Indian Institute of Science

Variational Methods in Mechanics and Design

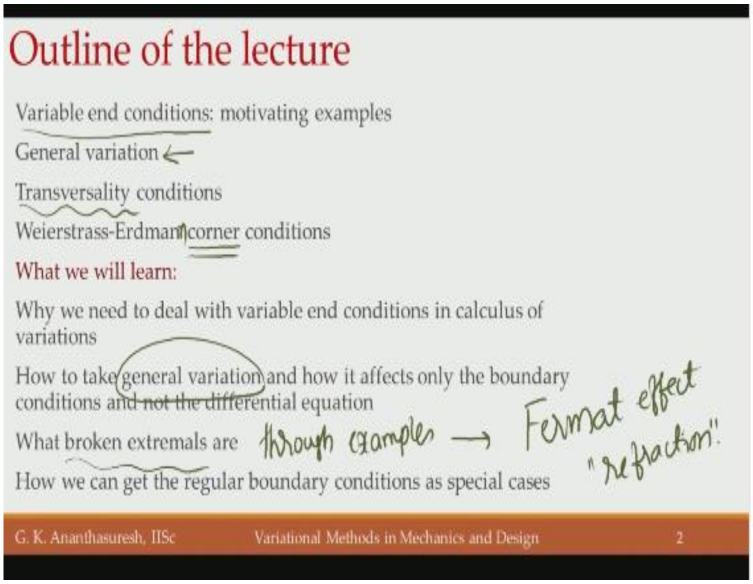
Prof. G. K. Ananthasuresh
Department of Mechanical Engineering
Indian Institute of Science, Bangalore

NPTEL Online Certification Course

Hello now we discuss one other small concept but a very profound concept in calculus of variations called general variation that leads to a number of interesting things such as what we call transversality conditions, broken extremals by broken what we mean is actually not made in two pieces but something that has a kink that instead of having a smooth curve there will be a kink such as this that is called broken extremal.

As if you have a continuous wire suddenly if you make a kink in it that you say wire is broken and that kind of broken extremals and we also look at corner conditions basically those broken extremal points or corners physically a corner in the curves we will discuss that and that also can be extended to surfaces that is one dimension two dimensions and three dimensions it has important implications in applying variational methods to mechanics will consider some of those examples as well.

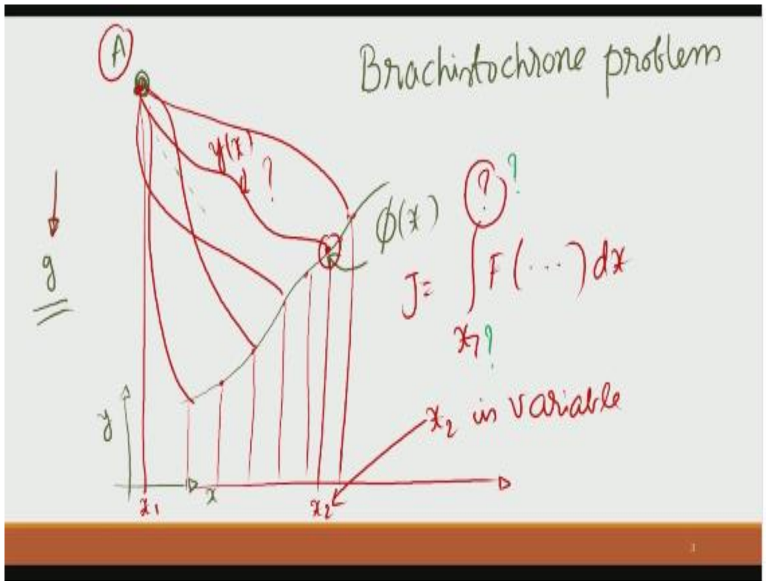
(Refer Slide Time: 01:25)



So let us begin with this concept of a general variation and look at outline of this part of the lecture so we will consider variable end conditions that is what leads to the concept of general variation and that leads to what we call transversality conditions and also this corner conditions named after we are stress and Edmund actually there will be another end here okay, what we learn here is how to deal with variable and conditions and how we can take this general variation which is more than eager to variation that we have studied.

So far and we see how broken extremal looks like through examples so we will do we look at some examples to look at this broken extremer in fact one example that we would take would be the very first problem we had discussed which is the Fermat effect refraction okay we can call it Fermat observation we can even call Fermat effect because he is the first one to explain refraction of light saying that light takes a minimum time path and not minimum distance path okay so that Is what is our agenda for this part of the lecture okay.

(Refer Slide Time: 02:49)



So as a motivation for this let us consider the famous problem in calculus of variations which is under the effect of gravity if I have point A and point B we say if I were to connect it with a curve and make a slide out of it if I leave a ball here and that ball will fall under the effect of gravity you should take the minimum time that is what we call Brachistochrone minimum time problem that is a famous problem this famous problem is solved more generally and that is actually general variation that we need to discuss.

