Indian Institute of Science

Variational Methods in Mechanics and Design

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NPTEL Online Certification Course

So we have four equations and four unknowns so let us look at them I try to solve okay.

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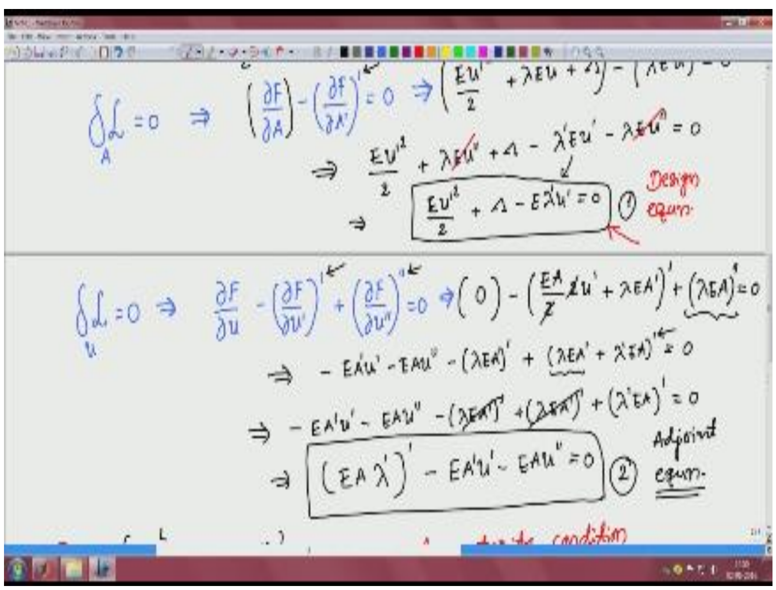

So let me first show the first equation look at this $EU'^2/2+\lambda - eE\lambda'u'=0$ so let us first write that equation here.

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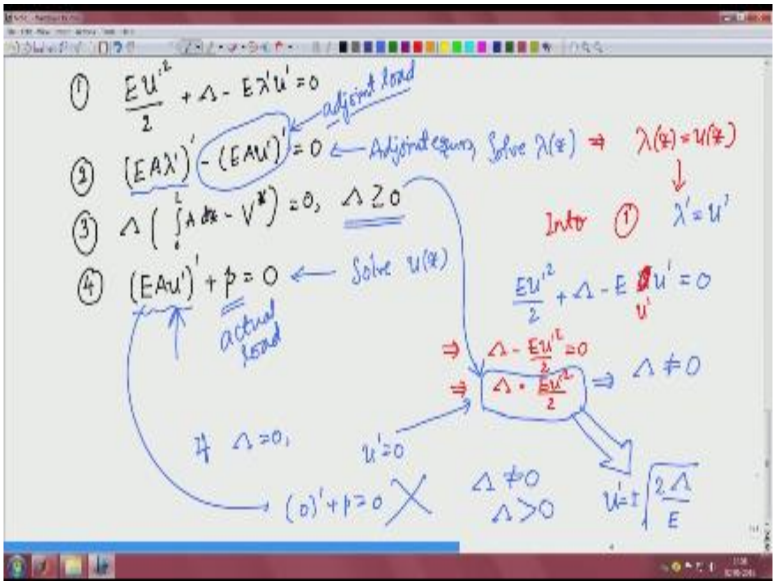

Equation 1 we have $EU'^2/2-\lambda+\lambda$, let us look at that $+\lambda -E\lambda'+\lambda-E\lambda'u'=0$ that is our first equation second equation is $(EA\lambda')'$- $EA'u'$ EAU'' in fact if we notice these two terms okay actually are I can write it as – $(EAU')'$ because if I expand this $EA'u'$ and EAU'' we get so it is EAU λ'' - EA'' okay, if I write the second equation which called adjoint equation okay, which is EAU'' rather I think we had the λ' first and then u' later, so this one is e a λ'' - $EAU'\,'=0$ okay, and then equation 3 which is our complementarily condition that is λx integral 0 to l A dx - $v^*=0$ and the condition that λ should be greater than equal to 0 and a fourth one which is $(EAU')'+p=0$ these are our four equations.

