Indian Institute of Science

Variational Methods in Mechanics and Design

Prof. G.K. Ananthasuresh
Department of Mechanical Engineering
Indian Institute of Science, Bangalore

NPTEL Online Certification Course

Today we are going to talk about the role of calculus of variations in mechanics we had discussed the beginning of this lecture but now that we have discussed calculus of variations in detail that is with constraints and then two types of constraints functional, constraints function type constraints or global and local constraints and the general variation. Now let us revisit that in the context of beams first let us look at something that we had discussed earlier about mechanics seen in two in three different ways.

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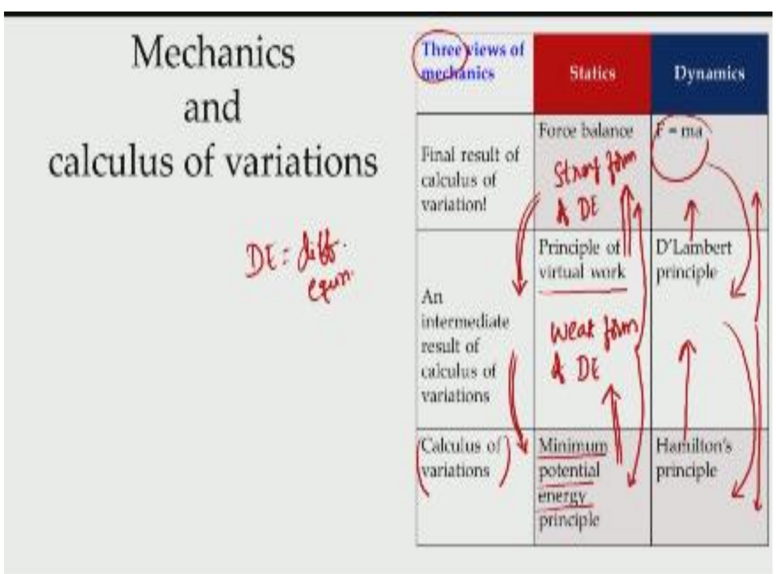

So if you look at mechanics in calculus of variations we have three views of mechanics so we have first one it is a three views of mechanics we have we are discussing both statics and dynamics, right. In the case of statics if you just want to use for force balance that is perfectly alright alternatively one can use principle of virtual work that and then if you want to find another alternative that is a minimum potential energy principle right, so this leads us to calculus of variations.

If we start from calculus of variations where we want to minimize potential energy where the variable is the deformation lot of possible deformations only one will be chosen which minimizes the potential energy that is the one that is the static equilibrium solution. If you start from here then you can move up and we will get principle of virtual work which is nothing but the weak form of differential equation which you get from the weak form we can get the strong form that becomes force balance, okay.

So if you start from minimum potentially principle you can get what we call weak form of the differential equation of the differential equation are force balance which we call strong form of the differential equation so DE when I write, I mean differential equation that governs the static equilibrium condition. If you start from force balance you can actually come here which we also is known as our real names D'Lambert method a mathematical method gives you the weak form and from there you can actually come to a minimization principle under certain conditions which will be discussing later.

So if you start over here you can come back that is if you start here you can come back to this and you can come back to this or if you start here you can go back to this and go back to this or if you start from the weak former principle of virtual work you can get the strong form or you can get this also, so all three of these are equivalent see if you know one you can get the other two so you can believe in any one of these and we will be fine this all for statics.

Similar thing exists for dynamics also so if you start from Newton's second law everybody knows this F=ma that is equivalent to D'Lambert principle which is like a weak form of that and from there you can come to something called Hamilton's principle right, are you can start here and go there and from here you go there are from this you can go to this or that these are three equivalent principles in mechanics and calculus of variations in a way ties all of them together.

Because when you take let us say statics minimal energy principle from that you can get the weak form or the principle of virtual work from there you can get the force balance which I intend to show using the example of a beam that you would have learned in your mechanics of materials course.