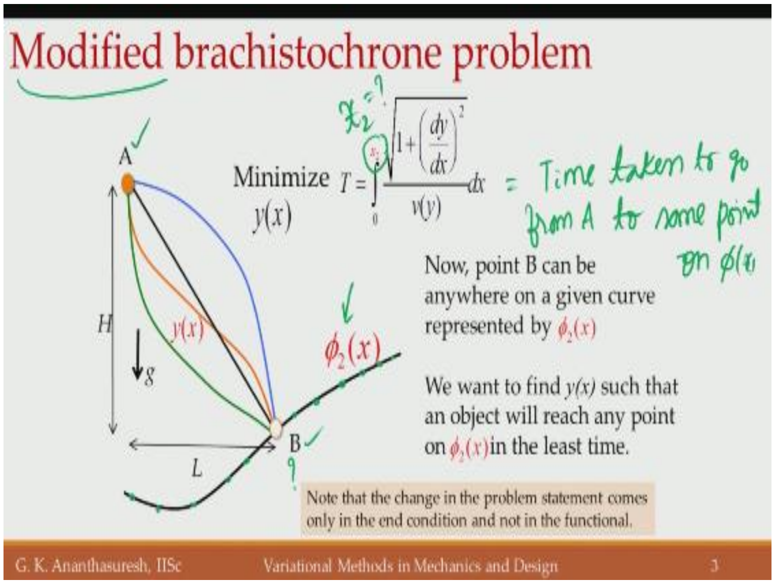
Now what we say is that let us say that this point B is not there okay instead what we say is that this point can lie anywhere on this curve okay we can take some coordinate system let us say x and y okay let us take a curve called Ø x okay that is this now we say we have a ball as it is shown over there and say that that ball has to come to some point on this we do not care which one okay.

You just come and touch this thing under the effect of gravity starting from this point right there are many curves right so we can go there we can go here we can go there so there are lots and lots of possibilities at another effect of gravity you make slides in each of the cases whether string this ball onto the wires you can just constrain to move there or we can make a slide so that you know goes and goes there which one of these will the ball take rather which curve we can say which curve if I say this is y of X right.

Which curve will minimize the time to go from point A to some point in on f of X on that given curve this is our given curve that is the modification we do what is the consequence of this consequence is that when you start let us say these our x axis if I do this I said this is x1 right and what is x2 we do not know it could be this it could be that it could be this it is variable right let us say y star is a solution then this comes x2.

But this x2 is variable meaning that we know x1 but we do not know that and yet we have to have the integrand with all its glory first derivative second derivative function many variables and whatever right so are many functions when it have the functional we do not know the bound right it need not be only the ending thing it can also be the beginning thing one of them we do not know is that is our real thing here we may not know this we may not know that we should be able to or maybe both of them we do not know that is also possible we can makeup problems where both of them are not there okay, that is a modification of the brackets token problem.

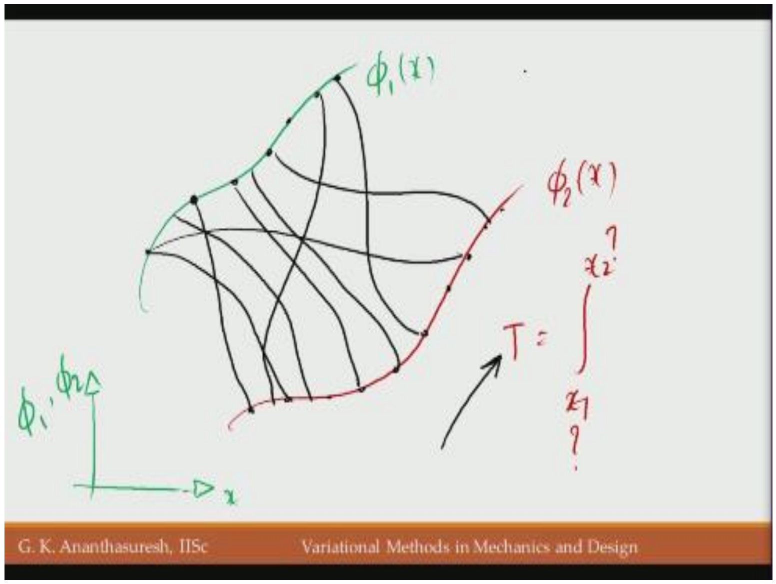
(Refer Slide Time: 06:34)



That is what is shown on this slide we specify point A all right but B can be anywhere on this curve and hence we have a variable condition of this x2 not specified so this is X 2that we do not know we have to find it as part of the problem you can actually see the generality now right because we are not just minimizing the functional that we have but if the functional is an integral the bounds are not known okay.

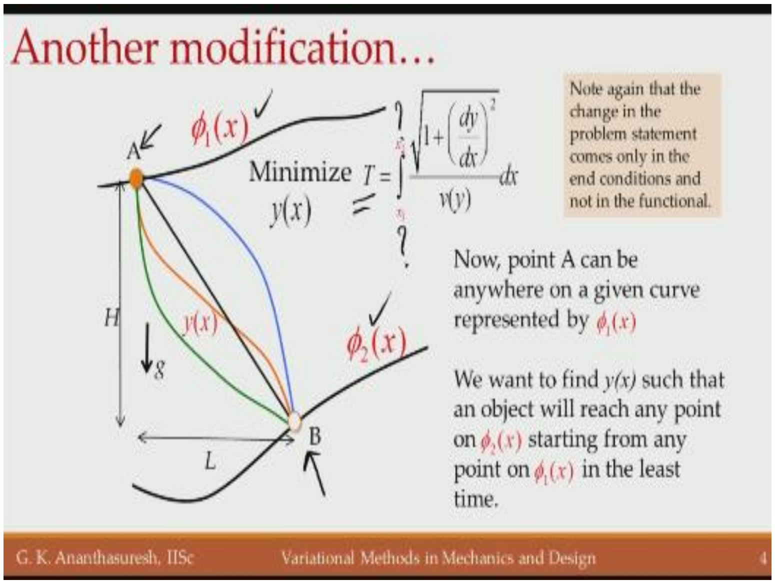
That is the modification of the blackest of one problem okay were Ø2 as a function is given here where point B has to lie right and this is the time taken this is the time taken by the B2 go from point A2 some point on the curve Ø 2x time taken to go under the effect of gravity from A to some point on Ø of x we do not know what B is that is an unknown well that is what we will solve today okay.

