Mechanics and Control of Robotic Manipulators
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Lecture 15
Examples Related to Inverse Kinematics

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Welcome back to mechanics and control of robotic manipulator. The last lecture we have seen introduction to inverse kinematics. Mainly we have seen what are the methods of solutions and what are the way we can start approach, but we did not see yet to see any example. So, this particular lecture is going to talk about what you call finding closed-form solutions. You have seen one such method of solution is closed-form. It will give the analytical relation between one to another. So, we will try to see few examples, and then see whether we can derive the closed-form solution.

So, as long, if the system is in planar, so three joints, so three equations we can find it. So, however, if you go for, multi-body, if it is go for a spatial system, then we have to see the necessary and sufficient condition. The necessary condition is you can see that given point supposed to be within the workspace and there are supposed to be 6 joints, in addition to the sufficient condition is, or you can say the necessary and sufficient condition, one is the workspace, the other one is 3 consecutive rotary axes supposed to be parallel or intersect at a point. So, these all we have seen.

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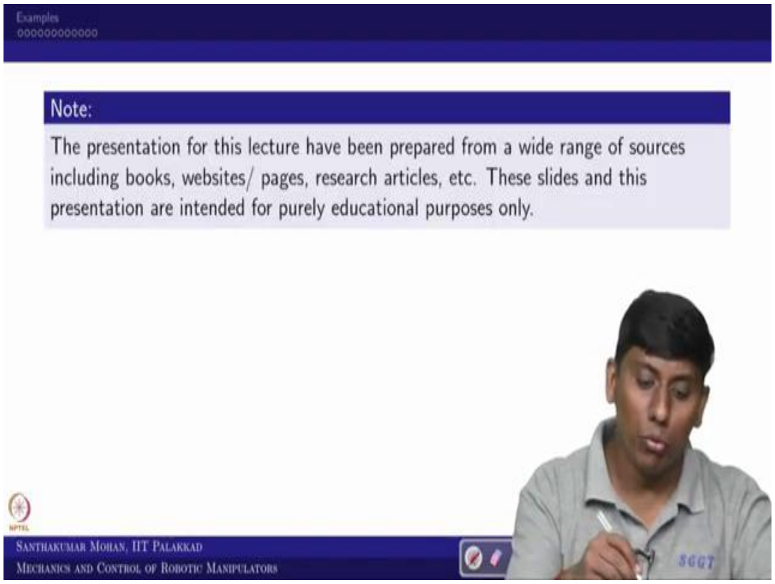

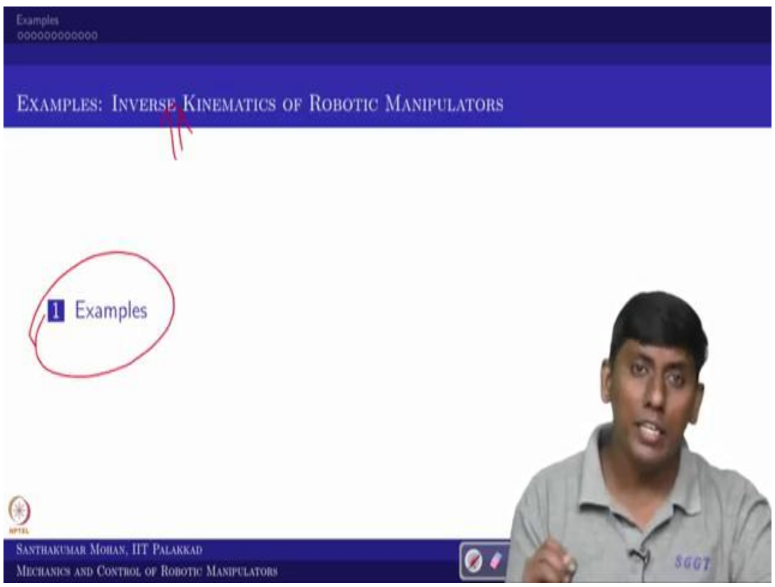

So, now, this particular lecture, what we are bounded. So, we are bounded to see some examples related to inverse kinematics. I have taken 3 such example, we will see, so how the thing goes.

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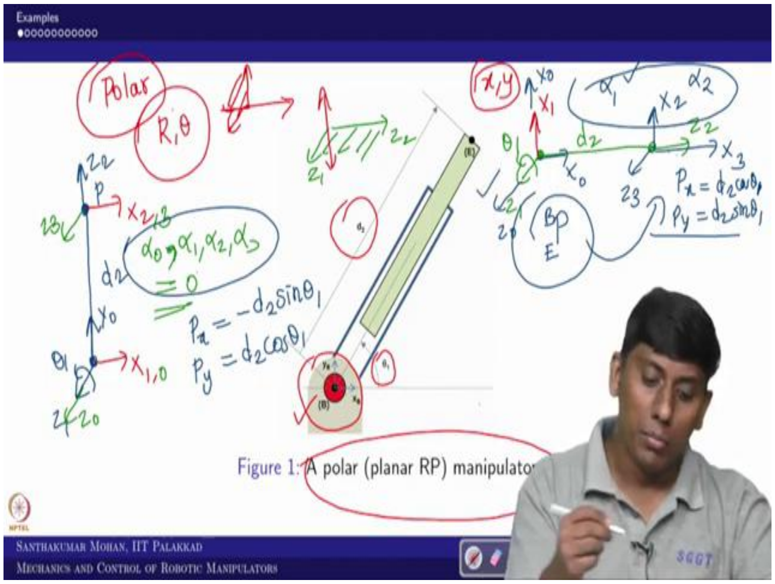

So, for that, what we are trying to take, first thing, we are taking as a polar manipulator, which means it is a rotary and prismatic planar serial manipulator. So, why it is called polar, because it is giving the polar coordinate, polar coordinate is R and theta. So, that kind of thing, it is going to give, that is why it is called polar manipulator. So, it is in a planar case where you want to get XY translations. So, it can be obtained by changing you call theta 1 and d2. So, this is the main benefit.

So, now this frame of reference, we can take it xb and yb, which is given. So, what we can see, we have to construct the frame arrangement for this given RP manipulator. So, there are several ways. So, two of the way, which normally people use. So, one way is we can start with the base. We can start with the first joint, which is Z1, which is having a rotation, which is called theta 1. And the second, which is translation, which you can write it. So, in the sense, so theta 1, d2 can be written this way. So, this is going to give a very much benefit. What benefit?

So, you can see your X1, which is we write perpendicular to the plane containing Z1 and Z2. So, here what plane it is containing. So, this is a plane containing, so your X can become outward or inward. In this case, I took it outward, in the sense X1 and X2 can be parallel. Further, what you can see, I have plenty of choice, my Z0 also can be parallel with Z1, and Z3 I can choose in such a way that it is parallel to Z0. I can take everything is parallel, there is no alpha, so alpha 0, alpha 1, alpha 2 and alpha 3, so all are zeros, whereas the other conventional method, which says you can start the Z1, which is giving theta 1 and Z2 which is conventional, very much conventional, Z2, which is d2.