Now one thing we notice over here which is adjoint equation if you compare that with our governing differential equation you see that they are of the same form if you look at equation 4 it is $(EAU')'+P=0$ and equation two is $(EA\lambda')'$ - $(EAU')'=0$ this part okay is called adjoint load, this is called adjoint load okay, because if you look at this term in this term they are the same except that U is replaced with λ for this one this is the actual load right, that is the given load actual load on the structure whereas this is called adjoint load.

This equation helps us solve U(x) if somebody gives us EA and P we can solve for U. Similarly, this adjoint equation we call this as adjoint equation okay, if somebody gives us the adjoint load and E and A we can solve for λ so using this we can solve for $\lambda(x)$ in fact here it is obvious to see because we have two terms $(EA\lambda')'$ - $(EAU')'$ that immediately tells us that in this particular case $\lambda(x)=U(x)$, so the displacement axial displacement to $U(x)=\lambda(x)$ which is a Lagrange multiplier function, okay.

Now knowing that if we substitute this result into one you put this result into one then what does one become we have $EU'^2/2 + \lambda - E\lambda' u'=0$ if λ is =you we also know that λ' will be equal to u' so let us replace this λ' with u'right, then we get u'^2 here right, here we have $EU'^2/2$ here EU'^2 overall what we get now will be $\lambda - EU'^2/2=0$ our $\lambda=EU'^2$ /2and this is an interesting result for us because we can actually compare this with what we have about λ look at that we had said λ should be greater than equal to 0 that is exactly.

What we got because E which is X modulus which is most often not negative right for a material most material that we use in general we will have equal to positive value there are cases very advanced cases where you actually design a material you can design materials you can have a negative x modulus for a brief period when if it is it not time but a range you can have but in generally is positive E is positive 2 is positive EU'^2 is positive respective of whether u' is negative or positive so overall we are satisfying this condition.

In fact now we can look at this complementarily conditioned λ times the volume equal to 0 we want to know whether $\lambda=0$ or the volume constraint part is equal to 0 here we can see that let us say λ equal to 0 in which case the constraint is not active consider not active meaning integral a dx -V* less than or equal to 0 is not equal to 0 instead of less than equal is not equal to 0 is strictly less than then λ is 0.

If λ is 0 what we find is that u' is 0 when u' is 0 you look at this equation when u' is 0 you take another derivative of it if u' is 0 identically from 0 to l the whole thing is 0 but +P should be equal to 0 P some that is given to us so we do not allow from here and the governing differential equation you conclude that λ is not equal to 0 if it is equal to 0 you would have okay, if let us write down if $\lambda=0$ okay, what we get is that from this equation from that equation we say that u' should be equal to 0.

Then from this equation we will have $0'+P=0$ which is not true because P is given to us so λ is not equal to 0 when it is not equal to 0 it should be positive so we know that λ should be greater than 0 in this problem, okay that is U' everywhere should be false way u' whatever it is $u'^2\lambda$ we have easy to see that this λ is positive and you also see u' is a function of X but then λ is a constant so what it tells you from here we conclude that u' which is equal to $2\lambda/E$ okay square root and + - it will be + or - both okay let me clean that up a little bit.

U' is equal to plus or minus because EU'^2 is there I am taking square roots so I should allow both plus and minus.