(Refer Slide Time: 07:54)



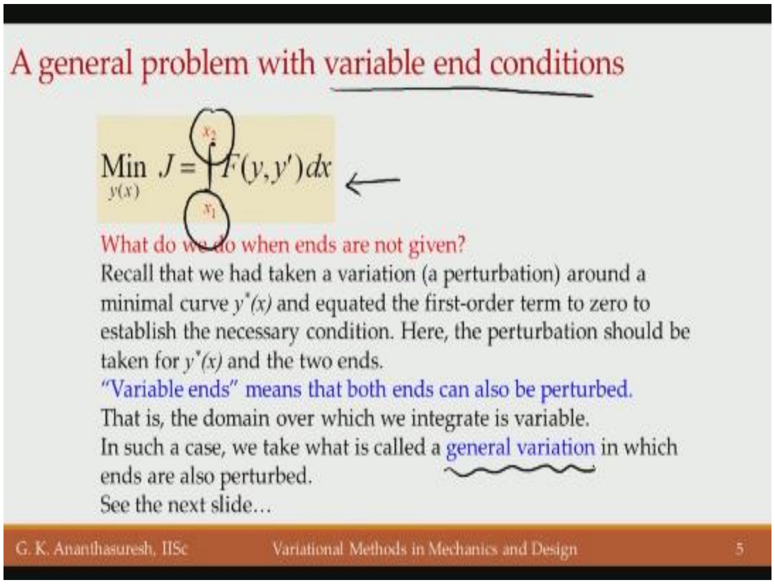
And if you wanted to make it even more general as I said I can have one function there let us call that Ø 1(X) again choose coordinate system this is x-axis and y-axis is Ø 1 we can also have another thing Ø2 okay let us say this is Ø2 that means that my integral of the time taken I would say x1 x2 but I do not know what x 1.is I do not know what x2 is both are variable because I can start from any point on this several points right. Many, many points on this and end up at any other point so now our such space has really gone so I can start from there and go here to go there or start from there and come here and right so possible curves that we have are too many right out of all of those things we should pick the one that means this T this is the variable and conditions general both end conditions are very okay and that is what we have shown here.

(Refer Slide Time: 09:08)



S this v Ø1x, Ø2x are given and we need to find a point A that lies on Ø 1x point B that lies on Ø 2x and minimize our time were both the ends are not known in fact whether one is known or both are not known does not matter one is not known both are not known does not matter once you understand how this type of problems can be solved okay.

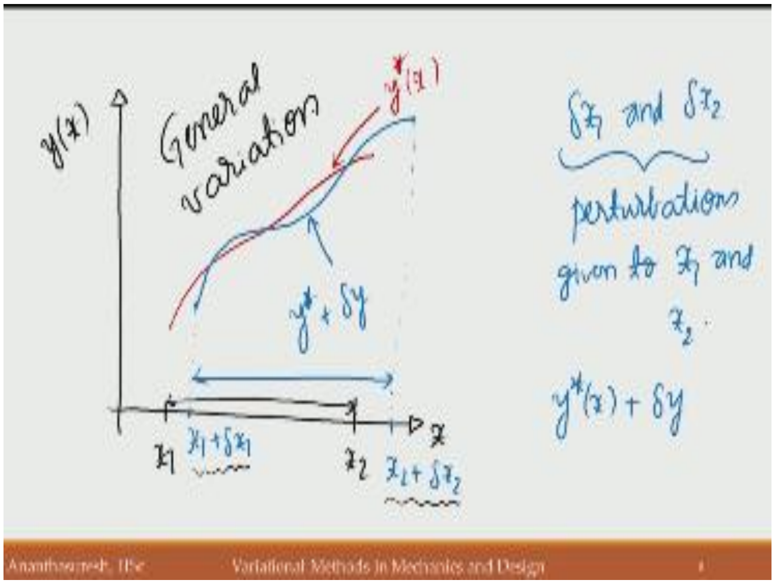
(Refer Slide Time: 09:40)



And so our general variation that is needed for the problems which have variable and conditions looks exactly like this the change is only in the ends those are variable ends we do not know what those are where to write them okay for that we need to do this general variation concept along with why we should arrive at x1 * x 2 *

So now we need to find so far just a function we have to find a function which we denote as y * X but here we also need to find x 1* and X 2* if they are not given sometimes one of them may be given other is not given but sometimes both may not be known you both are given that becomes a normal problem. If one of them is both of them are not given then we have this concept of general variation.