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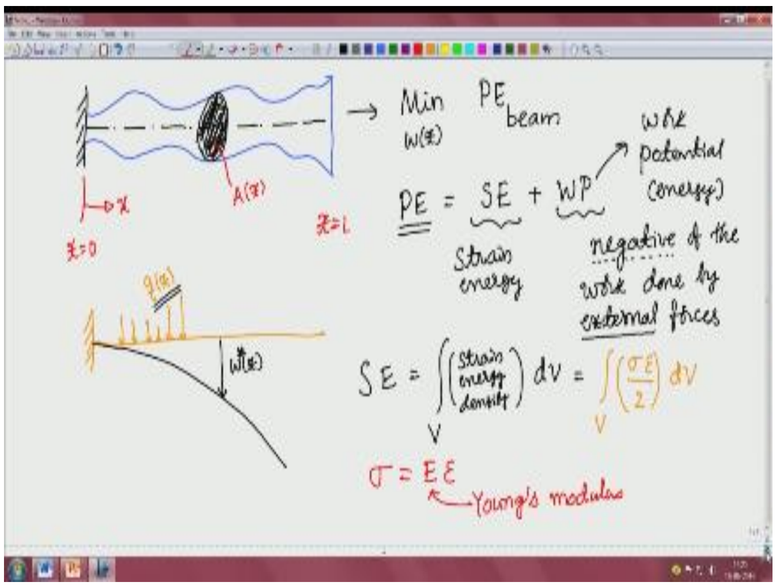

So where we have sorry, where is this can you just come I open this but I do not see the pen thing oh this is not that way okay, let us take the beam problem let us take the beam problem okay, in fact let us not worry about constant cross section beam we can take a variable cross section b I will only draw the neutral axis okay and as far as the cross section is concerned let us actually take intentionally a variable one which are normal mechanics of materials class you would not take, right.

So here we have at every point there is a cross-section whatever shape whatever size parameter they may be there let us say that is A(x) and our x starts here this is our x, x=0 0 here X =L and

A(x) is a function right, if you take up the beam problem let us start from minimizing the potential energy so if I say minimize the potential energy of this beam which has let us say some load acting on it okay, so if i say a line diagram now so I will just write the beam even it has cross section I will just show this that the force is acting on it in some fashion.

Let us call it q(x) is everywhere force acting as per the function q(x) and the boundary conditions do not matter I am just showing a cantilever beam but they will come out of calculus of variations calculation that we are going to do. If you have this so we want to minimize this potential energy with respect to W(x,) W(x) is that this beam is going to deform under the load and this displacement is W(x) okay that is a transverse displacement.

So when a beam has a loading it can deform in many other ways but it is going to take a particular deformation what we can call let us say W*(x) that is a solution of this problem minimize potential energy problem no constraints here just minimal pressure the beam with respect to W(x). Now let us say what this potential energy for a beam is in general potential energy is strain energy plus work potential as we had already stated once strain energy is this SE and work potential we can actually call it work potential energy if you just want to say that it is energies I also want to see energy explicitly stated there its work potential energy that is potential energy to the work done by the external force that is exactly what it is but it is negative of the work done by external forces here q(x) is the external force on the beam a lot of internal forces we do not consider that it is only the external forces okay.

Why is it negative, negative because in thermodynamic thinking work done on the system is negative that is how we put this as negative energy, negative potential energy directional forces restrain energy the energy stored in the structure, so this strain energy we can write or the cumulative for the oval overall elastic body the beam is an elastic body here so there we integrate over the volume okay, what do we integrate we integrate strain energy density strain energy density which is simply strain energy per unit volume.

This strain energy density is defined as the area under the stress strain curve, okay if you write σ versus ε around the curve for the linear analysis which is what we are discussing now he will be σε/2 okay, so we have strain energy density here we want to integrate the entire volume, so then we ask the question what is the relation between stress and strain so we have σ for a beam there is only one stress σ and there is one stress a one strain ε normal stress and normal strain they are related by E which is Young's modulus which is material property here, okay.

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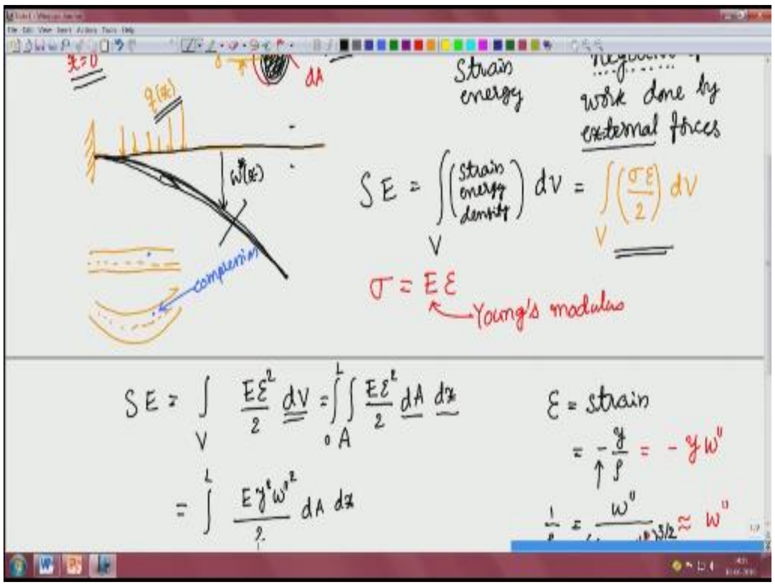

