concept of this Rayleigh quotient okay.

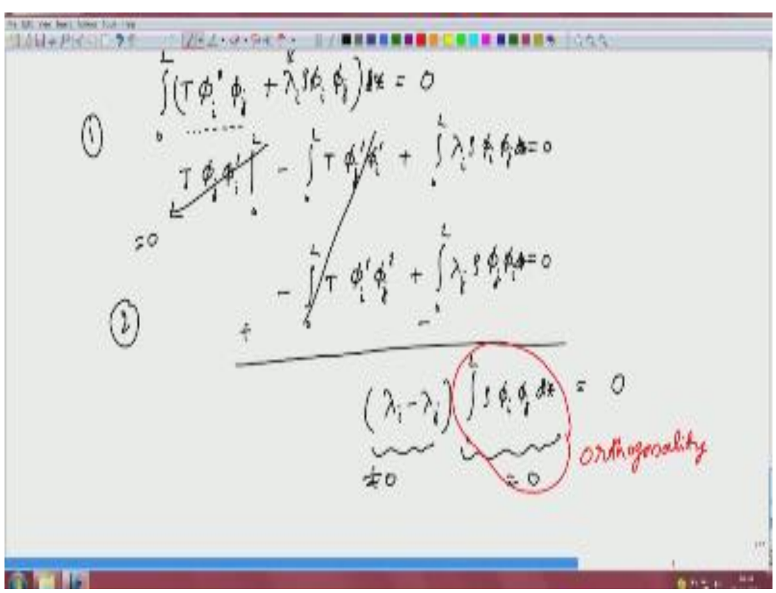
What we did was we multiplied by φ in integrated this is one trick that you should remember in this topic when you have equation u multiplied by something the same thing the whole equation integrate from 0 to L and then see what happens okay so we will take that so we will start with that is we want to prove orthogonality to prove orthorganality of φ I + φ J.

(Refer Slide Time: 02:21)



Again remember these are functions of X φ I(x) + φ J (Fx) j r as such what you want to do for starting point that equation that we have $T\varphi'' + \lambda \rho \varphi = 0$ will instantiate this equation for φ I and φ j because we said there are and infinite φ so we will write this as T φ I'' which case have to write λ I ρ φ I = 0, and other one t CJ'' + λ j ρ φ j = 0 okay. Now what we will do is this equation we will multiply by φ j everywhere and this equation will multiply by φ i okay and then we will integrate from 0 to l okay. That integration I will do it here that will become cluttered.

(Refer Slide Time: 03:38)



So the same thing I will write 0 to L T φ I’’ φ j + λ I ρ φ I φ j DX since right hand side is 0 even after integrating will remain 0 okay again we will do integration by parts for this okay when I do that I get t first function I will take φ J integral a second function I take φ I’’ so that becomes prime 0 to l minus derivative of first function that will become φ j ‘’ integral of second function φ I’ and then this is 0 to L λ ρ φ I φ j = 0 this is of course λ and this thing is going to be 0 anyway because φ I φ J if you have they have to satisfy the end conditions that goes away this is what I get with the first equation.

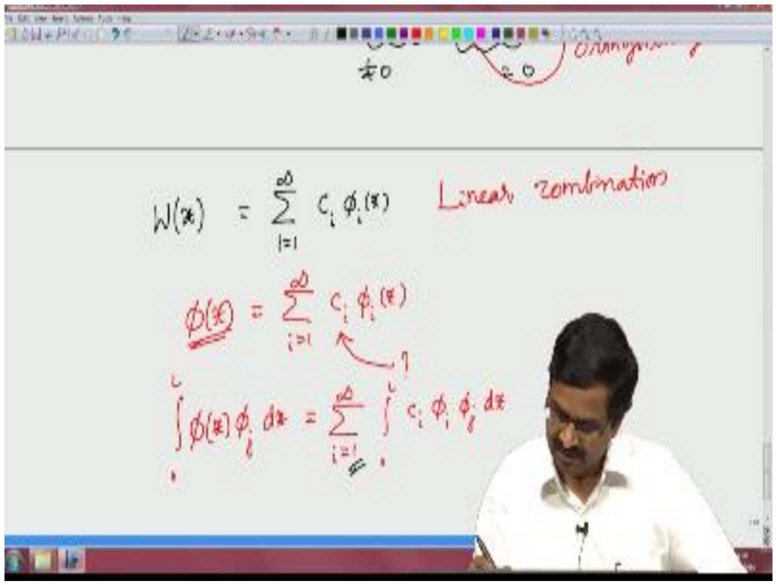
Similarly second equation but that is this one over there the first equation 1 integrate that is this you can integrate I have this thing okay now the second one if I do this since I and J are reversal I can just write it right so I will write -0 to L T wherever jay is all right I φ I ‘φ j’ actually this did not matter because it is the same thing plus 0 to l λ j I is there so I write j ρ φ φ j I can that did not matter right.