(Refer Slide Time: 10:38)



What is general variation we talk about let us discuss that first so let us say we take our x axis and this is our y ax is the function that we are trying to determine right so let us actually call this X 1and X 2 so let me actually draw a curve which might be say from here till herein some form okay that is what we call y of X which may be the solution y* also okay now in order to check if that is let us call this Y* which is the solution that minimizes a functional right.

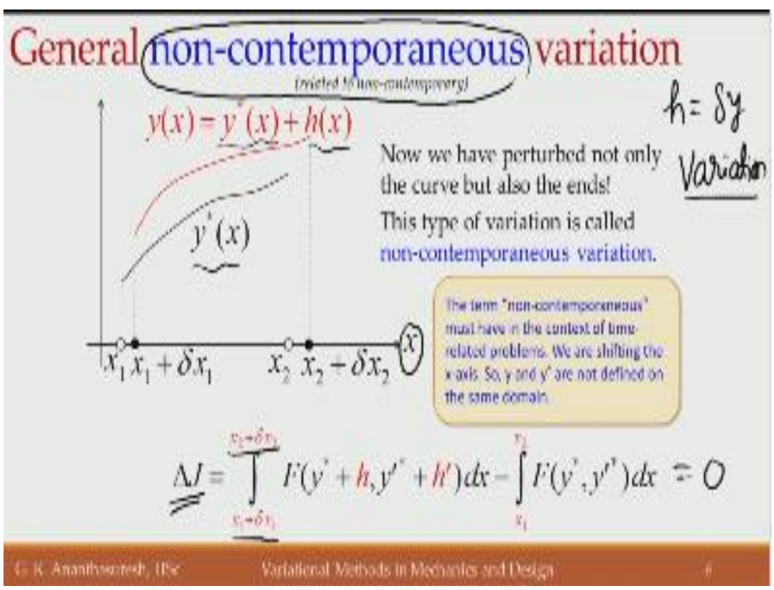
But now in order to check whether it is actually the minimum we always been doing perturbation right that perturbation is this Y * we need to part of it so that it is different now since X 1 and X 2 ends are also not known we have to put them part of them also let us say this becomes x 1 + δ X 1 and X 2 becomes x 2 + δ X 2 so δ X 1 and δ X 2 these are the perturbations in what perturbations two let us call it they are not perturbations in their perturbations given to the ends.

Which we do not know that is why we need to do see whenever we say local right from the beginning of this course you have been saying if somebody gives you an answer and say this is the minimizing quantity a number or a function we need to perturb it and see locally it actually gives the least value objective function with or without constraints so here also when we say that this is a minimum it is not enough to perturb y so Y * that we have right we can give δ Y for it perturbation right.

But we are not satisfied with that here because end conditions are also not nodes were part of the end so δ x1 x2 are those perturbation that means that I need to define a new function let us say from here that one and from here somewhere we have to do this the new function has to go from here to here something like that okay so from the red curve to blue curve this is the perturbed 1 this is y * that we have plus δ Y right notice that the domain of δ Y has changed that this is the new domain the old domain if you see the old domain is from here to only here x1 to x2.

But now it is from x1 plus δ X 1 to X 2 + δ X 2 that is what we should do and see what happens okay it may look very complex at first sight but once you understand what this general variation that we talked about right once you understand the answer is actually quite simple it is a profound concept but is a very simple concept right, we are perturbing the domain as well as the function.

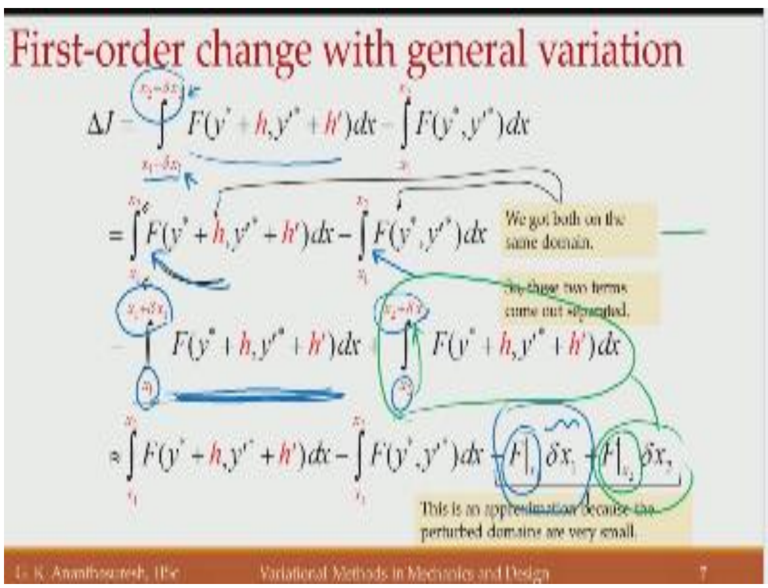
(Refer Slide Time: 14:32)



And that is what is shown here and we have a very fancy name to it called non contemporaneous variation okay what that means is that this concert was dealt mostly for dynamics problems we have the independent variable X was actually time T right when you are saying that I want to do from t1 to t2 your function defined now we have t1 + δ T 1 T2 + δ T to write because of that this non contempt that you are not do in the same time right it is not contemporary it is non contemporary that is what this is general non contaminants variation so we have our Y * that is given and then we have the perturbed one with this Y * +HX in our terminology our δ Y both.

