Indian Institute of Science

Variational Methods in Mechanics and Design

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Hello in today's lecture we will deal with the differential equation or a function type constraints because in the last class we dealt with constraints that are of functional type which we call also global constraints now we deal with local constraints which are function type constraints let us start with the problem that motivated us to consider the constraints that was the stiffest bar problem.

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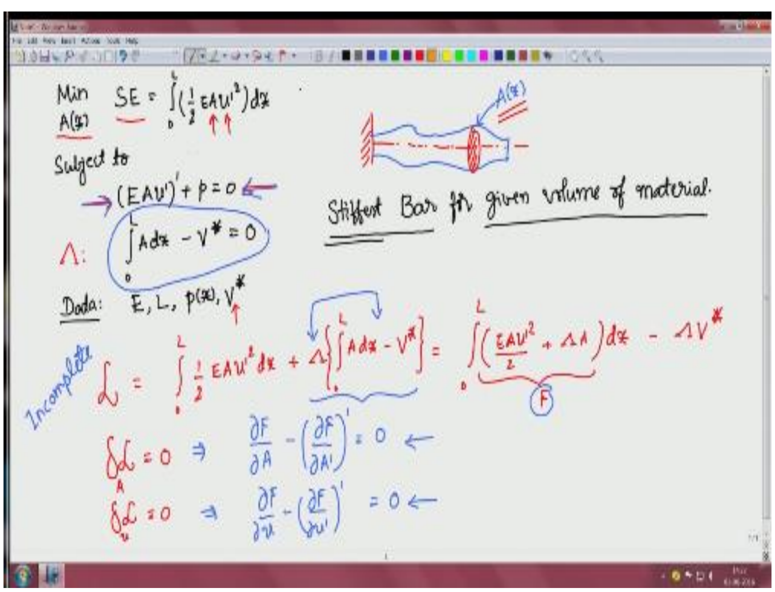

That you see here we need to find the area of cross section that is A (x) as a function that is our design variable here or optimization variable will determine the cross section that is cross section at any particular point x where we wanted to minimize the strain energy which we have written in terms of A (x) and u(x) and its derivative in this case it is a first derivative and where E is the young's modulus and again let us remind ourselves what is data here what is given and what we need to find what is given young's models material length of the bar and the loading p (x) which can be any function that varies with x and also v* t the amount of material.

So we are trying to do tend to design a stiffest bar the stiffest bar for a given volume of material so v *is given to us and we are trying to find v(x) in between we have U (x) which is the state variable and that is governed by this equation and in the last lecture we talked about how to deal with this type of constraints that is this type of constraints which are functional type constraints or global constraints because they pertain to the entire structure volume you talk about volume of the entire bar not a little place a small place in the bar.

So this is a global constraint or functional type of constraints where we also introduce the corresponding Lagrange multiplier so that we could write the Lagrangian for this problem where the Lagrangian is you write the objective function which is 0 to L ½ in fact object function strain energy so actually this should not be there then let me erase that so objective function is the strain energy okay that is Stein energy so we write that ½ EAU'^2 dx and whenever you have a constraint which is a functional type of constraint there is a corresponding multiplier which is this λ.

So we say λ times the constraints which is a dx - v * and then we could just take th eLagrangian and it becomes like unconstrained minimization not only with respect to a but also respect to u the state variable so here we to okay the variation Lagrangian with respect to A that is one equation and then variations are grains in respect to U that is another equation right that is what we had discussed earlier where the integrand in this case if I were to write this as an integrand we have 0 to L EAU'^2 ½ of it + ΛK okay all of this is integrate integrand and then I have - λ v* would just a constant b * is given λ is an unknown constant v*is a known constant it is a constant and this is our integrand normal F.

If I have that then using this F that is lagrangian when I say equal to zero that leads to us leads us to the Euler-Lagrange equations which we say $\partial F/\partial A - \partial F/\partial A'$ because actually A' is not here but I am just writing equal to 0 right so far it is not the lagrangian included only the global constraint but that is not complete our Lagrangian is not complete because this does not take into account our governing equation which is a function type constraint which is what we will discuss that also gets added to our Lagrangian with a similar setup where this is a constraint is a correspond- player we multiply them together additive Lagrangian.