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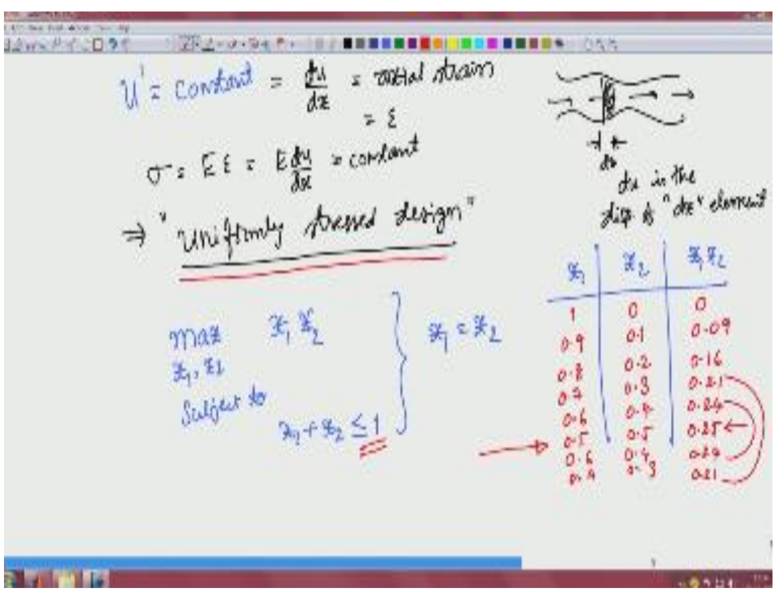

In any case what we have arrived at is that u′ is equal to constant because it depends on capital λ which is constant E which is a constant and 2 which is a constant right, u′ =constant what does it mean u′ for a bar which is du/dx is nothing but axial strain if we recall the definition strain change in length by original length so if I have a bar okay, which is a bar of some cross section if I take a section there let us say this distance originally is dx with this bar deforms due to actual loading but when the load is applied going to deform and if that is U(x) the function du will be the displacement du is a displacement of that little dx du is the incremental displacement of dx element, okay.

So based on the strain that we know very well du/dx if that is strain ok axial strain ε then the stress is E times epsilon that is Edu/dx since du/dx is constant stress is also constant so what we have here is a uniformly stressed design okay, σ equal to constant means that we have uniformly stressed design so optimal design here has the property that it is stressed equally everywhere that actually makes sense if you think about it, because we are trying to minimize the strain energy subject your volume constraint, okay.

Then in order to minimize to energy the material at every point in the domain 0 to l must contribute towards minimizing strain energy and how can that happen if you stress every point equally if some part is not contributing enough towards minimizing strain energy then that is not being fully utilized, right so uniformly stress design here gives you the optimal thing you go back and look at an example that we had taken where we had taken this example in finite very optimization maximize the product X 1 X1 times X to maximize the product x 1 xx 2 2 variables x 1 and x 2 subject to you can strain that x 1 + x 2 is equal to or less than or equal to 1 we can put right in this case when there is not equal to you might as up as well use up all of one that is given that is you will choose X 1 X 2 such that the sum equals 1 are not less okay.

So if you because you have to maximize the product of X 1 X 2 okay now if you see what is the answer for this it will be where x1is equal to actually x2 if you actually take different numbers I write x1 and x2and x1 x2 I can columns okay if I say this is 1 and 0 because some of these two should be equal to 1 that will be 0and then I take this to be a point nine and this is 0.1 will become 0.09 and then I put 0.82 each time making sure that I use up the resource that is one this will become 0.16 slightly more it is increasing and then we have 0.7 0.3 that will become 0.21 then 0.6 0.4 that is 0.24 and then I have 0.5, 0.5 0.25.

So far that is a maximum see what happens when I go to0.6 and 0.4 then I get back down like these two next one 0.7 and 0.3 if we go this is 0point seven and point three that will become 0 point to one just like asymmetric so you see when these two are equal that is fun because there is symmetry in this problem there is no difference when you put material at one location of the another in this case because we assumed a P of X that is given that load is acting everywhere you want to get uniformly stress if you want to get a stiffer structure given amount of material then you have to make it a uniformly stressed okay.