So now if I subtract one from the other right so what I get you know this is plus minus so these two things are the same they get canceled okay and what I get here is λ I - λ J x 0 to L ρ φ I φ J DX okay. I forgot DX here right so there should be DX, DX this is equal to 0 right hand side is on 0 rights. So now we said that these two are two different Eigen values so this is not equal to 0.

Then this must be equal to 0 that is the orthogonality. So this is the orthogonality property right and then if you also say ortho normality that is going back we said we want to normalize it like that sorry integral 0 to L ρ φ I^2 = 1 that thing we say there is a normalization condition put together we get ortho normality okay now when we have this the advantage of having functions that are perpendicular to each other is that any function this w this w of x that I get x and t okay that I can write as a summation of several of these functions okay.

In fact I do not have to first say time let us just tip to space for now w FX is φ I of XC I that can be Q I function of time we can do but let us stick to this CI right now want to infinity okay that is because if in vector calculus you have XYZ orthogonal coordinate system any vector can be expressed as a component of x y and z right.

(Refer Slide Time: 08:05)

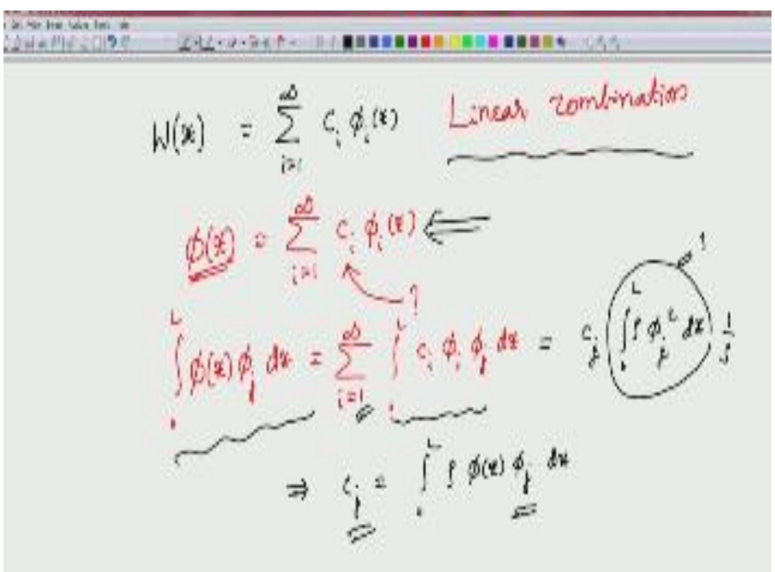


Similarly here we have orthogonal functions by taking a linear combination that is what we call it linear combination linear combination if you take you can write any function okay that is something that actually needs to be rigorously proved but we can intuitively understand that when you have orthogonal functions in this case we also made them orthonormal you can get this right that means that any W of X that you have or rather I can just call it since I can just put like φ of X φ of X any function I can write it as I = 1 to infinity CI φ I of X.

Who gives me all the C is what are those how do I compute easy the same trick in this business what is the thing we just multiply they take this φ and integrate it from 0 to M that is we take the function this φ is there right everything I want to know CI so what I do is let us a particular let me call it the CJ or something because C is already there let me take it as j φ J I want to take CJ okay.

So I do this and integrate it from 0 to M because I know this φ (x) somebody has given me I want to know what are the C is so I put 0 to l φ X φ j DX I do the same thing on the right hand side I = 1 to infinity, so integral 0 to l CI φ I φ j DX okay now remember our orthogonality property whenever this i is going from 1 to infinity 1 to infinity whenever it becomes j only then this integral that we have I can also put ρ if I if I like but ρ does not matter in this orthogonal relationship that we just did here okay ρ is just a constant.