So, in this case, what is the plane containing? So, Z1 and Z2, this is the plane containing. So, this is the plane containing, your X axis supposed to come outward or inward, it is supposed to go vertically up or down. So, here I am taking coming out, it is going vertically upward. So, now you can see like it is having some rotation is involved with X1. So, now I am taking for my convenient, I am taking as Z0 is parallel. So, X0 is here.

So, now X0 to X1, there is a rotation and Z0 to Z, you can say 1, there is no rotation, but Z1 to Z2, there is a rotation, alpha 1, there is exist. So, there is exist in alpha 1, and you can take it Z3 parallel to Z0. So, then you can see, you can take it X2 like this and X3 like this, again, you can see like a Z2 to Z3 there is another. So, alpha 2. So, alpha 1 and alpha 2 is coming into picture. So, which is non-zero, where in this case, these all are 0.

Further, if you want to think about your forward kinematics, I assume that this is the point P, the Px, what you can expect from here, because this is 0 degree. So, what would be the Px? So, Px is minus d2 sin theta 1 in this case. So, what would be Py? So, Py is d2 cos theta 1. So, that is the case, because your Y0 is coming this way. In this case, what you can see, the Y0 is going upward, so the Px would be so d2 cos theta 1 and Py is d2 sin theta 1.

So, we can get this, but this is what you call forward kinematics. So, we will see how formally we will do this, we will take this and frame arrangement, get the DH parameter, substitute that into the arm matrixes and then you can multiply and finally get you can start from P. You can say E with respect to base that is coming this or not, we will check it. So, for that, I am taking this representation because even you can take this, but I am taking one such example, because this is bounded for inverse kinematics.

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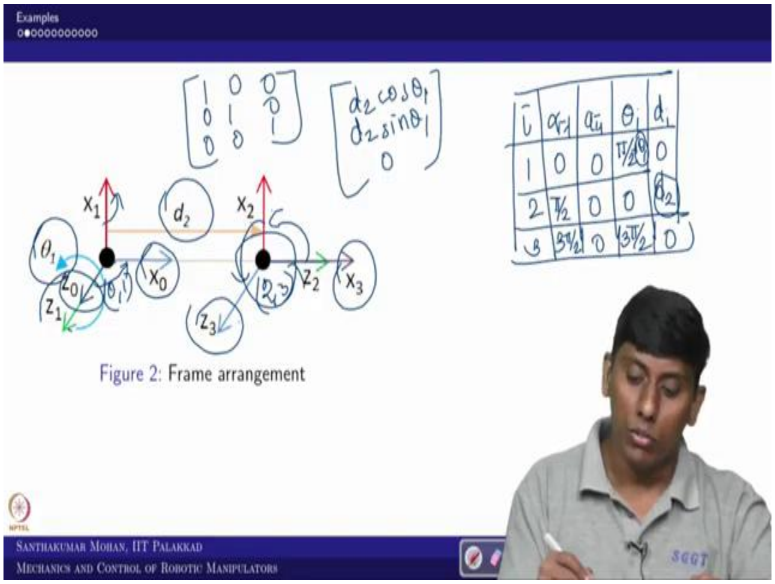

So, you can see, this is the frame arrangement, which I have drawn already. That is what we have taken. The theta 1 is one of the variables, which is a rotary joint, which is on the base and d2 is a piston cylinder kind of arrangement, which is sliding. So, now Z3 and Z0 are parallel, and X3 and X0 are also parallel, what you can expect your orientation matrix, it would be, so identity matrix. Your rotation matrix is identity. What would your position vector?

So, your position vector is d2, so cos theta 1 and d2 sin theta 1 and 0, whether we are getting this or not, we will see. For that you can write your DH parameters. So, what are the DH parameters you can write, i so alpha i minus 1, so a i minus 1, then theta i, and di, so here how many frames? There are 3 frames we have to bother 1, 2, 3. So, 0 to 1, is there any angle difference? Or you can say, with respect to X0, is there any angle between Z0 and Z1? No. Is there any distance between Z0 and Z1 along X0? No. So, what is about Z1? Is there any angle between X0 and X1? Yes, which is 90-degree. In addition to that, what? So, the Z1 is active. So, that you can write it. So, what you can see along Z1, there is no distance between 0 frame and 1. So, this distance is 0.

Now coming to the second one. So, this is here. So, now you can see about X1. So, there is a Z1 and Z2, there is a distance, which is positive, because it is rotate like this, the Z rotation direction, so it is Z1 to Z2 is positive, pi by 2 positive and there is no distance along X1, so 0, and you can see X1 and X2 parallel and about Z2 there is no angle. So, that is 0, and what you can see there is a distance along 1 and 2 frame, so that is what I call d2.

So, in this case, this is variable, and this is variable. So, now what else, so the third frame is actually like the displacement is nothing because 2 and 3 are coincide, so you can directly put that 0, but what you can see about X2, the Z2 is rotate 270-degree to may coincide 3pi by 2, and theta also there is an angle with respect to Z3, you can see that there is an angle. So, then you can see that is pi by 2. But here what, this is the rotation, it is Z, X, about Z3, X2 to X3, distance, the angle that is 3pi by 2.

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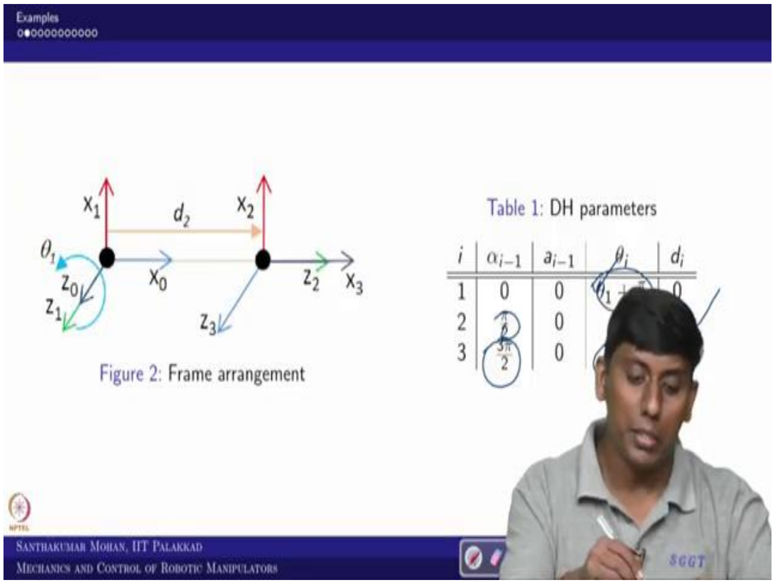

So, now this is the DH parameter, which is we obtained, so you can see which we have obtained. So, now what you can see, whatever we have written that all coming into a picture. So, this is the variable, and this is the variable. We can substitute this into individual arm matrixes and then we can see what is, you can call the DH representation giving finally forward kinematics.