So we have this feature of optima optimal design that uniformly stressed design and that is also fully stressed like we have to use up all the material here we will use up all the material and make it fully stressed as well then you get an optimal structure okay in fact this is just a qualitative feature optimal structures not only this bar when you take a beam or a plate or a continuous two-dimensional structure or three-dimensional structure when you want to maximize the stiffness or minimize a strain energy for given amount of material then you end up with this feature of uniformly stressed design throughout the structure you should have equal stress equals stress and equal strain but also equal strain energy density.

Meaning the stress-strain curve in you draw area under that which will be half Sigma Epsilon that will also be uniform throughout the structure okay.

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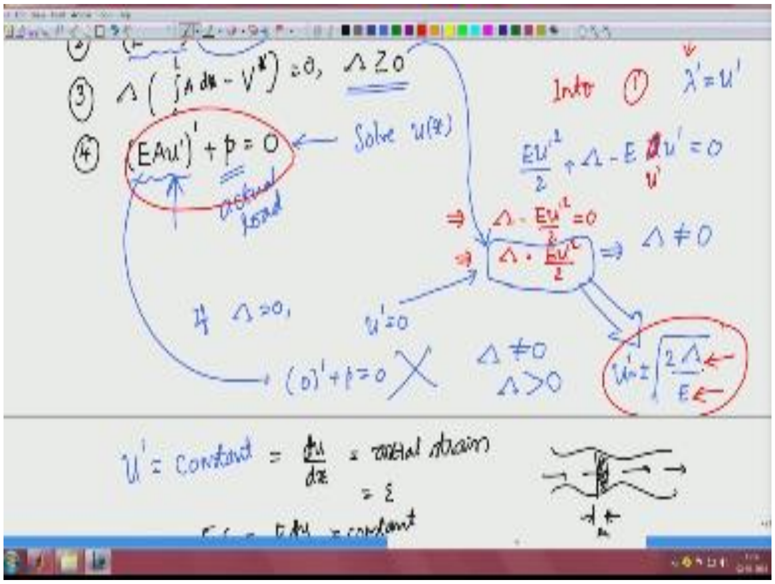

Now let us go back to our equations now we have solved λ = u already and you itself can be solved by sorry the governing equation now we have also found that ADX - v * has to be active meaning that that should be equal to 0 so Δ > 0 we know what Δ is and Δ is + or - √2 Δ /e we have done everything now we need to solve right we can solve because we know u’ which is over here right we substitute that into this equation then we will be able to solve a.

Now you can see the design equation right that is what gave us u’ that is what we started design equation even though there is no a it gave us the feature of optimality here which is uniformly stressed design and we got u’ and we substitute over here okay.

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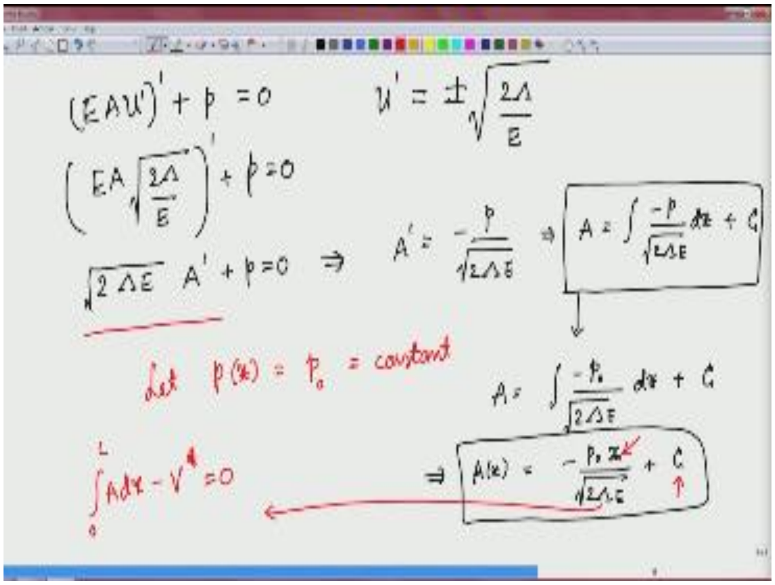