Now if we substitute over there for our strain energy okay, our strain energy here now this strain energy is integral V coming back from here σ if i substitute it will become e times ε another ε is there square by 2 or the volume okay, now looking at the beam at any given positions if I take let us say this one there is some cross section area here and the length dx let us I take, I want to cover the entire volume so I take a dx a slice of the beam and that if i look from this side it is going to be some cross section, right.

In this cross section I can take a little one something like that little square call it dA so I am splitting the dv that is the overall volume that I have this little disk okay, of the thickness dx this is dx that is dx, dx and then dA I will have a little thing that we are to integrate over the area first and then the length also so if I take that I can split this as two integrals one over the area another one over the length so that I can cover the entire beam okay.

If I want to cover the disk that we just do where integrate over the area so that is this disk we have to integrate over this entire area here and then we are integrate from x =0 to x=L then record entire volume of the beam so I put 0 to L and A over the area of the cross section and then we have $E\varepsilon^2/2$ dA and then dx so basically dv is split into dA and dx right, next thing we recall is what is ε that is a strain normal strain in the beam which if you recall from mechanics of materials is given by $-y/\rho$.

y is something that is measured from the neutral plane to the point under consideration that is I have taken that d at certain height if this is the point through which neutral axis close let us I put this in yellow right, so over here is neutral axis the dA is that in certain height maybe the other color is better, so this is y you know that if I take a point over here from the neutral axis to this at that little distance is y as the beam bends the way we take our convention to for the beam that is straight like this deforming like this is considered positive so if I neutral axis is the center like this if I take a point above the neutral axis that would be here these things will be under compression, compression and hence time will be negative and that is why we have the minus sign over here okay, another thing we know is a note that strain is linearly proportional to distance from the interlocks that is why it is $-y/\rho$ and why that is linear interfere the strain other thing we know is that $1/\rho$ is the w function that we have this second prime of it that is d^2w/dx^2 divided by $1+w'^2$ raise to 3/2 okay.

But in cases where w' is spa that is different the slope of the beam that is deforming is not significant we can neglect that so we can neglect, neglect this compared to one that we are adding to so this can be approximated as simply w'' okay, if I do that this is going to become minus y into w'' because $1/\rho$ is approximated as w'' right, so now we come back to this we can write this as 0 to l Eε is now square $y^2w''/2$ dA dx okay.

Now we will rearrange a few things to take out the things that do not depend on that do not change on the area which if you assume E is number two is of course and then the w''^2 also because that is something that does not change on the area of cross section because a one-dimensional model the beam one-dimensional model of the dimensional solid beam we just have a neutral axis so if we take a slice at a point particular X the W there is the same because if you recall the way we do this here.

We just show this we are not saying what exactly happens to the beam as it is width is there so across the width we do not say anything all of that deforms like you are only worried about a neutral axis deformation right.

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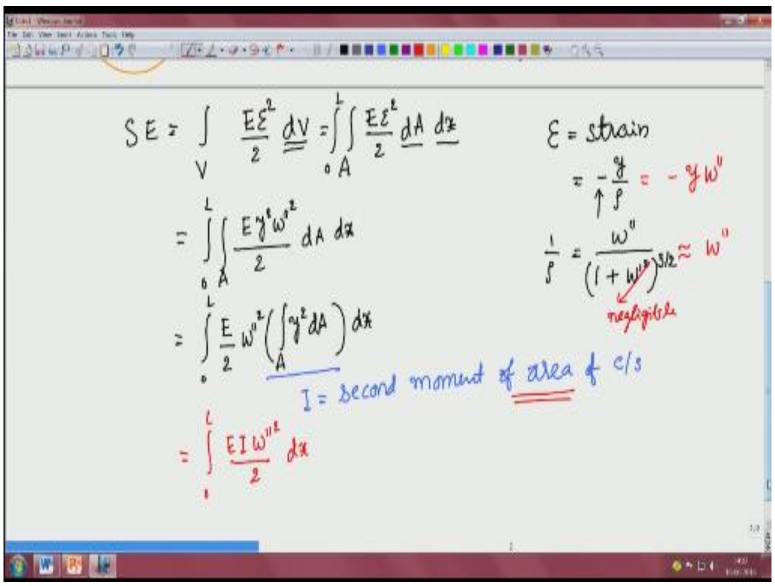