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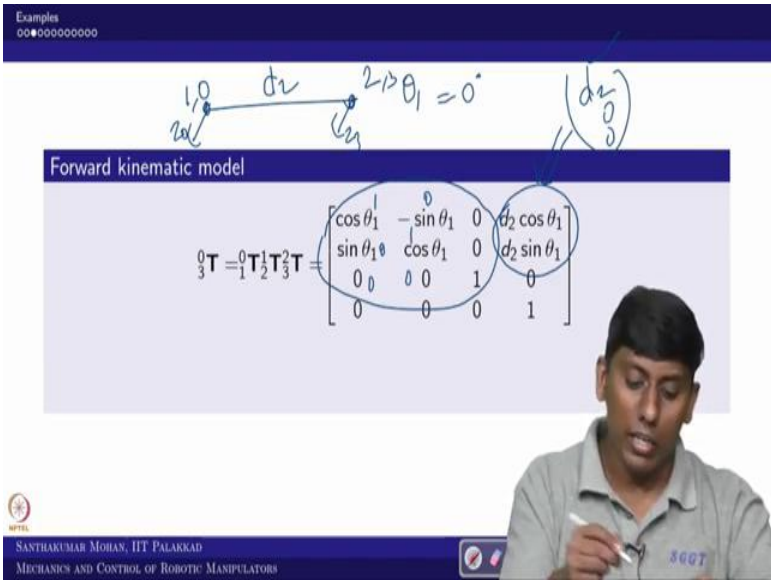

So, that we can substitute and finally multiply, you can see, this is the final multiplied matrix, what you get, so you can see that this is the position vector, which we have written, and this is the, you call the orientation or the rotation matrix. So, in the freezing point, theta 1 is actually like 0 degree. So, if you put 0 degree, what it is giving, so it is giving d2, 0, 0, that is what X displacement only there. So, you can recall.

So, this is 1 and 2. This is d2 and this is 0 also. So, in this case, you can see this displacement along X. So, d2 is coming. So, you are getting it. And if you substitute this, this become 1, 0, 0, this is 0, 1, 0 and 0, 0, 1. So, in this case you can see it is identity matrix. That is also matching.

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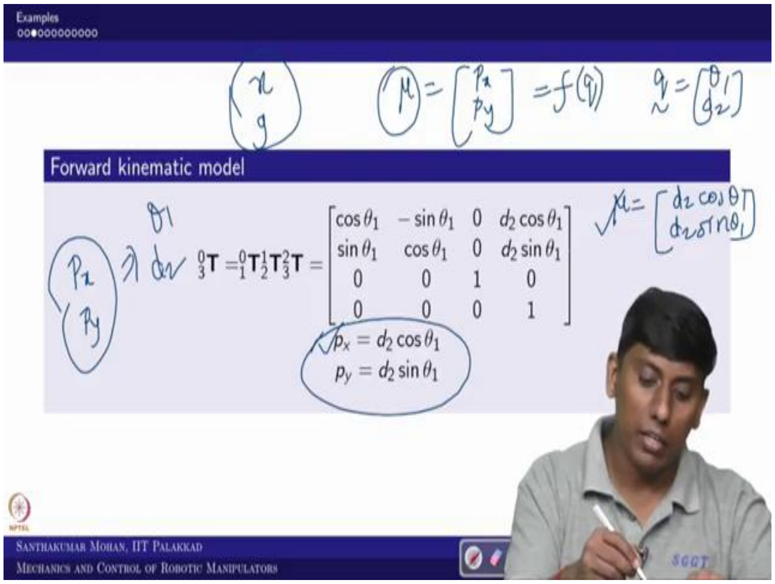

So, now what one can find it, so you can see this is the position vector, since it is a 2 DoF system, so we can see only two variable can be task space. So, here the task space variable is only X and Y. So, that is what we call the Px, Py, so this is mu here is only Px, Py. So, this is the task space vector, that I can write as a function of q. So, q is vector here. So, theta 1 and d2. So, now what that, so you can write mu as, so d2 cos theta 1 and d2 sin theta 1. So, this is what you call forward kinematics. You got it.

Now if I give Px and Py, can I find theta 1 and d2? Yes, in this case, it is straightforward, you square and add what it will come, it will give only d2 squared, so you can take it that square root, so that would be your d2 and you take Py divided by Px, that would give tan theta 1.

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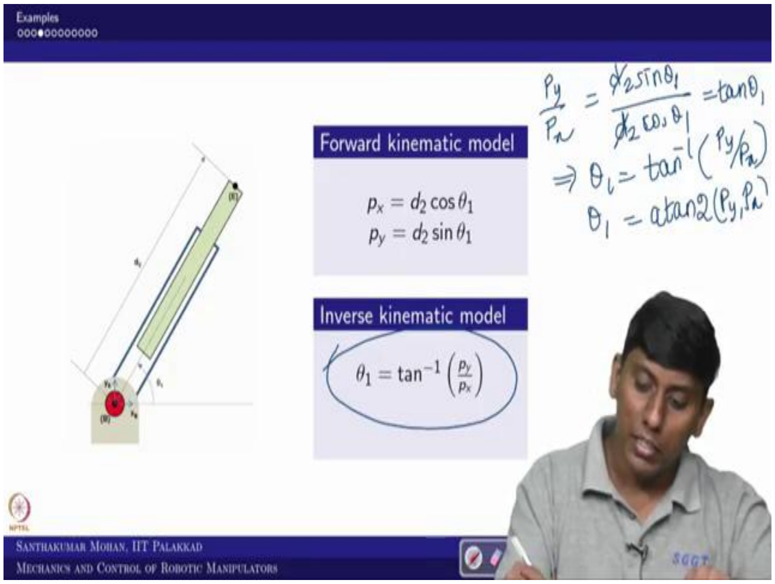

So, that is what we are saying. This is the forward kinematics for given PR manipulator. So, if you take this Py by Px, so that gives what it is equivalent so d2 sin theta 1 divided by d2 cos theta 1. So, which case this get canceled, this will give me tan theta 1, so theta 1 is tan inverse of, so Py by Px, and what we say, it is quadrant dependent, so you take theta 1 as, so a tan in the sense arc tan 2 argument Py and Px, the sin of Px and Py makes sense.

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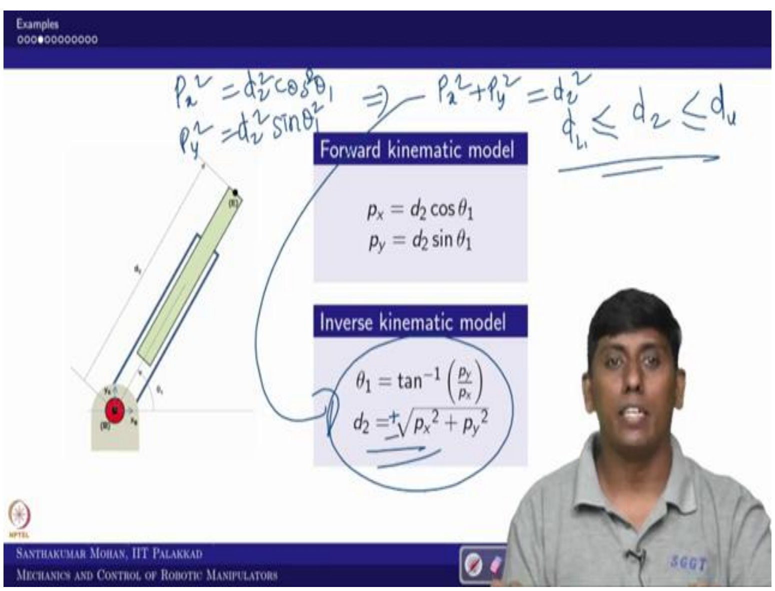

So, now the square and add, so that will give d2, so square and add means what, so Px squared equal to, so d2 squared cos theta 1 squared. So, Py squared is d2 squared sin, so theta 1 squared. So, if you these two add, what it will come Px square plus Py square equal to d2 square, so from there, you can get this relation.