So into the governing equation which is e a u'+ P = 0 and we have u'+ or - √ it was 2Δ / e right let us look at that √2 Δ x e + or- we can take either that depends on the kind of boundary condition that we will take that will come later for now let us take positive sign so I can put e a u' I will take positive sign so I will put √2 Δ / E and '+ p = 0 so what I get this e in the square root over here that goes so what I get is 2 Δ e √ x a' because all candy is constant Δ is constant to is constant this 'applies only to a + P = 0 okay.

How do you find area we simply integrate x so I can write a 'is- p x √2 Δ / e and I can integrate to get a which is integral 0 to l - p /√ 2 Δ e actually not definite in to the indefinite integral here DX okay because a is a function right so a we do not know what if any every point +some constant right if I integrate this I will be able to find area of cross section optimal area of cross section okay.

Now p of x is something is given to us if we know p I can integrate let us take a special case okay of let P of X it is actually a function that can very mature where you want but let us take it to be some p 0 some constant that is everywhere we are applying the same force P_0 right if it is constant then from here we can see a will simply be integral – P_0 it is not P of X now just constant and 2 Δ e and the √ DX + C okay so we can integrate this right just everything is constant that will be this implies a of X will be – P_0 X / √2 Δ e + C .

That will be our area of cross section okay which is a linearly tapering down cross section because this just constant okay, c is just some constant and p_0 is constant that is given to us to Δ e or constant we do not know Δ yet but we can find it how do we find we have the volume constraint for it okay so if you go back to everything we have not it used this constraint equation 3 we only have concluded Δ should not be equal to 0 here and should be positive and we found the Δ value over here we are not used this integral a DX - v * should be equal to 0 if you put that we can find Δ provided to get the boundary conditions.

When new boundary condition we know what C is in order to find Δ we would use the volume constraint after this in order to find Δ we need to use zero to tell a DX- v * equal to 0 so Δ will contain this v-* substitute back you already have expression it varies linearly with a negative slope okay we have a solution for the problem okay.

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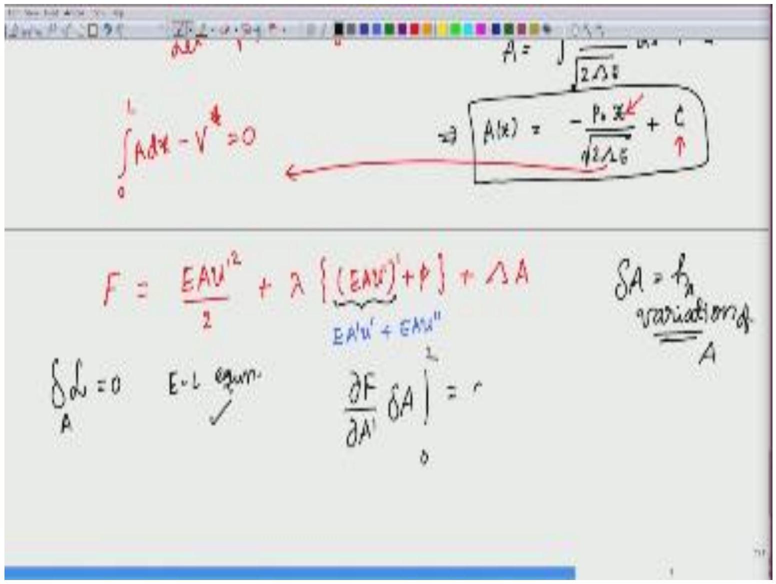

Now we have to look at the boundary conditions also in this problem okay so for that we have to remember what our integrand is right integrand f that we have in this problem in the Lagrangian we had started with the strain energy that was e a u'^2 / 2 and then governing equation had these terms which is Δ times e ay u' + P and then we had in the constraint Δ a DX I am leaving out Δ - v* right this is the integrand that we had.