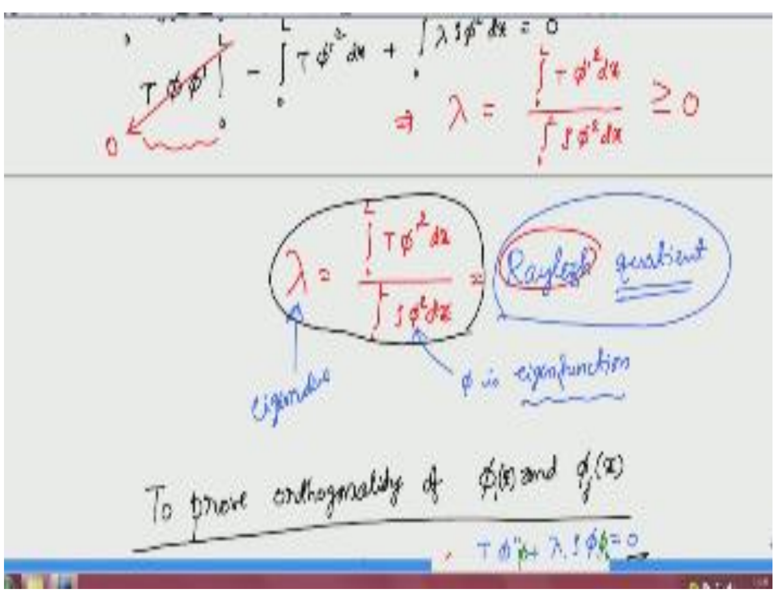
(Refer Slide Time: 10:22)



So whenever I is not equal to j this integral is going to 0 so only non zero term that will be there will be CI 0 to l φ I^2 DX r VJ square here is φ J because I am taking j only when i equal to j then it is not 0 and we had said that ρ included is one r if i take this okay x ρ and then divided by ρ in other words is equal to 1 so i get CJ equal to this are what i get CJ is integral 0 to L it is this portion and then ρ 1 over ρ goes there ρ φ of x times φj DX.

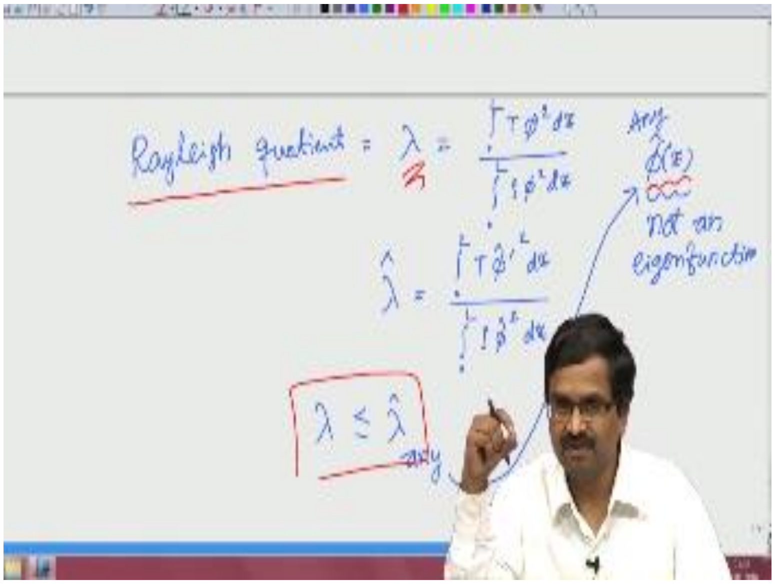
So if I know the Eigen function φ J I can get the corresponding coefficient that is CJ that is going to be in this linear combination okay, so the lot of nice things about Eigen value problems but let us come back to the Rayleigh quotient concept that we had discussed.

(Refer Slide Time: 11:35)



This λ has this particular ratio okay so this one has even more significance because as we already said even when the φ is approximate λ we come very close but with an interesting property so let me write that Rayleigh quotient one more time.