So, now you can see this will give plus or minus. So, it is giving a multiple solution, but this multiple solution is not really possible, why? Because it is only positive, because the d2 have restriction. So, this is lower limit and this the upper limit. So, in the sense, it can be equal. But in that case the positive value only will come, that is why I did not put plus or minus. So, now this is one such example, which is very, very easy.

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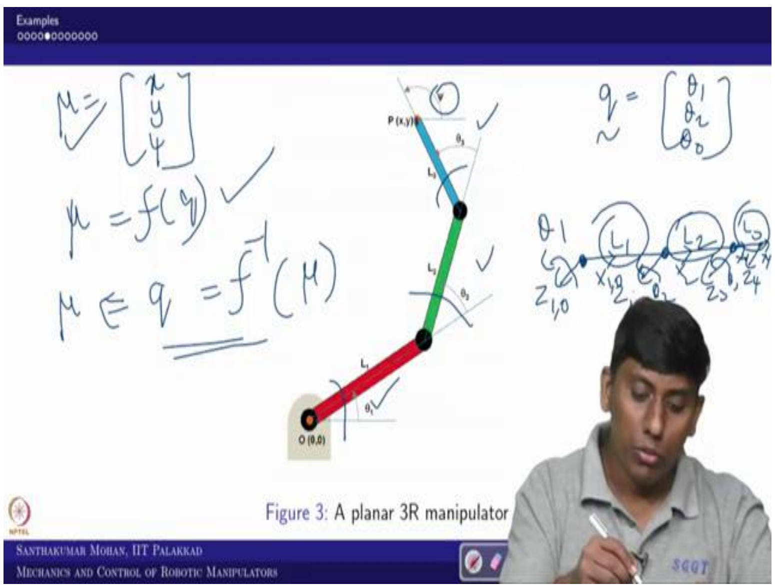

We will take one another example, what another example, you still in planar, but you have, you can say 3 rotary joints. So, this is one rotary and this is second rotary and this is third rotary where theta 1, theta 2, theta 3 are the joint variable, your q vector would be theta 1, theta 2 and theta 3, so in this case, the mu vector, so it is in a plane. So, X translation, Y translation, and you can say Z orientation. The P x y and the angle, with respect to X axis of 0 to nth axis, that is what you call psi.

So, now what you can see the mu can be written as function of q. So, this is what you call forward kinematics. If I give mu, can I actually like find q? So, what that, so f inverse of you can say this as mu, so this can be found. This is what you call inverse kinematics. So, for that what first one supposed to do, we have to fix the frame arrangement. So, since it is a planar, what you can do, you can start the Z1, you can do it either way.

So, I can say this is Z2 and this Z3. So, then even I can extend that I can put as Z4. This will be L1, this is L2, and this is L3 and even I put it Z0 also common, then it will go, X1, this goes X2 and this goes X3 and this goes X4. Like that I can actually draw and then I can find these distances are now coming and here theta 1, this is theta 2 and this is theta3, like that I can fix a frame.

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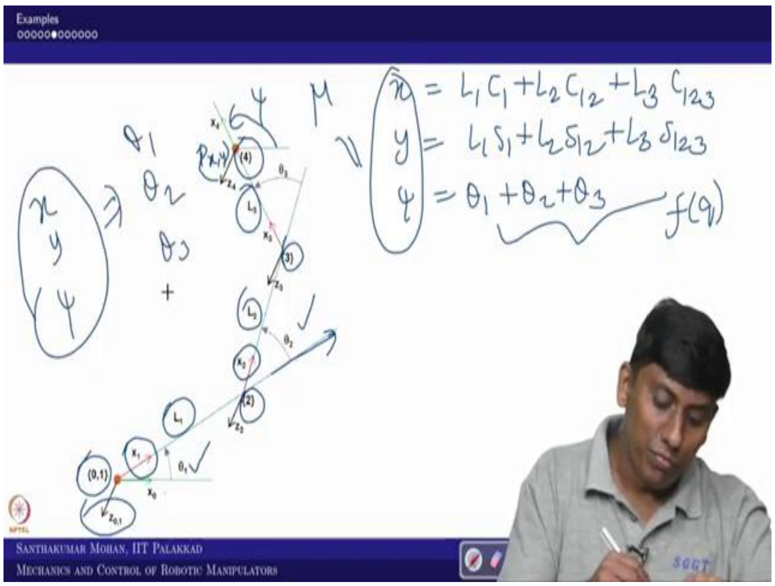

So, for benefit, I have just drawn that with some angle, you can see, this is the; you can say Z0 and Z1, where this is the frame 0 and 1. This is the frame 2, frame 3 and frame 4, where you can see each X axis aligned with the link and along this length, you got link length, L1, L2 and L3. And the angle between X0 and X1 about Z1 as theta 1, then angle between X1 and X2 as theta 2, X3 and X2 as theta 3.

And finally, this angle called psi. So, now this position is X and Y. So, now can we do it? Yes. It is straightforward. So, X I can write L1, C1 means cos theta 1. So, L2 C12 means, so cos theta 1 plus theta 2. So, this is L3 C123 means cos theta 1 plus theta 2 plus theta 3. So, the Y is L1 S1 plus L2 S12 plus L3 S123. So, in this case, we really do not require the DH parameter. But still, we can do it.

Then the psi is theta 1 plus theta, 2, plus theta 3. So, what is this? So, this is related to mu. What is this? This is function of q. So, forward kinematics, we have done. So, what we need? We need, if I give X, Y and psi, can I find theta 1, theta 2 and theta 3. Can I find it? Yes, we can do.