We want to show that when you have function type constraints or local type constraints it also leaves the concept Lagrangian once the Lagrangian then we can do this gives us one equation and similarly if I take with respect to you and $\partial f/\partial U''$ if there were to be U'' you will also add that and equal to 0 you get differential equations one equation to solve for a other equation solver the adjoint variable that we would introduce and we will find out what that is right.

So our task today is to understand the role of the function type constraint that we have these are the one that we need to take.

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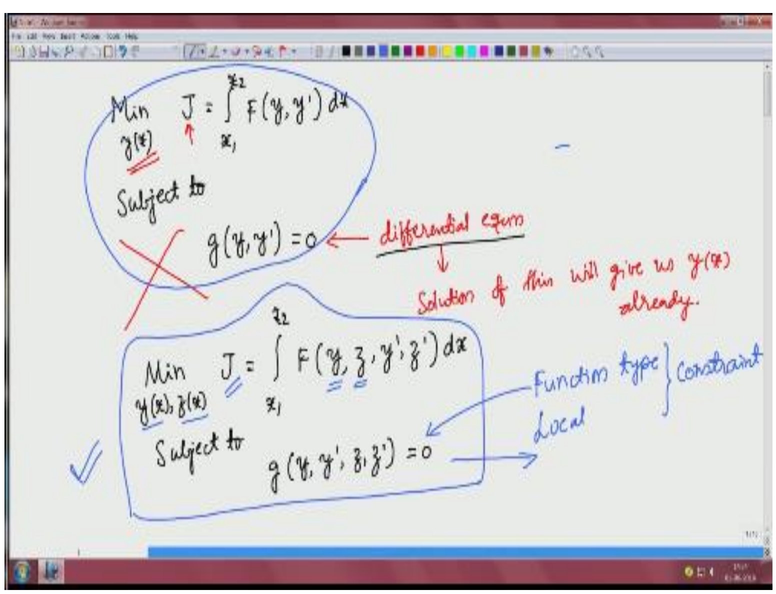

So let us start writing a general problem of minimizing J a functional from x1 to x2 where F we have as a function of let us say y and y ´dx where y(x) is our unknown if I now write a constraint which is function type or differential equation type of constraint which let us call it is G as a function of g and g y ´ equal to 0 if you look at this problem and ask is it a properly post problem the answer would be no because a constraint inequality constraint which is a differential equation this is a differential equation when there is a differential equation the solution of this solution of this if we solve solution of this will give us y(x) already right.

So constraint itself is giving us the solution because y (x) is our unknown we already got the solution for y (x) right then what are we minimizing we want to minimize another functional J with an integrand F that depends on y and y ´so this is not a properly posed problem because we do not have enough search space to search and find the minimum of our objective function which is J what do it in that case we need to introduce into the functional okay our J an integrand which depends on two functions.

Let us say there is a y and there is a z and then there could be y ´there could be z´ and then dx then the unknown functions will be y (x) and z (x) much like in our bar optimization problem we have A (x) and also U(x) there should be two functions then only we can impose a differential equation type of constraint so now I have to say subject to a differential equation type of constraint which depends on y and y´ and z and z ´equal to 0 now this is a properly post problem.

So the problem that we have here is proper because we have an integrand in the functional objective functional which depends on two functions y and z and their derivatives there can be any number that generalization we have already discussed likewise we got one function type constraint this is function type constraint or we can also call it local constraint there are different names to it sometimes people call it finite subsidiary constraint but let us understand that this constraint has to be valid over the entire domain.

So this constraint is valid over the entire domain so this constraint is valid or the entire domain so here G which is a function of y y ´z z´is a differential equation but involves two functions that is y(x) as well as z(x) when you have two functions with one differential equation we cannot determine both of them so then there is a space in which we can search to minimize our J so this is a properly post problem whereas this one is not this one is not a problem whereas this one is so we need to have two functions when you have a function type constraints.