So H is δ Y we have been interchange in that basically that is the variation here we call it general variation this is variation we are called general variation because of change is the domain also so we take this δ J to see after perturbation the functional value increases or decreases or two first order it should be equal to 0 that is our criterion to first order the function should not change and then we will say whether it is low minimum or maximum so we do from δ x 1 + x 1+ δ X 1 to X 2 + δ X to subtract from it our usual functional both original domain okay.

(Refer Slide Time: 16:10)

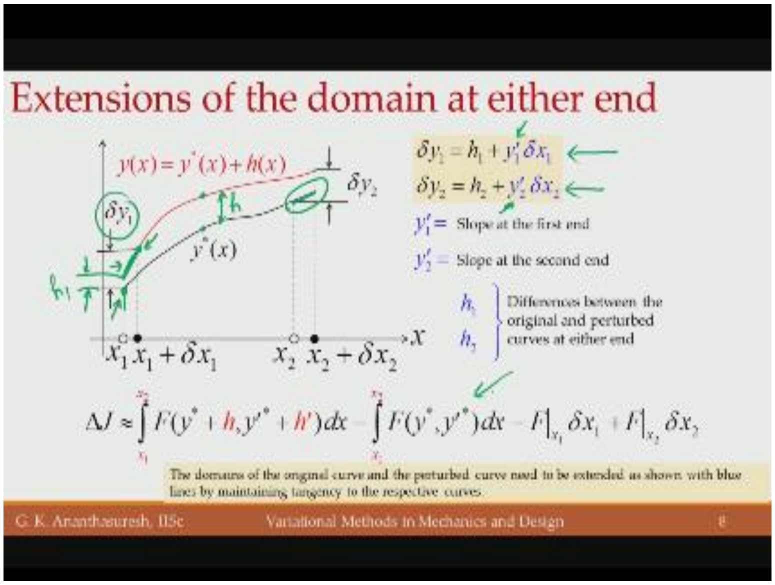


And that is what we have taken but we rearrange a little bit so that our limits are x1 tox2 for the functional with H and without H if you look at this integrand okay let me change the color of the ink okay if we look at this over there and this over here both have the same domain but we know that originally this one that is integrand the depends on the function and the perturbation okay that one we have it was from δ X 1/ δ x 1to x 2 + δ X 2 that is from here to here so we need to subtract and add so that it goes from x 1 to x 2 okay.

So this part is subtracted because this is included when you two when you go from X1 whereas it is starting from X 1 is δ x 1 so X 1 to X 1 δ X 1 is subtracted and x 2 δ x 2 was there here okay originally we have this for further that has to be explicitly added so that everything is the same now is the critical thing very simple but very important one if you look at these limits it is going from x 1 to x 1 + δ X so that means that it is going over a very small distance δ X 1because small distance δ X 1 we can simply write that portion as f value at X 1 times δ X okay so this particular integral that I am underlining is nothing but this little term here.

Because the integrand is not going to change much because you are going to make δ x1 a perturbation δ x 1 δ x 2 as small as possible that becomes F evaluate that point x1 times that and likewise the second one we take a different color and say this one right that basically amounts to this term because in that little range from x 2 to x 2 / δ x2does not change it is enough to take FA at that point time δ x two. Other two things the first and second term that is what they areas it is that are in this line is usually what we had earlier okay that is the difference.

(Refer Slide Time: 18:36)

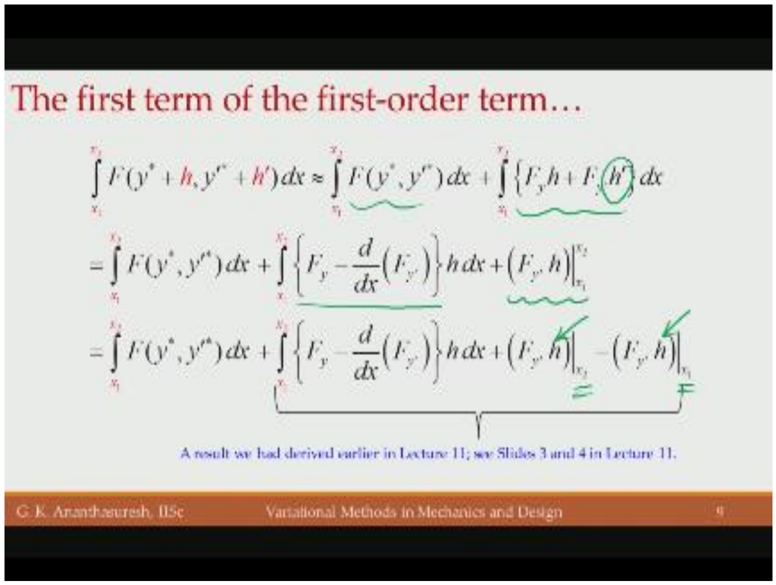


One other difference if you see what we have here is the age that we are writing okay I just said an earlier slide that age is δ Y but here there is a small difference okay the difference is what we will find out now right if you see what is perturbation H versus what is perturbation δ Y okay let us understand δ Y if I say at the left end δ y1 it is from a point that is here to the new point that is there okay that is δ y1 but what is actually h that is H will be the between the two functions if I take any point here that will be H okay.

That will be H here whereas δ is the corresponding points the contemporaneous we can call it that way because this points actually correspond to this point by the end right again this point okay is correspond to this phone if you put arrows so this point and this point we are comparing that is δ y1 whereas at the same point if I take and same one of the curve that is H perturbation the function where a delta y is actual variation including the domain condition there okay.