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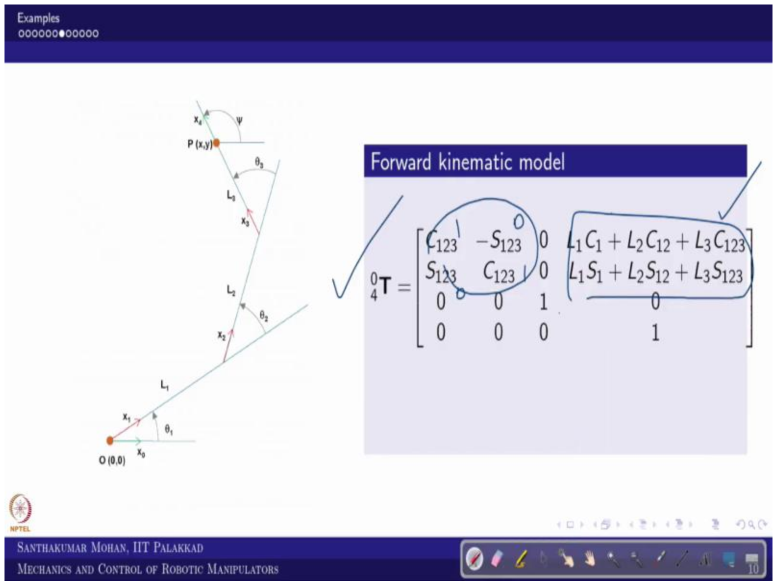

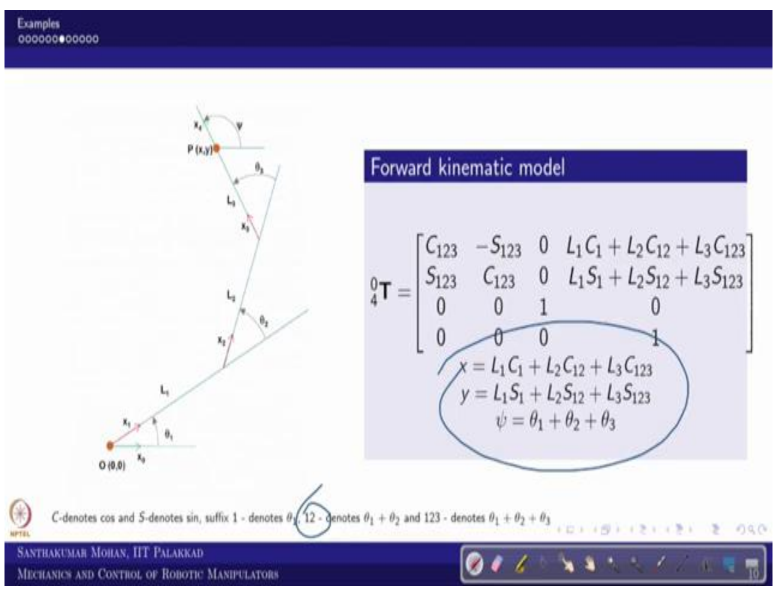

So, for that we will do the DH parameter, we will get these transformation matrices. And then we will let you see, this is the position vector. And this is the orientation. We got it. So, now even you cross-check, you put all thetas as 0, so what you will get, so in X axis. L1 plus L2 plus L3 in Y0, and is that obviously 0, because it is in a planar. And if that is the case, so Z4 and Z, you can say 0 are parallel, in that sense, you can see it is supposed to be identity. That is also getting it.

So, this is 0, 0. So, we are not bothering about the frame arrangement, or we are not bothering about the position vector. What we are bothering about, if I give the position vector an orientational information, can I find theta 1, theta 2, theta 3 individually? So, for that, what one can see, so we can rewrite this equation. And again, I write, so 1 2 means theta, 1 plus 2; 1 2 3 means theta 1 plus theta 2 plus theta 3.

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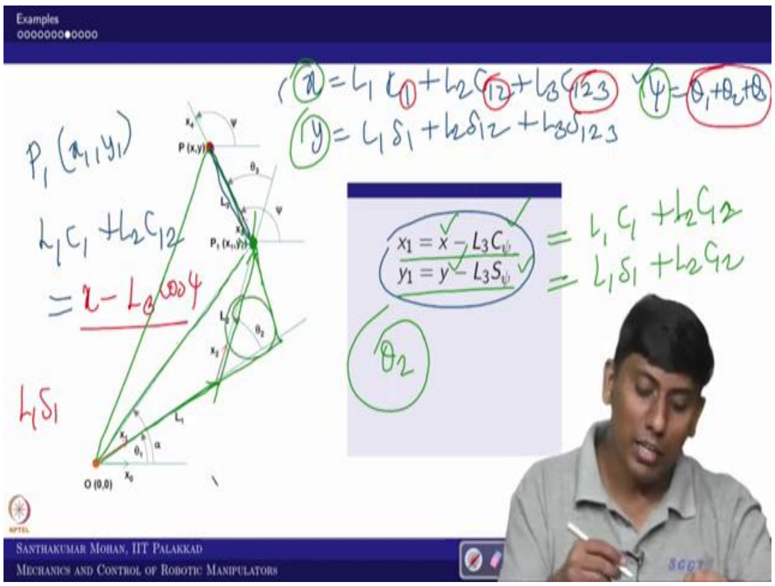

So, now what you have, so you have this equation. So, what equation, X1 and X2, you have X. So, L1 C1 plus L2 C12 plus L3 C123. So, Y is L1 S1 plus L2 S12 plus L3 is S123. So, what you have, you have this, and psi is so theta 1 plus theta 2 plus theta 3. So, this relation you have, so what one we are expecting. So, based on this, these are given, I am marking in green color, so these are given, can I find, so this, can I find this? So, how I can do? You can see there are several ways.

So, one is, if it is psi is given, what one can see from this point, so you can come back this point. What that means? So, if you write P1, so P1 is where you can X1 and Y1, so that can be written as what you can write, so it can be written as, so L1 C1 plus L2 C12, because that is the point. So, that can be equal to what? So, you have this distance, you subtract this. So, you can see that this Px or you can write as, so you can write as X minus L3, so cos psi will give this.

Similarly, L1 S1, this vector is equivalent to, from this. In the sense, I can go this vector addition, or I can go this way. Both are representing this. So, I can write X1 and Y1 in this form. So, now what benefit I will get? So, this c phi and s phi are known and L1, L2, L3 are geometric parameter, which are again already known and X, Y are known. So, what you can see this, I can equate as so L1 C1 plus L2 C12 this equal to Ll S1 plus L2 C12, what I can do, I can square and add. So, what you can see these two equations will give one equation, but that equation will give only theta 2 as a function. So, what I can see, I can get it. So, that is what I am trying to see.

So, what that mean? So, I am squaring and adding means what I can see. I am taking this as this length. So, now you can see, that is what trying to find and that to further simplify, I will get this theta 2. Once I get this, then I will come back to theta 1, how, that I will show you.

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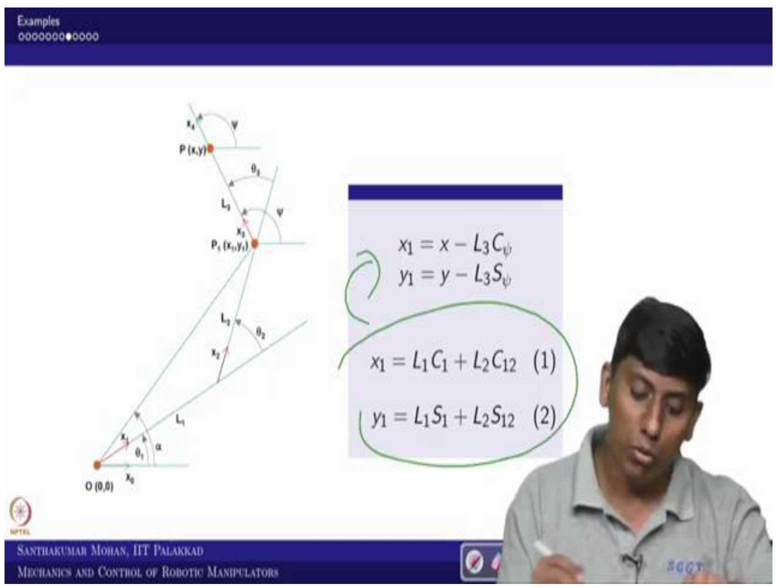