So we can even look at that once that we had written earlier to be sure okay on the way up where we had written this EA u' over here e a u'^2 / 2 Δ times e a'+ P + Δ a let us see if we wrote the same thing here right this is digit now in order to write the boundary conditions when you take variation with respect to of the Lagrangian with respect to a that is equal to zero we had the Euler-Lagrange equation which we call design equation that is done the boundary condition in this case this contains again we have to expand this.

Let me do that this is this will be e a' u'+ e a u'' so we have a and a' both terms are there in such a case if you recall our boundary condition was ∂ f / ∂ a' into we had that age okay that variation which I am just denoting with δ a now this is this δ a is edge or the variation once we had put a subscript I am just denoting that with δ a so this is δ a which is ache is a variation of a that is area profile perturbing.

It a little bit that at either end should be equal to 0 that was our condition here is where the importance of calculus of variations comes through it is not enough just to know how to read all arranged equations we need to know the boundary conditions in fact later on we will find some more subtleties in calculus of variations where we need to figure out how to write this properly so that we can solve any problem okay this is ∂ f by ∂ a ‘ Δ a 0 to L equal to 0 that.

That in this particular case to f by ∂ a ‘ where is it, it is over here right that is the term that has a ‘ that gives us λ Eu ‘ that will be do f by ∂ a‘ times Δ a at either end should be equal to zero again we also know that λ is equal to you in this problem so this becomes you e u ‘times Δ a at either end should be equal to 0 so first let us consider at X equal to 0 and then consider X equal to L okay now we need to talk about boundary conditions so far we suggest a bar a bar boundary condition can be free, free it can be fixed free or free fixed.

Let us take the bar that we are trying to design to be a fixed free bar I wrote to area profiles I am NOT drawing it so I say this is fixed here and then free there okay now if you look at this x equal to 0 u is equal to 0 right so it is satisfied this boundary condition is satisfied because you is specified that is 0 right and let us look at the right hand at the right hand you is not 0because that is free what about U ‘ in fact the boundary condition of the governing equation here would tell you that u ‘ is actually also equal to 0.

There because that is a natural boundary condition Nyman boundary condition so if when a bar is free if you read the boundary conditions for the governing differential equation along with that you would say that u ‘ at x equal to l is 0 so this point equation is satisfied all right so there is nothing much that we need to know over there right now if we go for that again I would need this integrand take it again here where the boundary can even take various respect to you this is EAU ‘ square by 2 plus λ EA ‘ u ‘ that is a first term that is over that then we also have λ EAU ∂ble ‘ plus λ A.

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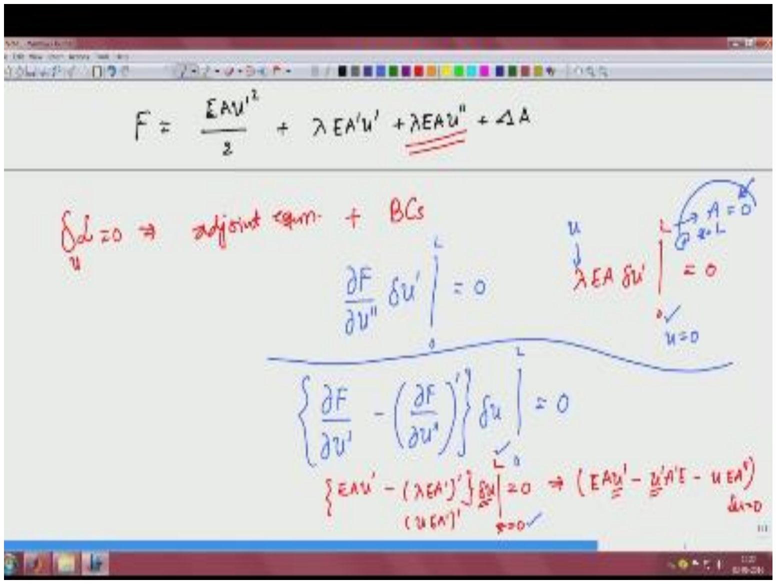