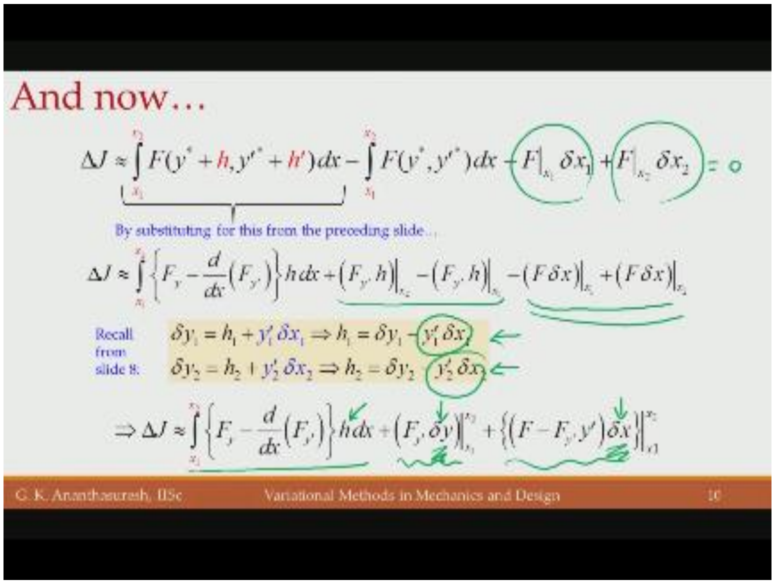
So we have now delta and δ y 2 and those relationships are given here by extending the curve if you see we do not have anything here if I want to know what is H here at this point x1 what is H we do not have a point there does not come so we have extend the red curve like this right I am or blue I am writing this extended curve the same slope and that is what is given here y 1' times Δ X 1 similarly Y to parent Δ X 2 because we are extending that a little bit over there like that because we do not have want to say what is H there and what is Δ y1that we have from here to here it is h1that is from here to here that is h1plus that part which is y 1 ‘ Δ X 1.

(Refer Slide Time: 21:06)



Similarly Δ y 2 is h2 plus y 2 ‘ Δ X 2 and that is what we will use when we try to solve this for now we have h1 h2 we replace that in terms of Δ y1 and Δ y 2 okay if we do that what we end up getting is our then we have to do the usual thing that is go to variation equivalent here when we have this and this okay f y star Y ‘ star we have Y sub y h y, y ‘ H ‘ okay and in order to get rid of this H ‘ we do integration by parts that gives us the differential equation part of it integration by parts will give you d by DX the usual thing or you also get the boundary condition okay.

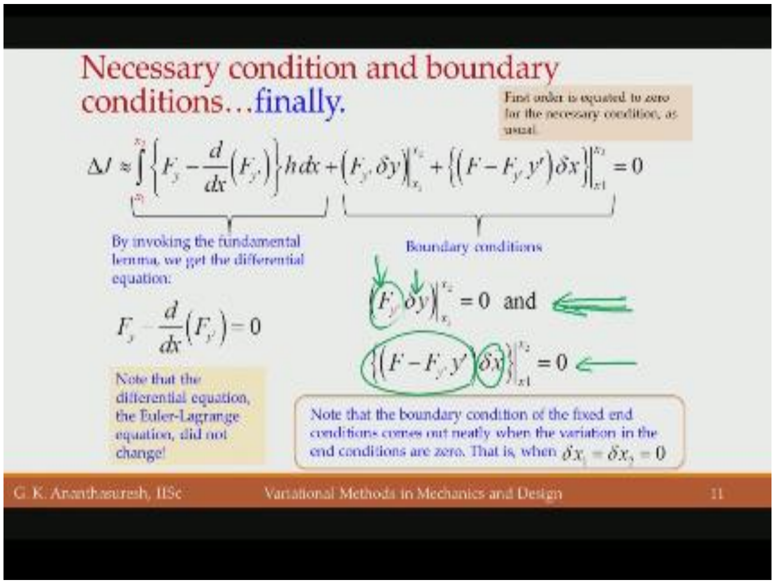
(Refer Slide Time: 22:10)



And that is what is shown here with the boundary condition put at x2 and x1 now what we do is for this H we replace in terms of our Δ Y and Y ‘ and so forth right if we substitute there and also remember that Δ Z Δ J that should be equal to 0 had these terms also okay now overall what we get in Δ J substitute everything boundary conditions come these are the usual boundary conditions is a special boundary condition that we got because of the perturbation of the domain itself that when we substitute what we had for Δ y 1 and Δ y 2 in terms of h1h2 what we end up will be this integral to which we can apply fundamental m of calculus of variations because H is arbitrary.