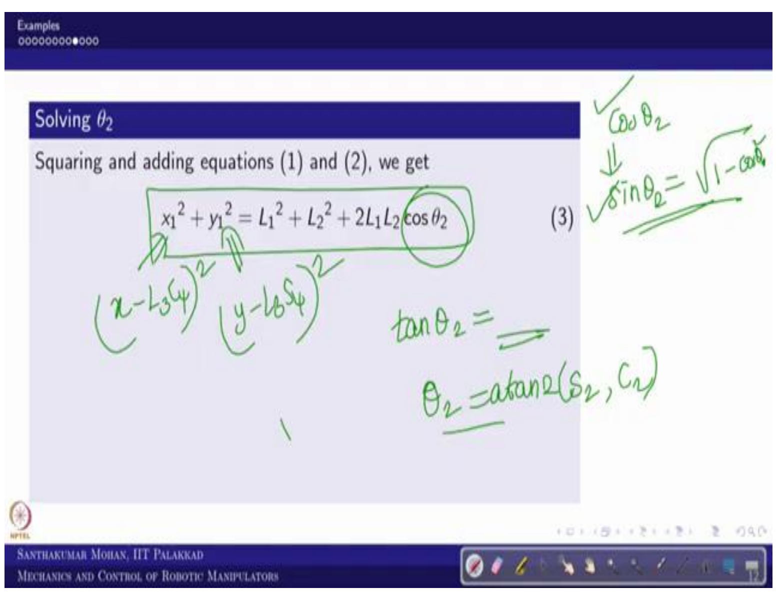

So, right now these two are equal, I am rewriting into my own benefit, so now I am squaring and adding, so this X1 squared, Y1 squared which is I am adding. So, this would be equal, and this. So, now what I will get. So, this is the final equation.

So, now here I substitute X minus L3 cos phi whole squared. This I can Y minus L3 sin psi whole squared I can substitute, then you can see the left-hand side is known. So, what is unknown, only this cos theta 2, if I find cos theta 2, I can easily find sin theta 2, what that, it is square root of, so 1 minus cos, so cos theta 2 square, because we know sin theta squared plus cos theta 2 squared equal to 1. So, based on that I can find this.

So, once I know sin theta 2 and cos theta 2, what I can do, I can take tan theta 2, so then I can write theta 2 as a tan 2 as like this, I put s2 as sin theta 2 c2 is cos theta 2. So, like that, I can mark it. So, what you can see, so the theta 2, I can get it. So, once I get theta 2, what I can do, that I will show you.

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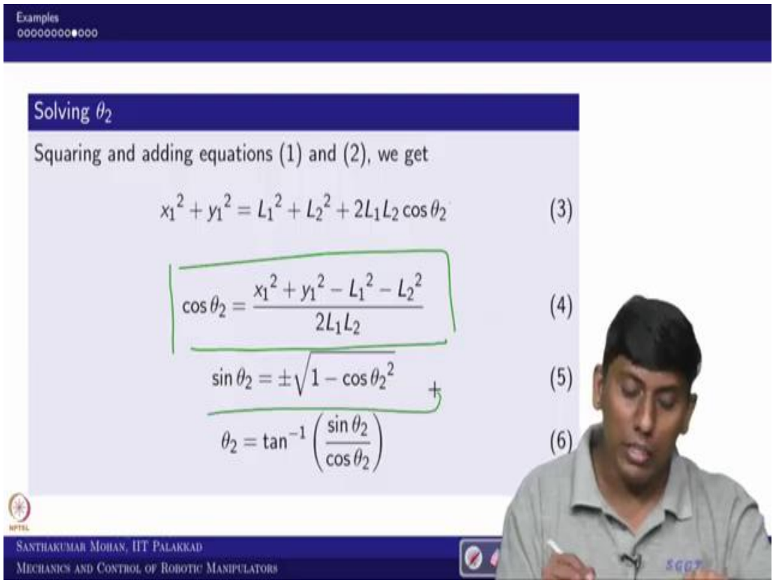

So, what you can look at it, so now you can see here in this particular slide, we have taken cos theta 2 from this big equation. So, this equation we modified, and we have found sin theta 2, then we have found the theta 2.

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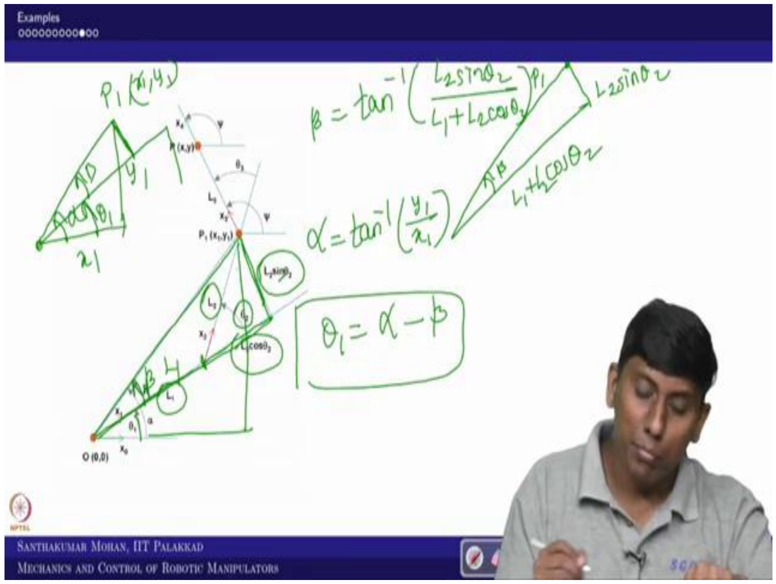

Once this is known, you come back to this picture. What is known? So, this L2 is known and this theta 2 is known. So, what one can see this triangle, we can make it. So, what that? So, this theta 2 is known, this I can mark it as L2 cos theta 2 and this L2 sin theta 2, but what is this distance? It is a link length L1. So, this is L1. So, now this total length, what I can get, I can make this triangle. So, then, what I can see, I can find some angle called beta, so I can see. So, this is L2 sin theta 2 and this is L2 cos theta 2. This angle I call beta.

Further, what I know, this point is P1. The same P1, I can mark it like this triangle. So, what that? So, this P1 and I mark it this, so what that? So, this is X1 comma Y1. So, this is Y1 and this is X1 and this angle, I can call alpha. So, now you can superimpose these two triangles. What you can see that the theta 1 is something is coming. This is a superimposed. So, now this angle is beta. So, this is theta 1. So, for that, what one can do it.

So, we can use the relation, which we have. So, what we can see this is all already known. So, the beta I can find as tan, so tan inverse of what, so L2 sin theta 2, so divided by L1 plus L2 cos theta 2. So, what is alpha? Alpha is tan inverse of, so Y1 divided by X1. What is theta 1? Theta 1 is alpha minus beta. So, this relation I am bringing it now.