(Refer Slide Time: 12:01)



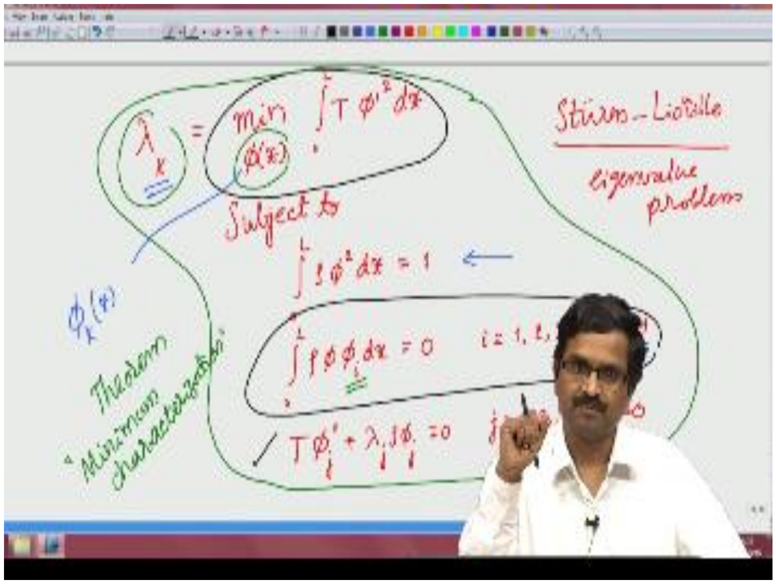
So really quotient for this problem okay so that in this case this λ is integral 0 to L T P prime square DX divided by 0 to L ρ φ square DX whatever φ function that you take okay, even let us say P hat not an Eigen function ok let us say it is not and I can function that is it is not the solution of our Eigen value problem which is TP double prime plus λ ρ φ = 0 it is not a solution of that.

Even then when I get this λ hat value let us say 0 to l d I have φ hat instead of that we had prime square DX divided by 0 to L ρ φ hat square dx okay this will be greater than zero generator than zero but what we find is that any φ that you take this λ is going to be less than or equal to less than or equal to λ hat no matter this is any λ hat meaning any φ hat function you take any φ that function you take this will hold good that is minimum characterization of the Eigen value problems okay.

So when you have Rayleigh quotient you take what you can guess so mfy you will get some λ that λ you can consider is larger than the real λ fundamental frequency that will have okay that λ it take but then there are several λ, so we can propose a theorem that considers the fact that there are several λ okay but if you are in the vicinity of a particular Eigen value no matter what function you take for φ the one that you evaluate in using this Rayleigh quotient is going to give is going to be larger than the real value very.

In other words Rayleigh quotient is minimum in the vicinity of that Eigen function if you consider Eigen axis if you say Eigen function axis whenever I can function of this φ the φ axis whenever φ becomes an Eigen function corresponding λ will be minimum and that is a real λ Eigen value okay that is the significance of Rayleigh quotient okay.

(Refer Slide Time: 14:57)



Now let us suppose a theorem that takes care of this so what we will say is that λ k is equal to minimum Rayleigh quotient will come out of this but we will just write it as a theorem minimum with respect to some P of X again calculus of variations problem that is why we are discussing this because important in mechanics for this vibration problems or which are also known as Liouville problems okay.

That is well let me write it because this buildings are pronunciations are different to UV problems okay I think there is a you also Liouville Louie some new problems okay Eigen value problems that is easy to remember I even value problems okay in all of those cases we can derive this orthogonal Eigen functions and all of that and then this minimum characterization here what do you minimize you minimize in this particular case I will write T integral0 to L T and φ'2 DX okay there is a quantity that we will minimize subject to some conditions actually this is not just un constrained problem subject to some conditions and which are first that normalization condition that is ρφ2v DX = 1 put that then we have orthogonality that we consider several other Eigen functions.

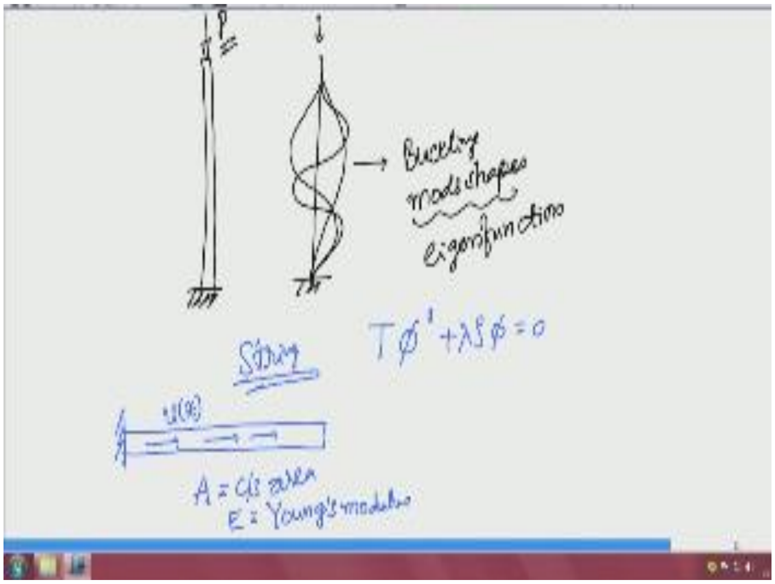
So we will need to write a row see that we are considering times let us say V I DX equal to 0 where I goes from 1 to 3 so forth up to k minus 1 only okay because we are asking for K^{th} Eigen value we put this orthogonality up to k minus 1 and then there is ortho normality that we are ensuring with that that new function that φ when you get that that φ turns out to be φk x when we find the solution that turns out to be φ k okay.