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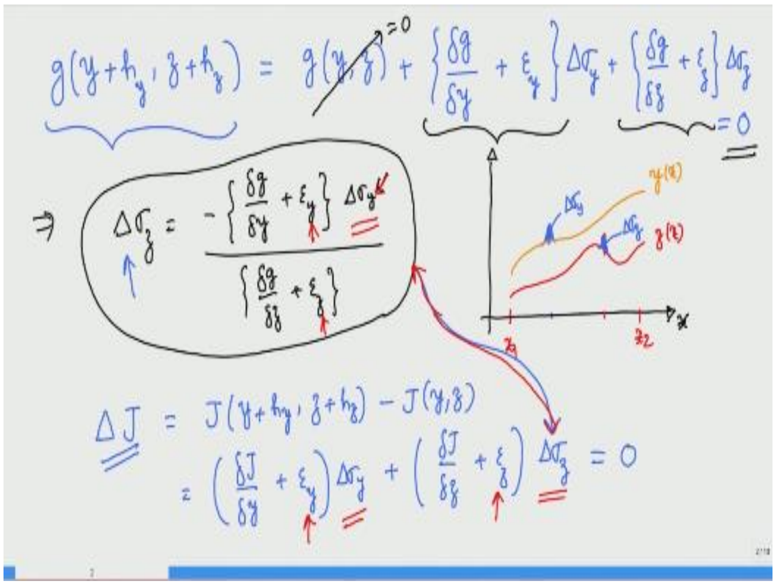

So let us try to think about this problem now we have a function type thing which is why again whenever we want to establish necessary conditions we need to put up the functions so we will be perturbing y let me call that hy and then z let me call that hz so this h that we have is a variation or a perturbation now since we have two functions I amusing subscript y and z differentiate the perturbation for y perturbation for z if we have this how do we write this to first order so this will be equal to whatever it is at y and z whatever differential equation or an algebraic equation that may be there okay.

It is a function type constraint + we want to write the first order term for the first order term we can take the variational derivative something we had discussed that is if we have an expression a differential equation or a functional what happens to the value of that function or functional when there is a perturbation in the function here we have two functions y and z so if I put up so I will have the variational derivative of g with respect to y okay and then there will be some errorre lated to the perturbation in you K times Δσ y that if you want term using the relation derivative which we had used when we dealt with the global constraints another name for it functional functional type constraint.

Likewise we would have here variational derivative of this with respect to z our second function and some mirror times σz okay again to recall what we mean by this Δσ y Δσ z what we mentioned is that if this is x and if I say this is from x1 to x2 let us say that I have a function let us say this is our y (x) okay then there will be a perturbation around it right at some point let us say I choose an arbitrary point here I give some perturbation.

So I will have a little blip like that and area of that is that one we said is Δσ y okay similarly I can have a perturbation for z also let me draw for the same domain is that let us say this is our set of X at this point if there is a perturbation then there also I will have a little blip area of that right east δ σz okay what we say is that this ν function value with the changes of little thing at for y a little thing for z when we do that this function value should remain equal to zero.

The constraint was there at g equal to 0 even after perturbation so we have the first order perturbation due to variations in Y and Z at some points we are just only at one point we are perturbing they can be different points then we have this right when you have such a thing and it is equal to zero we can express one perturbation in terms of the other so what we say here is this any way is equal to zero.

Because y and z are minimizes they must satisfy the constraint, constraint is g equal to 0 that will be satisfied okay and that leaves us that we can express δ ΣZ in terms of δ ΣY so that will give us minus δ g δ y that is a variation derivative for now we can write later the same errors go to 0 into δ σy and then variational derivative this is not why this is δ g δ z plus εz right.

This is what we get here because of the fact that after perturbation the constraint must be satisfied that is why we got okay now we use this fact that we can have one perturbation freely okay that is this one we can have whatever we want then this other one gets determined we use this fact and look at the change our change in the objective functional that is our interest right that δ j which again will be j with y plus h y and z plus h z minus j @ y z okay.

So that one to first order term we would say it is the variation derivative of a with respect to y plus some error which we can call ε this error this I will not be the same but using the same symbol does not hurt you understand from the context okay so this times δ σ y plus due to change in the function z that little blip we talked about this is ε said δσ set that is what we have right.

Now what we do is into this δ Z we substitute what we got from the constraint because constraint again must be satisfied even after perturbation and this first order change this should be equal to 0 also that gives us the necessary condition so what we will do now is substitute for δ σz what is over here what is over here and everything then we will be interms of δ ΣY because that is what is here then we say that is arbitrary one part of an arbitrary then what multiplies will be equal to 0 and will also not worry about this small errors that was there that are there.

Then they will all go to 0 as our perturbation becomes smaller and smaller which is what we use the as a fact test of necessary condition right.