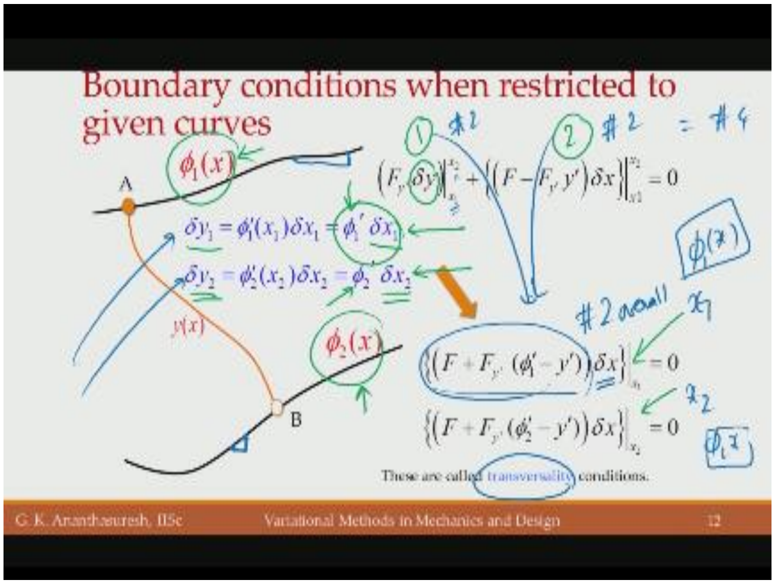
And hence we get Lagrange equation as it is but now we get perturbation in Δ Y and perturbation Δ X okay because additional one comes y ‘ because of these terms right this notice that now we have perturbation Δ Y function and domain independently we get boundary conditions were both okay.

(Refer Slide Time: 23:10)



And that is what we have we get in the function perturbation this is domain perturbation if domain is not perturb it is fixed let us say that endpoints are given x1 x2 are given then this will be 0 they can be satisfied so that is why it is called in general it is valid for this and the specific case we discussed so far in collective variations where ends are given when ends are given Δ X is zero and so it is satisfied if it is not given what will be 0 is that f minus f ‘ x y ‘ is equal to 0 okay we can actually get it out from the general thing we can get the specific one that we have discussed so far.

(Refer Slide Time: 24:09)

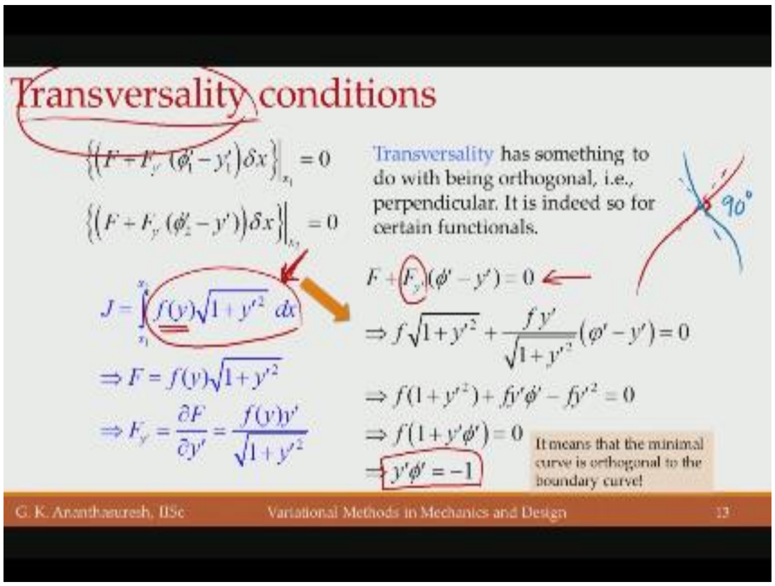


And for that boundary condition is exactly what is shown here though f by ∂ y ‘ which is that x Δ y equal to 0 Y is specified that is 0 Y is not specified that is 0 okay that remains the same okay now so far general variation we said X 1 X 2 just not known whereas now we are saying they have to lie on some curves fee one on this side fee one on that side v two on this side what does that mean that means that when you take this into account and observe these relations right Δ y1now cannot be anything because he has to satisfy between Δ X 1 Δ y1 we have the derivative of fee 1 and then Δ y 2 and Δ x 2.

We have derivative of V 2 unit substitutes earlier this boundary condition this boundary condition will separate now we can replace Δ Y in terms of that and for both ends then we get separately forx1 and x2 see that one and two together let me change the color of the ink yeah one and two together become of course these were from x1 to x2 right now we have written one condition another condition X 1 X 2 because we have two different curves so here the curve involved was fee 1 X right here the curve involved is fee 2 x okay one and two actually were at x1 and x2 there were actually four conditions two conditions here two conditions here that becomes actually only two conditions overall okay.

Here 2 plus 2 total four conditions are there x1 x2 this condition X 1 X 2the other condition for but now we have need to because we have found a relation between Δ Y Δ X because once I am here if I have to be here when I move when I move Δ x1 I also have to move Δ y 2 so those are constrained okay let me draw a little bigger may be somewhere here if I move from there to here if I move so much at necessarily move Δ Y okay that is what are the relations okay and these are called transversality conditions right.