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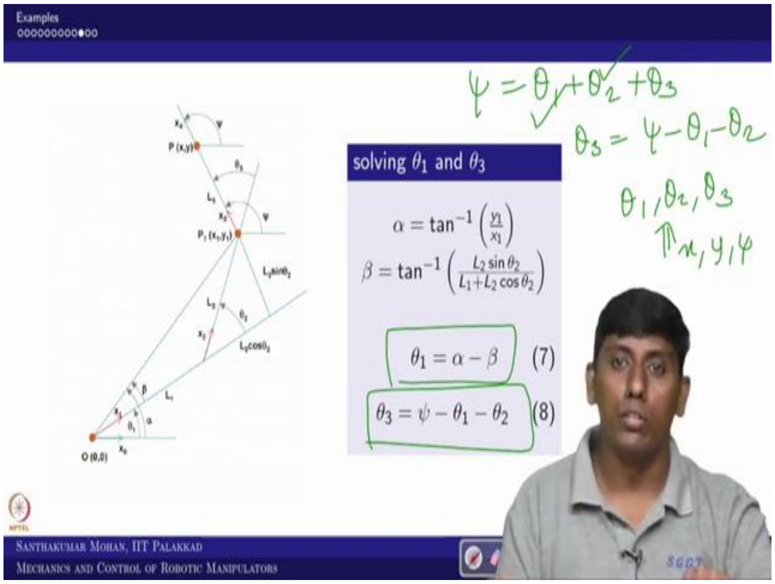

So, you can see that alpha is tan inverse of Y1 divided by X1 and beta I am writing as this form. So, now what is theta 1? So, theta 1 is alpha minus beta, but what is you have written psi, psi is theta 1 plus theta 2 plus theta 3, already I know this. I know this. So, what is theta 3? So, psi minus theta 1 minus theta 2. So, you get all unknowns for given X Y and psi.

So, now you got what you call inverse kinematics, as in you found theta 1, theta 2, theta 3 from the help of X Y and psi. You can see this same thing you can solve it in a number of ways, so I have shown one simple way, but there is no unique way. This is the only procedure you have to follow. Even you can go with the numerical solution, all those things.

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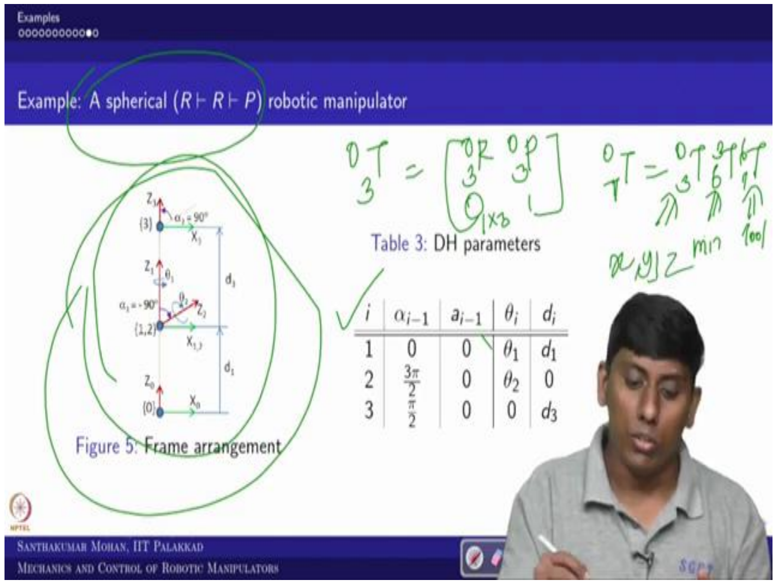

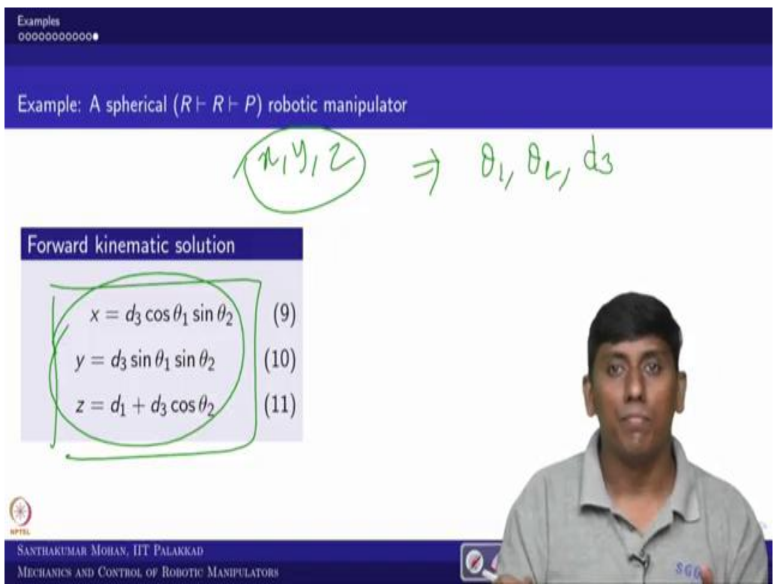

So, now I will just give only one slide further which is self-explanatory. So, you can see that the spherical robot, we have already, you can say draw the frame arrangement and we have found the DH parameter, and what we got, so we got finally based on the final transformation matrix of 4 with respect to 0. So, in this case, the T3 with respect to 0 will give rotational information of 3 with respect to 0 and positional information of 3 with respect to 0, so these all we have seen.

So, now since it is 3 DOF system and it is going to give a major axis because we have seen that transformation matrix of 7 with respect to 0, we can simplify it as 0 to 3 and we can say that 3 to 6 and 6 to 7, this we call tool orientation vector or matrix, and this is minor axis transformation.

And this is major axis. What that mean? So, your risk needs to be positioned on the space. So, you need 3 coordinate transformations. So, you need to have this particle position. You need to have it, you need to have X Y Z. So, this is major. So, that is why the first 3 joints we call major axis. So, the next 3 joints is orient your wrist, so we call minor axis.

The last you can say final axis, which is just a tool orientation. If it is a 6 DOF robotic manipulator, if it is other than that 6, if it is 7 or 8, so then these 2 axes we call redundant axis. So, now you are clear in that. So, based on that, the spherical robot, what it is giving, it is giving 3 axes. These 3 axes called major axes. So, based on that you can see this is corresponding to X, Y and Z.

So, what we are interested, the position of the end-effector is what you call forward kinematics. So, X, Y, Z can be obtained. So, this is we have done already with the previous-to-previous lecture. So, now based on this, if X, Y and Z as given, can I find theta 1, theta 2 and d3 in this case? Yes, we can do it.