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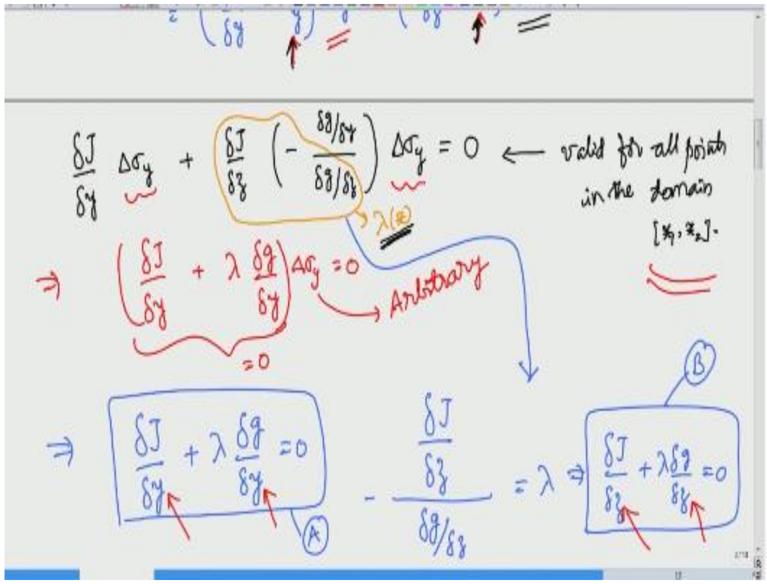

So now looking at this all right so we are going to continue with this and say that δ j x δ y I will not put the error so this error I am NOT going to right now that is δ σy plus and then δ g plus z again i am not going to write this error that goes to 0 as perturbation becomes smaller and smaller but we will make this chain δ σz here we would like to substitute what we have over there okay.

Which is δ G δ Y with a minus sign right so that we will write minus δ G δ Y divided by δ G δ Z we are not writing the errors anymore and this will be multiplied by δ σy also we eliminated δ σz with this and make it equal to 0 because two first order objective function should also not change and since this thing is arbitrary we say what multiplies it is equal to 0so this implies δ J plus δ Y now we will make that trick or whatever notation that we would use earlier include in the negative sign.

So this whole thing okay I would call it lambda of X earlier it was a capital lambda and we said it is a constant unknown constant where now we are making it an unknown function that is because whatever we have written is valid for all points in the domain because the point that we chose if we go back yeah at this point that we chose here are here go chosen anywhere so this should be true for all points in the domain.

So this equation is valid for not just globally for the whole thing this is the local constraints consequence valid for all points in the domain all points in the domain X 1 to X2 in the domain X 1 to X 2 okay, so consequently this becomes a function okay so now we would write this as lambda times δ g x δ y equal well this times we have δ σy equal to 0 since this is arbitrary we can choose our perturbation one perturbation any which way you want then that makes this equal to 0 okay.

So we get the variational derivative of the objective function plus lambda times variation derivative of the constraint equal to 0 that is one equation ok now if you look at this one which is in this orange thing here okay so what does that say that says δ J δ Z variation of j with respect to Z okay divided with the minus sign divided very negative G with respect to Z okay we are calling this lambda of X right.

So this gives us variation is way to go with respect to Z plus because this minus is there and we are taken to the other side okay so that becomes lambda times δ G δ Z equal to 0 now if you look at if we call this equation awe call this equation B you see this equations a and B are identical in the sense that we have Y here whereas here we have Z otherwise their form is similar so when you have two functions at all points in the domain we should have both of these true right that gives us the necessary condition that also tells us that if we were to that.

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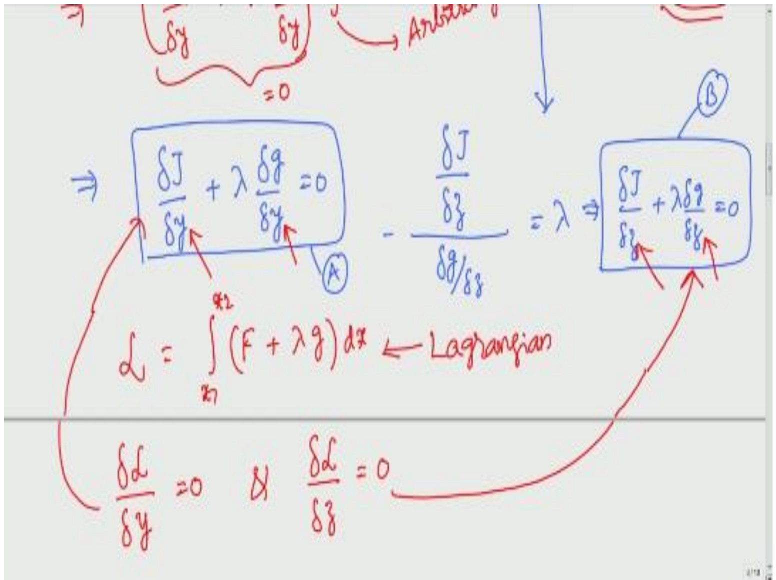