That the turns out to be φ K of X right so there is one more condition we have to put and that is our Eigen thing itself this is φJ'' λ ρφ J = 1 j = 1 2 3 any number infinity right we do not limit it ok so this whole thing is our theorem so this whole statement we have is our theorem what is this theorem it is minimum characterization minimum characterization theorem for Eigen value problems okay.

So if you already have with you k minus 1 Eigen functions because that is what we are including here k minus 1 then you can find the k^{th} Eigen value and of course the corresponding Eigen function by solving this calculus of variations problem this can be proved rather easily without using a Lagrange equations that takes some time even it is not complicated several steps are there simple steps not unlike what we have already discussed to see how all λ are non-negative ortho normality of Eigen functions similar steps when you do that you can prove this theorem.

And it is numerically very important because we are trying to we have we are able to define the Eigen value that is a k^{th} Eigen value okay if you want to get the first Eigen value it becomes simple because then we will not have these things this K minus 1 k equal to 1 none of them will be there we will just have this Eigen function relationship and this will be able to do that will be useful for a structural design which is what we will consider in the context of buckling buckling is also an Eigen value problem.

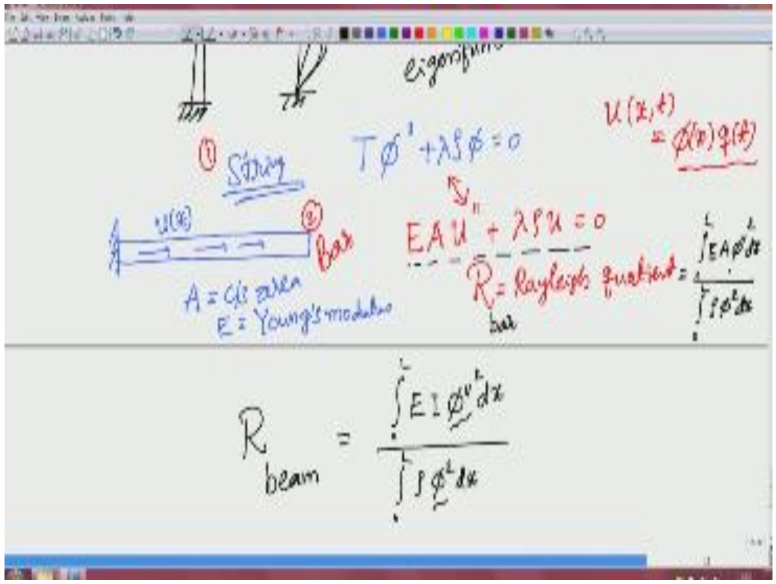
(Refer Slide Time: 19:53)



Because when you take when you take a let us say beam or a column as we call it you take that okay when we take a compressive load this column will become un stable beyond a certain value of this compressive load we all have the familiar things when you take this column when apply the compressive load on it at some point it will just bend like this okay, or it can bend like this also or it can bend like this also are many other ways so there are minimal buckling mode shapes buckling mode shapes are Eigen functions.

So mode shapes or Eigen functions of a corresponding Eigen value problem in this particular of a string problem our equation is $T \varphi'' + \lambda \rho \varphi = 0$ and similarly if I were to take a bar before we go to columns let us take an axially this is for a string with tension T okay, if I take a bar where everything deforms only axially and it has a cross section area this is cross section area and E is Aang's modulus Thanks modulus and this axial de formation we denote by UX that also leads to this same φ okay.