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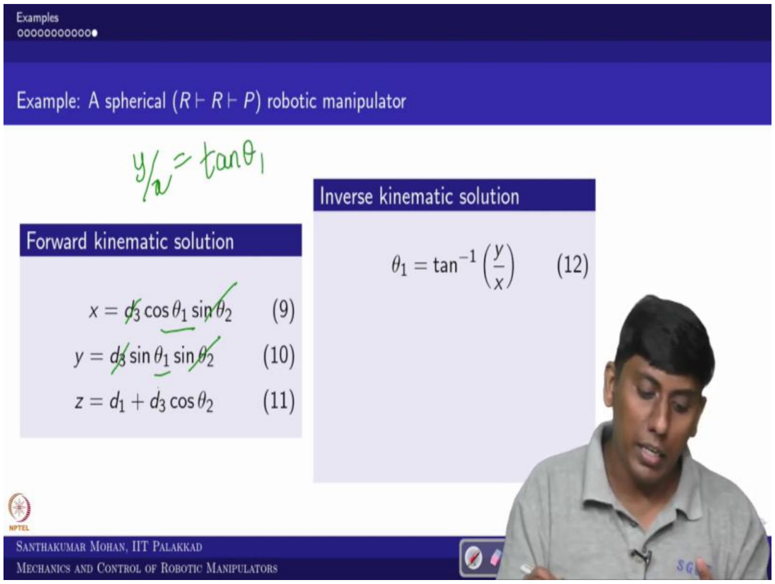

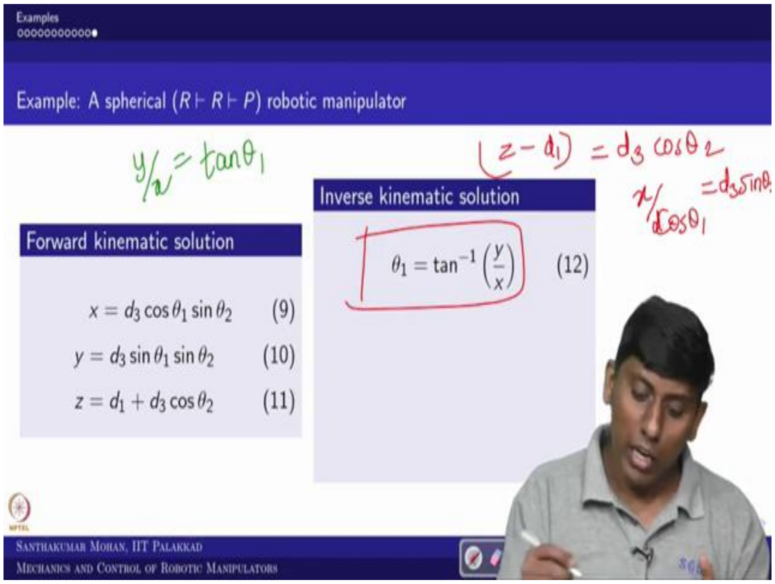

So, for that what one can do. So, you can see here. So, the theta 1 is common. And if I divide, so Y divided by X, what it is giving. So, this d3, d3 will go, and sin theta 2 will go, so what it gives, sin theta 1 divided by cos theta 1 it is giving tan theta 1. So, what you can see, so if you take tan inverse of Y by X, that will give theta 1. So, for theta 2, what one can do, so you can do it this way, so Z minus d1 is giving d3 cos theta 2 and the other way around you can see X divided by cos theta 1 is d3 sin theta 2. So, now you divide by X divided by cos theta 1 divided by Z minus d1, what it will give? It will give tan theta 2. So, that way we can calculate.

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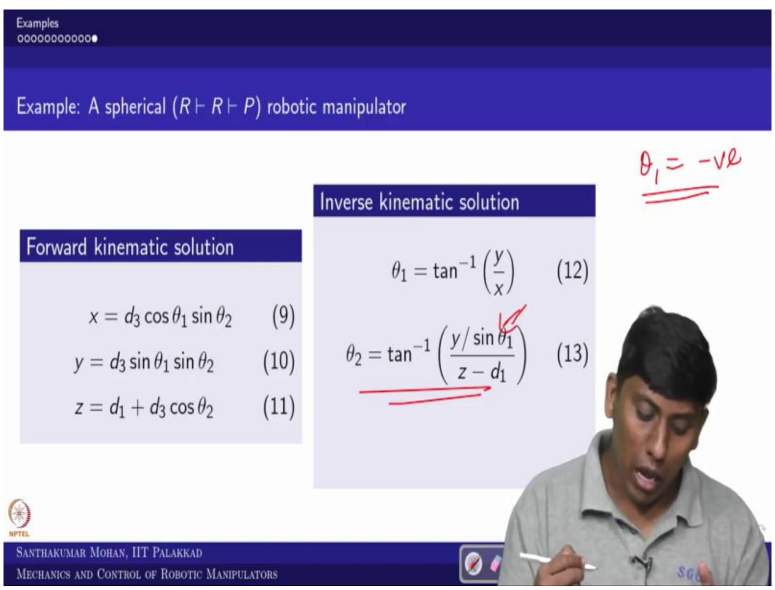

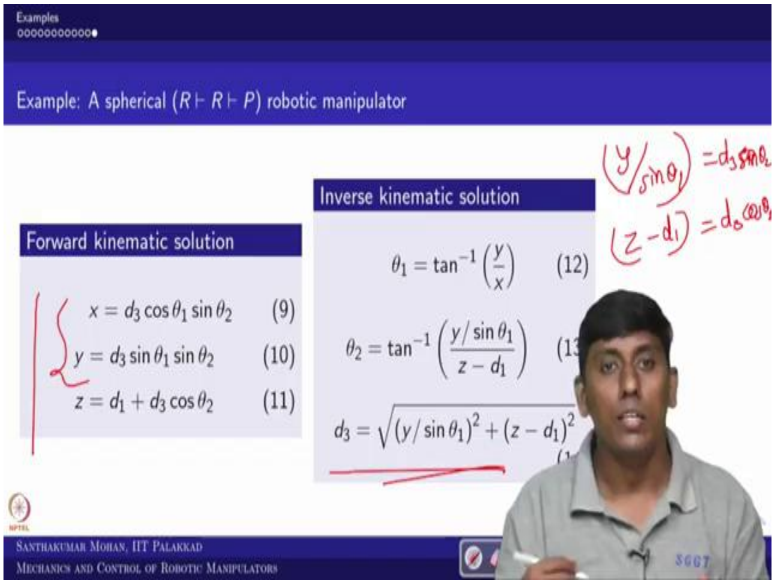

So, here I have taken Y divided by sin theta 1, anything you can take. So, this we have taken because this theta 1 negative sign will play. If you have theta 1 is a negative value. If we take cos theta 1, that will not give the sign change. So, that is why we have taken sin theta 1. So, now what would be the d3 you need to have, so d3 can be calculated in a number of ways, so one of the ways you can square and add these two, so that would be like what you are doing here. So, you can take Y divided by sin theta 1.

So, that is equal to, so d3 I think d3 sin theta 2 and Z minus d1 would give d3 cos theta 2. So, you square and add, so then you take a square root, this is what coming. So, now we can see like if you have a forward kinematics solution and you have number of unknowns equal to number of equations, you can solve it.

So, the next lecture would be on numerical method. So, how to get the inverse kinematics through numerical method, one of the popular methods we are going to call as in Newton-Raphson method we will be taking it. So, with that, this particular lecture is ending. See you then in the next lecture. Thank you. Bye.