So that can be taken out what is left here we will have y^2 da over so here also di is there so I should put over the area carrying there so this is over the area of cross section okay and then we have dx now if you see this quantity that we have is what we denote as I our second moment of area second moment of area of the cross section sometimes it is called moment of inertia that is actually incorrect because there is no mass associated with this is purely a geometric property so calling it second moment of area is appropriate.

So if you had learnt it as inertia change it to call it second moment of area or not inertia okay so this gives us 0 to l will call this I now W''^{2} / 2 dx okay all this we discussing again to potential energy we have written versa energies.

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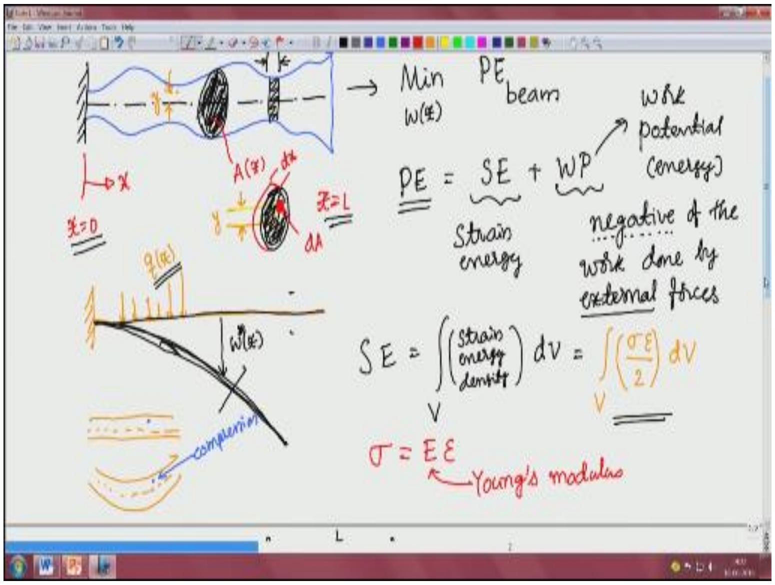

Work potential both of that we do.

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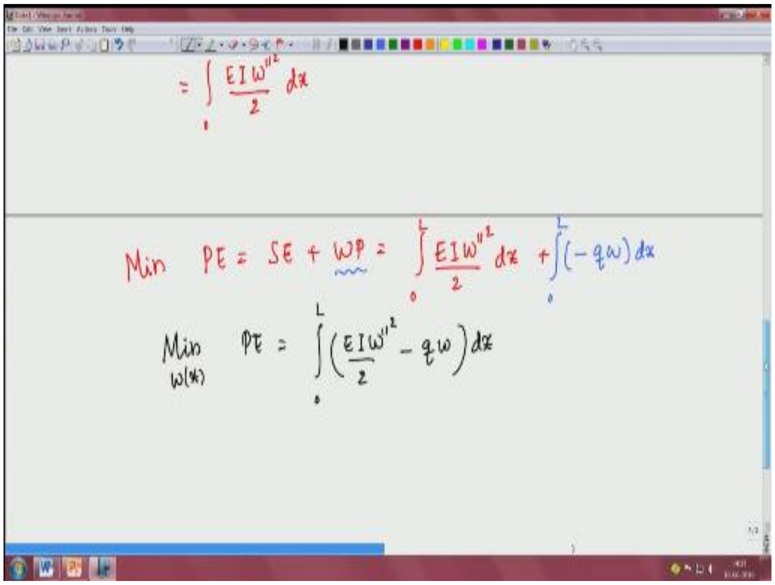

So we get so I want I would now right minimize potential energy which is strain 80-pluswork potential strain energy as we have just derived 0 to L EI W ´´/ 2 DX thurston energy plus work potential so let us write work potential in blue color that is negative or the work done by external force external force is here q and then it is being acted upon through a distance w that is the transverse displacement so this will be the work potential this also goes from 0 to L dx because q(x) is acting w of x is the consequent deformation product of these two so together we can write it as one functional so we have problem minimization with respect to W(x) of potential energy which we can write as 0 to L ELW $^{\prime\prime 2}$ / 2- q w dx.