I am just writing for convenience as we go down to the next thing we can right now here we want to write Lagrangian rivet to u right then we get what we call a joint equation which we already dealt with a Lagrange equation plus our boundary conditions okay those boundary conditions will be this so that one will be ∂ f by ∂ u double ‘ into Δ u ‘ is equal to 0 ok and we have the next one which is ∂ f by ∂ u ‘ minus∂ f by ∂ u double ‘ ‘ this whole thing times Δ u at either end should be equal to 0 there are two sets of boundary cancer two boundary conditions I say sets because they are to be valid at X equal to 0 as well as X equal to L okay.

Now let us write them down by looking at this F there so what does this tell us this one ∂ f by ∂ u double ‘ where is it, it is over here so that gives me λ EA right that is it x Δ u ‘ at 0 L should be equal to 0 also note that this λ here is nothing but you write that is what is true so x equal to0 u equal to 0 it is satisfied u is equal to 0 besides for the boundary condition now are the other end right where we have X equal to L you is not 0because free he is of course not 0 and Δ u ‘ is also not fixed because u ‘ that we have there is where it becomes 0 it is like a Neumann boundary condition or natural boundary condition.

We are not specifying it there we say A must be 0 so this one in order to satisfy we conclude that a at X equal to L should be equal to 0 a at X equal to L should be 0 in order to satisfy this boundary condition okay that makes sense because we already saw that area profile is tapering down when tapering down we say it should become 0 that if we specify when I say over here a should be equal to 0 it just happens unless you put another constraint that you should not become 0 okay for now we have no constraint a can become 0 to satisfy this all right.

Now similarly we can write for the next one right so this F is there this is done we got something interesting what we need there although so here we if we write the second one ∂ f by ∂ u ' that will be EA half and that goes will become u ' and then minus the second part will have λ EA ' and then ' of this that is all we have and then that x Δ u is equal to zero again we know that λ is nothing but you so we will have e ay u ' minus UAE ', ' that will give us so this particular thing is Eau'.

And then ' will take derivative so I will have u ' A ' E minus UEA double ' okay this whole thing into Δ u equal to zero now we have this is a ' anything gets cancelled EI u ' u ' a ' e they do not get cancelled but you can see what happens at x equal to 0 and x equal to L and find out how this condition can be satisfied 1 x equal to 0 if you look at it we have Δ u Δ u means that if you is specified to be 0 Δ is 0 so at X equal to L X equal to 0 we have no problem this is automatic satisfied because that is equal to 0 at the other end X equal to L u ' is 0 that is gone.

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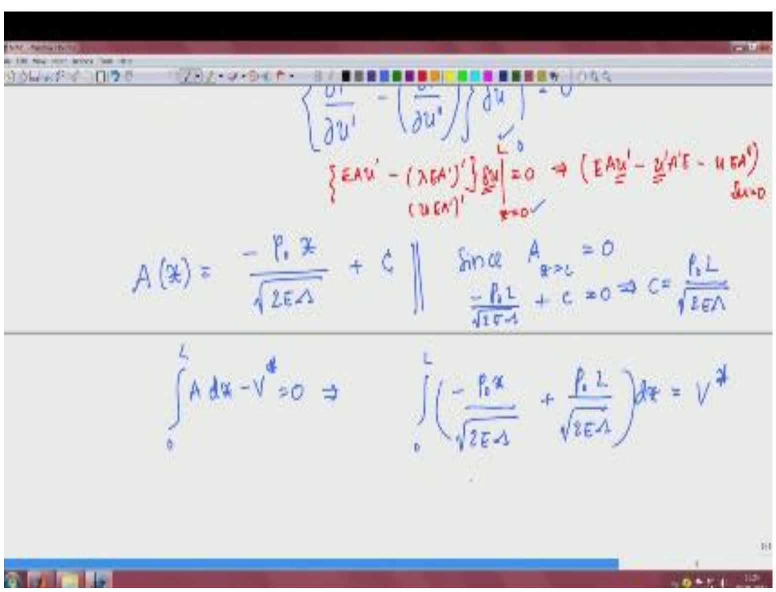