If we were to add to the Lagrangian okay so whatever we had from X 1 to X2 I had F right to that F I can now add lambda times G if I define it like this okay now if I write the variation derivative of this with respect to Y I get one equation because now we have extremeization in of the Lagrangian we get that equation and then I get variation derivative of that with respect to Z equal to 0 or Euler Lagrange equations or variation if you do that this gives us that equation okay.

And this gives us this equation okay so we have both of these equations coming if we write this Lagrangian okay so this way we can deal with local or function type constraint this local function G can be just an algebraic function it can be a differential equation it can be an integral equation as long as it is local when its integral equation I do not mean a functional but there could be integrals in it but valid at points with integral algebraic equation so something an equation that is valid for every point in the domain okay.

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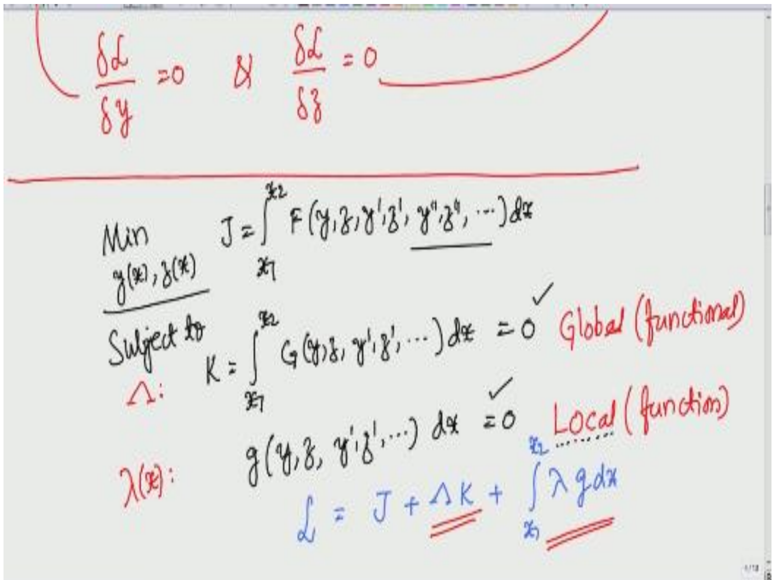

So with this let us recapitulate what we have discussed so far for constraints so if I have a general problem minimize Y of X Z of X two functions and a functional J which is fit depends on Y Z Y prime Z prime Y double prime Z double Prime and so on any number of derivatives does not matter okay subject to subject to constraints let us say if I have a functional type constraints which you are called k with integrand J that depends on Y Z Y prime Z Prime and so forth any number of derivatives equal to 0.

And now we have also discussed a case where we have algebraic or differential equation type of constraints again any number of derivatives what we need to do is first denote with an uppercase Greek letter like lambda for global constraint that is an unknown constant whereas unknown function for a local constraint once against his is a global constraint this is a local constraint okay global or functional type functional type and this is a function type constraint right.

So how do we deal with this it is very easy all you have to do is write the Lagrangian so that we have J plus lambda times k plus I should not say lambda times g if I write lambda times g it will be for a particular value of x r to integrate it actually will be from x 1to x 2 lambda times g times DX okay that is something we should note the difference between our global constraints and local consider to be integrated because their point wise constraints.

They are local they are at different points so we need to integrate it okay, once we have this Lagrangian we can directly write a Lagrange equations for the Lagrangian and we can write boundary conditions and that we takeover so any number of derivatives are fine any number of functions is fine and any number of global constraints are fine and some number of local cans local constraints should be number of recognition should be one less than the number of functions that we have otherwise we cannot solve okay, we will pause here and come back and solve a problem.