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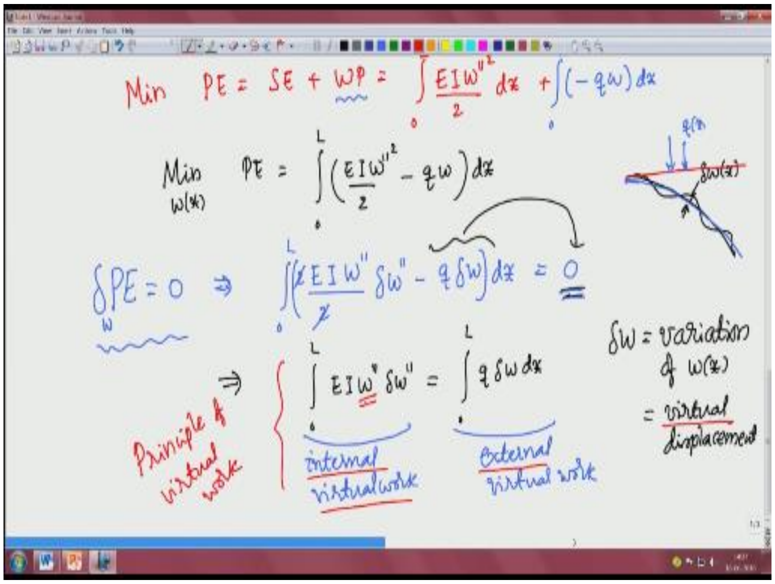

Now we got a calculus of variation problem or other minimization of potential energy that is a principle so here what do we do first thing we do is to take the variation of potential energy that is our functional here with respect to W equate it to 0 if we do that what we get is that 0 to L I want to take variation so that will become EI W´´ this square and ½ will go away so that to here and the two that you get wΔ^2 this to these two will get cancelled into Δ W ´´ - q and then we are taking variation so Δ W will be there this whole thing dx should be equal to 0 okay.

Now this particular thing if you see if I if I just write it in a slightly different form I will say0 to l IW ´´ δ W ´´ equal to because the minus is there I take this to the other side q into δ w dx okay because that is equal to zero I have taken this term to the other side right so now we see what this δ W δ W is for us from the calculus of variation it is the variation of W of x δ W of x is a variation of Wx meaning if the beam has deformed here we are perturbing around that that perturbation little perturbation that we have here that is our δ W (x) right but now we can also think of this as virtual displacement virtual displacement W (X) is the real displacement δ W(x) the virtual displacement.

So what is this quantity here this quantity here if δ W is virtual displacement this is the because q (x) is external force this is external virtual work okay so what we are saying is that somebody says that this particular thing for a straight beam if this is the equilibrium deformation under some loading q(x) okay we want to verify is it really we go back to Chris Berman operation energy and imagine a virtual displacement as if it is perturbed when it is perturbed to first order it should be equal to 0 that is exactly what our variation predations respect w says sectional work and this portion is actually internal virtual work internal virtual work okay.

So these two being equal to is nothing but the principle of virtual work principle of virtual work that is if you have a structure if you believe that a particular deformation is in equilibrium deformation then if you imagine a virtual displacement it is a virtual not real then the imaginary virtual work due to external forces is equal to the internal virtual work because this is nothing but strain right that is what we had just written for abeam.