(Refer Slide Time: 21:43)



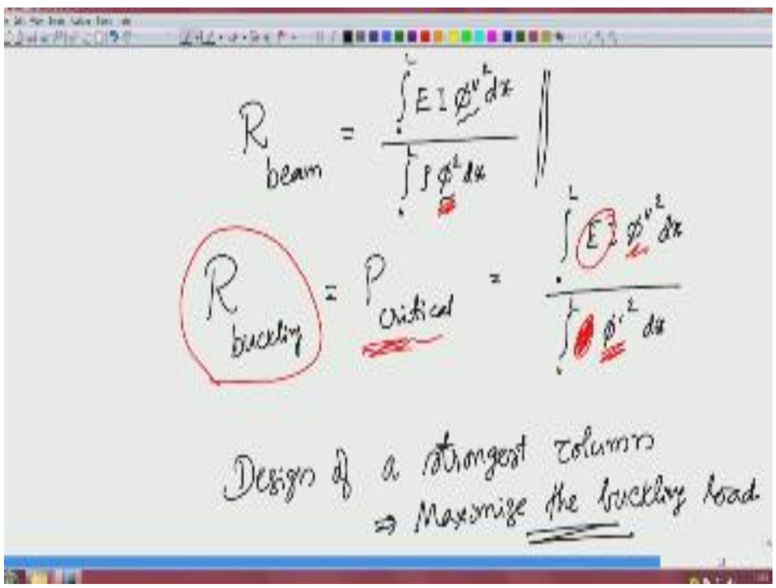
The Eigen value problem that we right there will be e ay u double prime plus λ the ρ will be there mass density for it also has been at length into u equal to 0 okay young's modulus area of cross section you get that so if you see the similarity between these two equations okay again if you assume that u is a function of X and T you can break it up into φ of X Q of T just like we have done for the string you can do for the bar also.

So we have one is bar and then this string and bar we can do this here also you can get Eigen values Eigen functions you get really quotient okay real a quotient in this case which we denote by R really quotient following the same thing in this separation of variables and substituting into the equation and so forth how do you get the equation to begin with this equation just use Hamilton's principle your right shin energy for the bar you know kinetic energy of the bar how to write it you follow Hamilton principle by Lagrange equations you get that take everything the same you can do for the beam.

You can do for the column as well okay every one of them will Rayleigh quotient and you will get a minimum characterization theorem such as this for all of them okay first of all there are infinite Eigen values and hence infinite Eigen functions and there is a this minimum value okay for a beam for a bar everything okay so real a quotient if you were to work it out for a beam again you can write kinetic energy potential energy for a beam or strain energy that you need perfume abrasions if you do this Rayleigh quotient for a beam turns out b/e I is X modulus I second moment of area φ''^{2} DX 0 to l / 0 to L ρ φ2 square deck this for a beam.

I should write it for the bar right because you are discussing bar will be similar with a ratio it is a quotient 0 to l φ a φ'^{2} DX instead of T you have EA at ease tension and this will be EA will have a similar equivalent thing then we will have ρ L and will have ρ T Square D X okay, so Rayleigh quotient can be no written but painting is this is φ''^{2} φ ^{2}square this φ'2 C2 over there similarly for a column for a buckling load.

(Refer Slide Time: 24:54)



Because that is also is something that is Eigen value problem are buckling la quotient for buckling that also becomes the P critical the critical buckling load. That if you do the same calculation you will get 0 to L this will be e I φ''2 DX / 0 to l here it will be ρ φ2 notice that for a beam it is just φ square it is V prime square so you can get the buckling equation in this form right.

So what we need to look at this function the constants mean these things do not matter no for us the value of P critical you need it otherwise you get so actually density will not be there so this row shouldn't be there for the buckling egg again I am writing the form of the equations but you have to work it out by writing what is the strain energy when any column buckles and what is the strain energy what is the kinetic energy you everything we get these equations okay you can get the critical values the these things will be I had values for this thing okay.