And you ‘ here that is gone then you have EA A double ‘ so for that to be because u Δ U is not 0the other end this should be 0 e is not0 a 0 a double ‘ should be equal to0 which we have because we got a to be linear profile when you take wonder it will be constant second derivative I take it is 0 so that is also satisfied so both this end and that end these are satisfied what we got out of that is this thing a equal to 0 so if I were to go back and say what my area profile for this case is we got minus V naught X divided by square root of 2 e λ plus C was there now.

We got the condition that since A at X equal to L equal to 0 then we can find see what happens then it will be P not L by square root of 2 e λ plus C equal to 0 that will be C so that gives us c equal to P not L by square root of2 e λ okay once we have this much then we can go back to our volume constraint 0 2 L A DX equal to 0 then sorry not that is not equal to 0 that is minus v star equal to 0 then we substitute this a that we got which is integral 0 to L minus V naught X divided by square root of 2 e λ plus C that we just got now p naught l-let square root of 2 e λ DX is equal to v star from here we can get this λ you substitute there you cannot answer.

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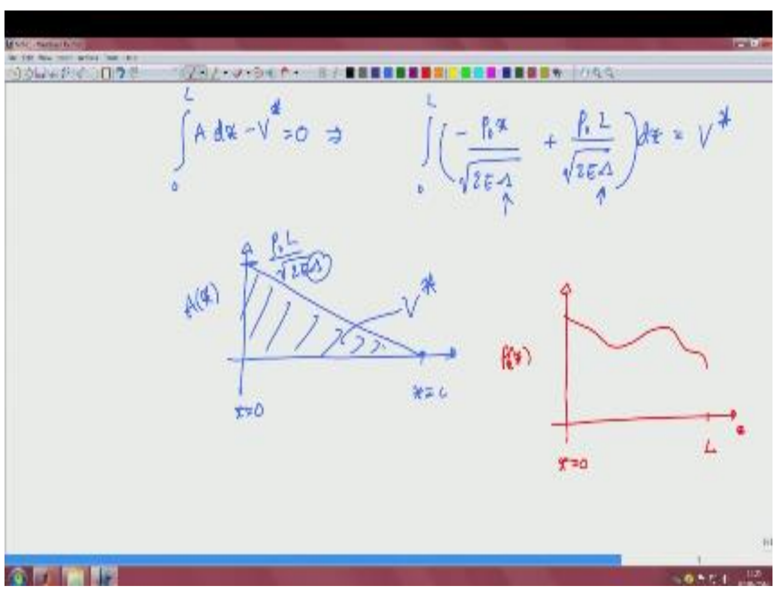

But qualitatively we already know that it is a linearly opening profile if I say X equal to 0 to X equal to L here area profile that is what we want is 0 and then linearly tapering it will go something like this at X equal to 0 if you look at our expression for area of cross section C is there this part goes jvc so that value there is going to be P not L by square root of 2 e λ and the volume you integrate a DX whatever that is area under this should be equal to v star and that gives you the value of this λ okay.

So we have solved the problem completely including the boundary conditions for a special case of P being equal 2p not constant what if P is some arbitrary things let us say I say P0 is given like a arbitrary function so P of X here is not constant as we assumed but let it is something like that from X equal to 0 to X equal to F then we have to integrate and do it for different boundary conditions we have to have boundary condition already we have to make sure that everything is satisfied impose them and try to get that which we will consider in the next lecture.