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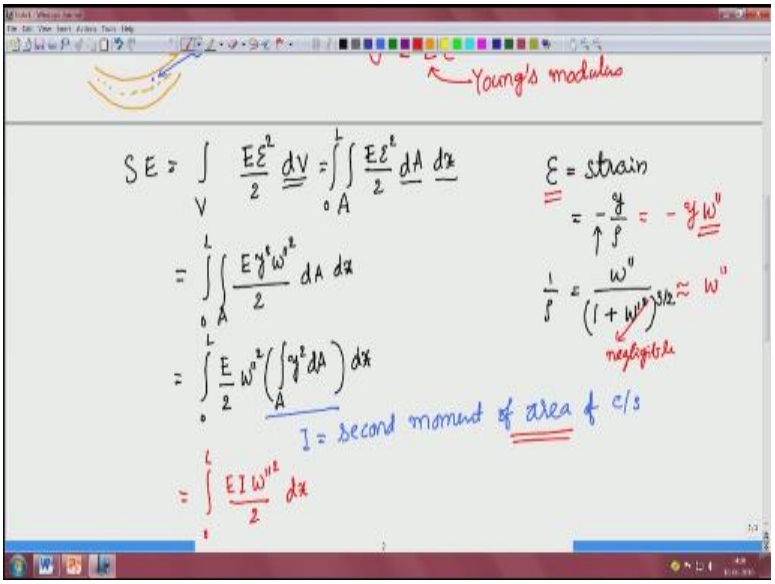

that ε is proportional to W ´´ here so the ε is w´´ y is there of course for a beam across the cross section strain varies linearly that is y is there otherwise it is proportional to δ W ´.

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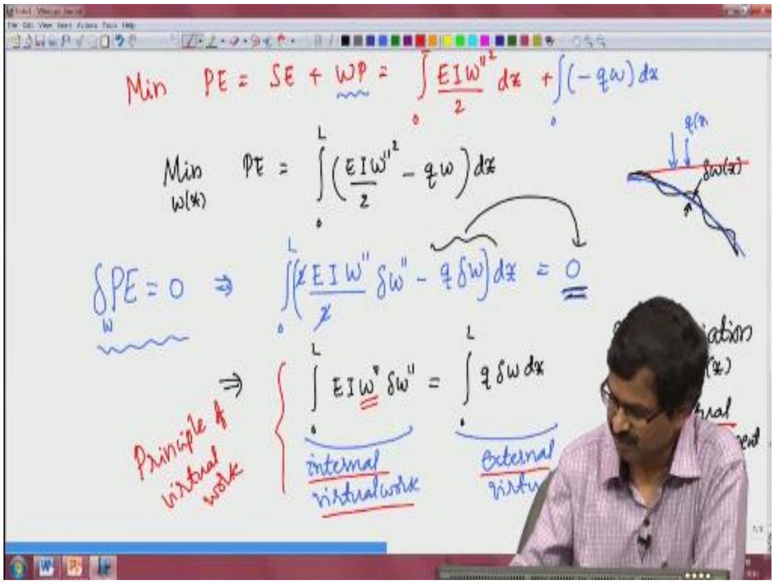

So that is like real strain this real strain and δ W ´´is the virtual strain okay so if we imagine δ W virtual displacement they will be strained so real strain real stress and then we have of course this here because stress and strain if you multiply you get the internal virtual work basically force and stress are equivalent displacement and strain or equivalent if force times displacement has units of energy stressed I am strain also use of energy so this is the work energy both have units of joules right.

So internal virtual work external watch work what you will work that is what we got so this one tells us.

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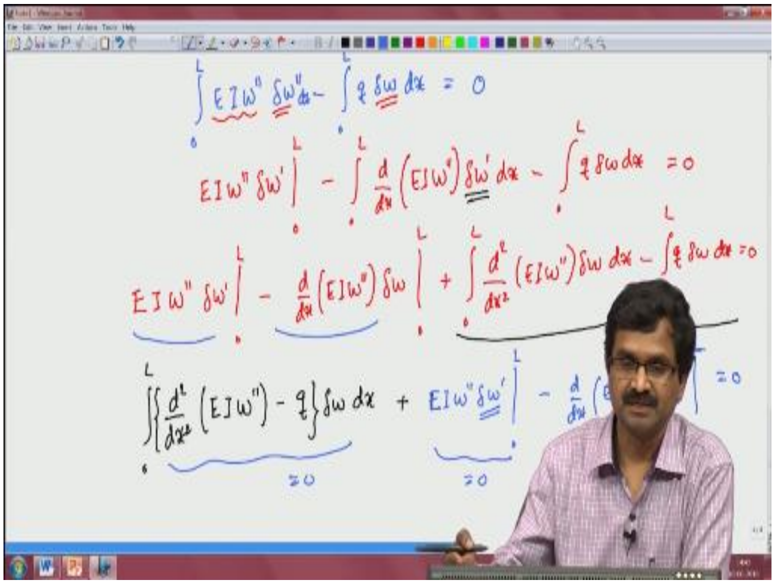

That from principle of minimal energy we can derive the principle of virtual work if you go one step further which was our integration by parts so if I start from principle of virtual work 0 to L ELW´´ δ W´´ - I'm bringing it to the left side again 0 to L q δ w well I have to write dx right so we should not forget that let me whenever have integration with respect to what right sinter is respect to x here so if we have this here so dx equal to zero now we need to because δ W is the one that is arbitrary.