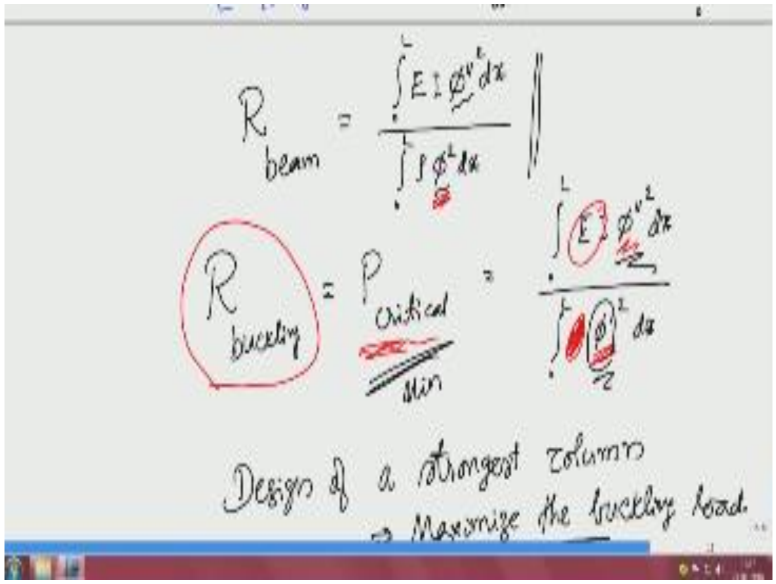
So what we will do is we will use this Rayleigh quotient concept to design the strongest column that is a column that will not buckle that is the buckling load will maximize so for that this concept of Rayleigh quotient is helpful okay we did it in detail for a string but I encouraged to it for a bar and a beam also for a column okay to see how this critical things come about all right so what we will do next is to look at the we have discussed mechanics now.

We will discuss design when I set design of a strongest column region of strongest column that is we maxi the buckling load we maximize the buckling load and that is what we will do ok so the highlights of what we have discussed today is that by starting point of Hamilton's principle we can write the equation of motion for elastic systems that are vibrating under no external force this free vibrations okay and then we got Eigen value problem for those where again values and then Eigen functions.

That is Eigen equation we got it we saw like an equation by assuming separation of variables that is space that the X variable dependence and time we separated out like φ of x times Q of T that led us to something that is constant okay some that depends only on space the term depends only on time they both are equal will be constant we call it minus λ and got two equations and so there are infinite Eigen values and hence infinite Eigen functions when we prove that Eigen values cannot be negative they are non-negative and we also proved that Eigen functions are orthogonal the perpendicular to one another.

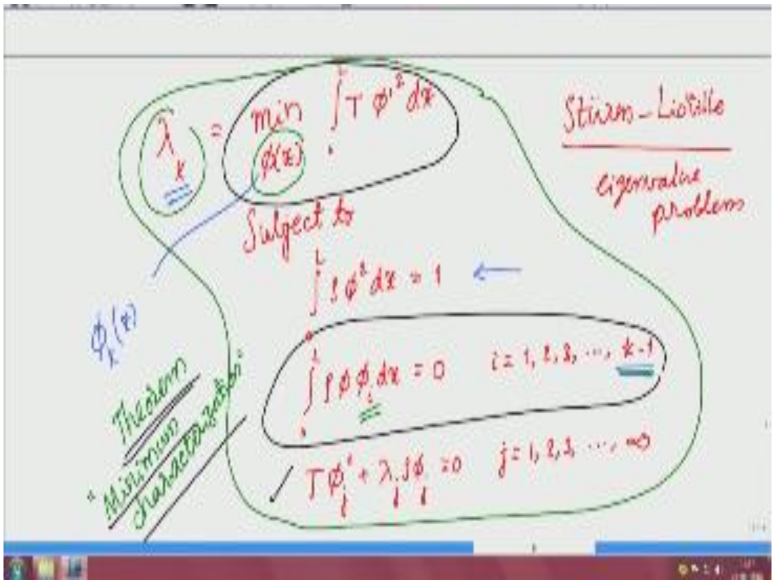
So that gave us the ability to write the Rayleigh quotient and also express any function in terms of these orthogonal functions the Rayleigh quotient is important.

(Refer Slide Time: 28:41)



Because that will give you the value will SP critical correct value even when this φ that we have is not very accurate and that it is going to be is going to be the minimum value no matter what you change for these things that is why we pose this minimum characterization theorem for Eigen value problems and that they all go under.

(Refer Slide Time: 28:59)



What will be different is just the operator that is we are taking a function doing some operation and then finding that that is just equal to the function times a constant that is the nature of this Eigen value problems as to Louilloe problems okay next class next lecture we will discuss the design of optimal design of a strong column or rather strongest column thank you.