(Refer Slide Time: 26:55)



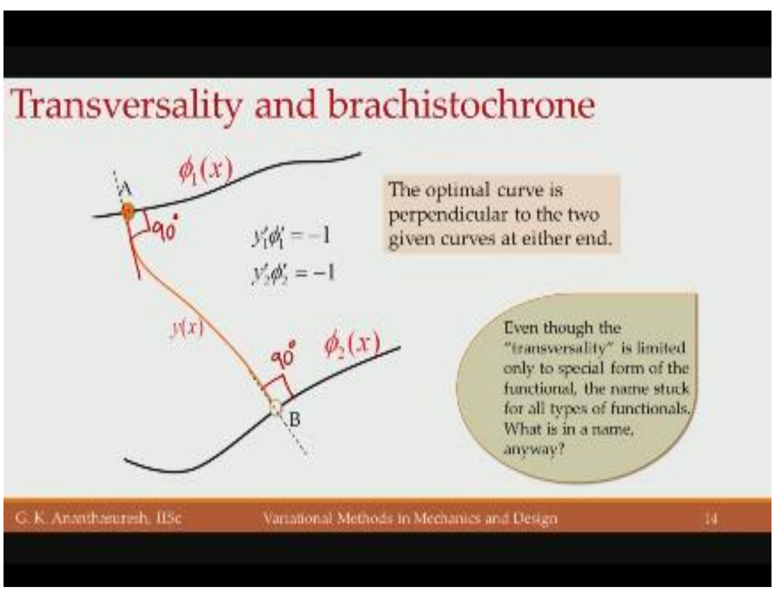
Why the name we will examine all right so what the conditions look like that when the end is fixed then Δ x is 0 then this must be 0 then this can be need not be 0when it is not fixed that that must be 0those are the transversality conditions we have we understand what those okay so transversality actually comes in the following way if we take a problem our integrand is of a particular form because historically and that is how it got developed the integrand here is of this form f of y and then y ‘ has a specific form of square root of 1 plus y ‘ square as you see this you recognize that basically d/s or the arc length on a curve.

That is square root of1 plus y ‘ square DX squared DX square DX square plus dy square square root when you take DX out that gives you1 plus y ‘ square and the F ‘ FF of Y can be anything if either that form now we substitute into the transverse condition that we derived in the last slide if you substitute for F Y ‘ because that is what we have then we get F which is simply F times square root of 1 plus y ‘ square and here we have ∂ f by ∂ y ‘ that gives you this if you simplify as it is done here you would end up seeing this Y ‘ fee ‘ is equal to minus 1 what does that mean we are looking at fee of X curve and then y FX curve their derivatives at that endpoint are their product of the derivatives is equal to minus 1.

That is true other curves which are perpendicular to each other so if I have one curve let us say there is another curve let me take a different color for it that is I have another curve like this at this point okay it is90 degree slope ok there are orthogonal two curves at that point right how do you say we have to draw a tangent to this and let me draw the tangent to this in its color tangent to here between those two we have 90 degrees so are orthogonal that is the thing that is their transversal right that transfers to each other that is why the name transversality comes in okay.

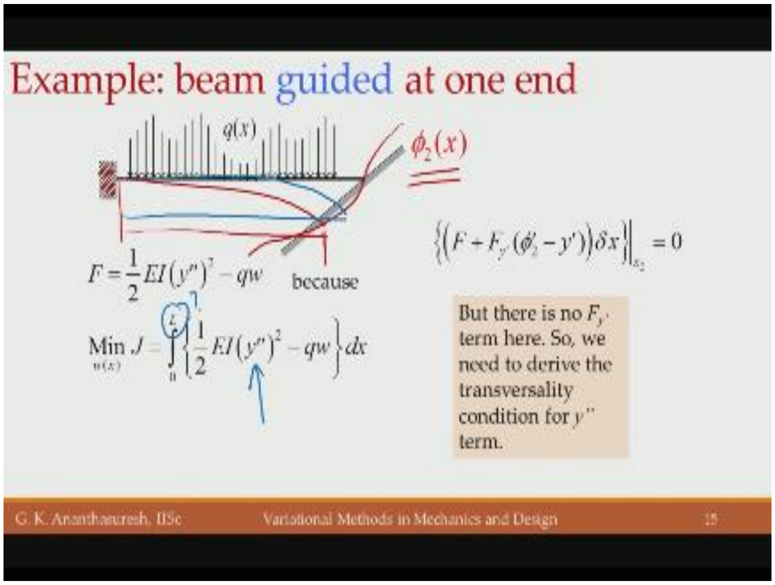
That is transversality alright that is only for specialty but the name has stuck as it happens in, in any field somebody calls it something that names get stuck and here even when the integrand is not of this special form we still call them transversality conditions okay.

(Refer Slide Time: 29:32)



And four brackets talk alone if I take fee 1 x and fee 2 x then what you would know is that if I choose the point a then it has to be perpendicular to the curve there, there should be 90 degrees and likewise at this point also should be 90 degrees whatever point you take go there ok transversality here exactly matches because you will have y ‘ in the same way here okay.

(Refer Slide Time: 29:57)



And let us look at an example it is a an example that requires little bit more work for us to apply what we have discussed there is a general variation consider a beam from mechanics it is our, our normal beam we have fixed one end right that is no problem the cantilever connection right side we are saying that there is a function fee to X here I am showing like a line but can be a general curve also like that fee2x and this right end of the beam can be anywhere this is a physical problem right you have put something and you are propped on the surface as you apply the load it is going to slide on it right.

So what is x2 for this beam we do not know because if the beam bends like this that is from here to here only right in another hand if the beam if I use a different color another hand if the beam bends let us say like that then the domain is from there to here so variable and condition so we do not know that end point we do not know that but the difference here is that we have now y double ‘ we discuss the previous slides when the integrand has y ‘ we discuss now to discuss what happen is y double ‘ what happens when you try triple ‘ and so forth okay which is what we will do in the next part of the lecture thank you.