So we have to get rid of this meaning we have to do integration by parts we have to do twice and that gives us the boundary conditions as we had seen earlier right so if I do integration by parts once then I will get the first function that is this thing integral a second function the heater w ´ that is that minus derivative of first function d/ dx of EI´ the cyber return d / dx our unit right prime here okay d /dx of that integral of the second function dx and this stays as it is that is Q δ W dx equal to zero.

Now we need to get rid of this by doing another integration by parts so when I do that so I have the same thing here EI W´´ times δ W ´at 0 to L minus here again the first function if I take that is d / dx of EIW´´ okay times integral a second function w 0 to L minus of minus plus minus of the minus thing of the integral of the derivative of the first function that will become second derivative r / dx^2 EI W´ integral second function that will be δ W x minus 0 to L q δ w dx = 0 now we get the boundary conditions and the differential equation so what for differential equation we take this and this I will write it in black 0 to L d^2 / dx^2 of EI W ´´ okay minus q this whole thing multiplied by δ w dx plus we have the boundary condition which I shall read in blue this one and that one.

So that will be EI W´´ δ W ´ that is a virtual straying are variations strain minus d / dx of EL W times δ W 0 to L all of them are 0 so individually we say this is equal to 0 this is equal to 0 and this is equal o 0 right unless somehow δ W ´ and LW are dependent on each other you can make up problems of that kind then you have to put them together and say it 0 otherwise you say individually things are 0 okay so if you make them individually 0.

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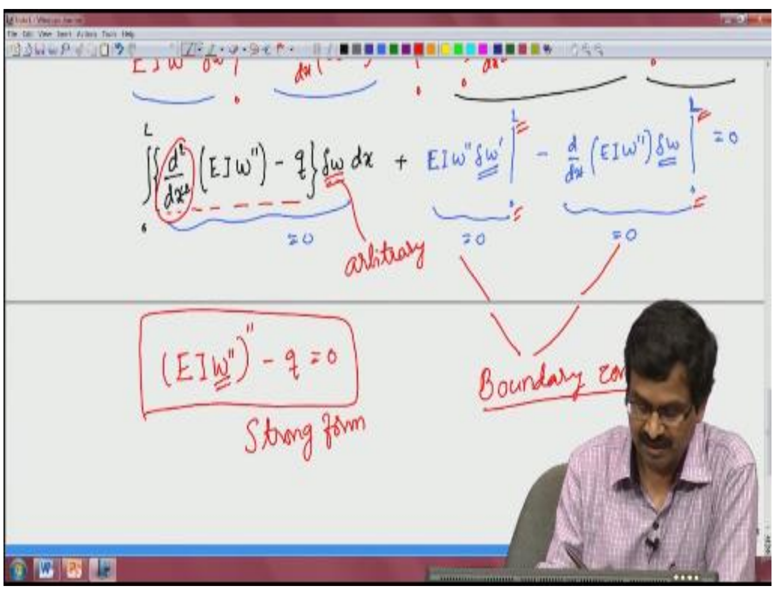

And the fundamental lemma of calculus of variation say that this is arbitrary for if you make that individually 0 which we have to hear for boundary conditions and differential equation part that is arbitrary fundamental lemma tells us that what multiplies that is this thing should be equal to 0 if I write EI´´ and this d^2 / dx^2 I will just pull it as two primes okay minus q = 0 this is what we get so if I were to be uniform that is it does not vary with x then that will become EI 40w is equal to q r - q equal to 0 that is what ever learnt in mechanics of materials when you discussed learnt beam theory but this is more general this is for variable cross section.

And then we have the these two are the boundary conditions okay these are the boundary conditions because we have forth our differential equation so we need enough boundary condition to solve for the four constants that are there w let us say e and I are not dependent on x then you get the fourth derivative of W when integrated you need four constants we have two sets of boundary conditions at 0 and L and 0 and L.

So you get four of those to solve for these boundary conditions so you can see the weak form gives rise to strong form this is strong form y is a strong form not strong worm it is strong form strong form why is this wrong form because where needs to be differentiable four times whereas if you go back to our principle of virtual work here w needs to be differentiated only two times that is why it is called weak form or the principle of virtual work.

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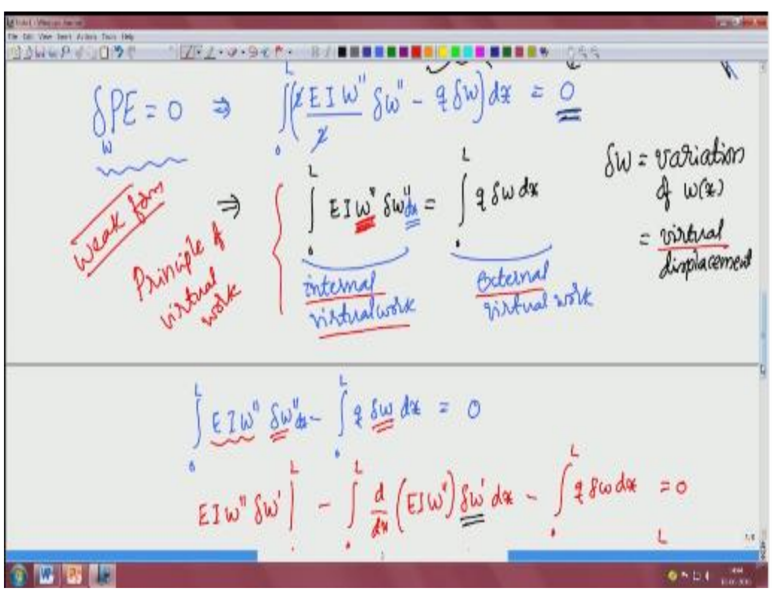

Whereas the other one is strong form okay so what we see now with the example of a be

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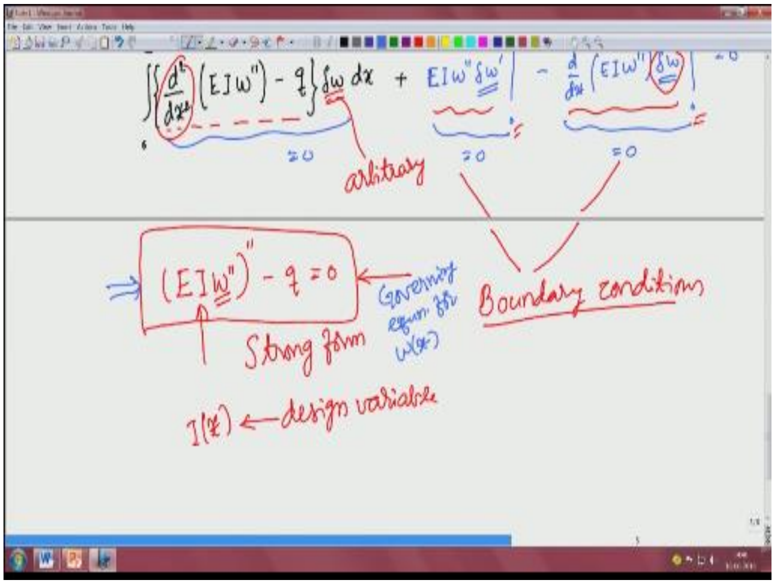

That we can use principle of minimum potential energy go back to week four more the principle of virtual work from there we can go to force balance our differential equation along with the boundary conditions so what our boundary conditions are there whether w specified or slope is specified we can go back to these conditions and write out our boundary conditions deliciously essential are normal or natural so when W is not specified this thing is not zero.

So EIW ´´´ should be zero or d/ dx of EIW´´ B 0 where I can be version of x as we have taken and this equation is very important for us because now when we want to design in a beam we actually make W (x) as a state variable and look at I the cross-section as a function of x is what we have taken that is what we design that becomes the design variable and this equation that we have becomes the governing equation for the state variable which is W (x) governing equation for W W(X) cannot be anything it has to obey this equation.

Because that is what controls are started similarly for dynamics if you do the Hamiltons principle will give you the dynamic equation which we will discuss in a later lecture when we look at the calculus variations role in dynamics thank you.