Computational Fluid Dynamics Using Finite Volume Method
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Lecture – 06
Review of governing equations: classification of PDEs

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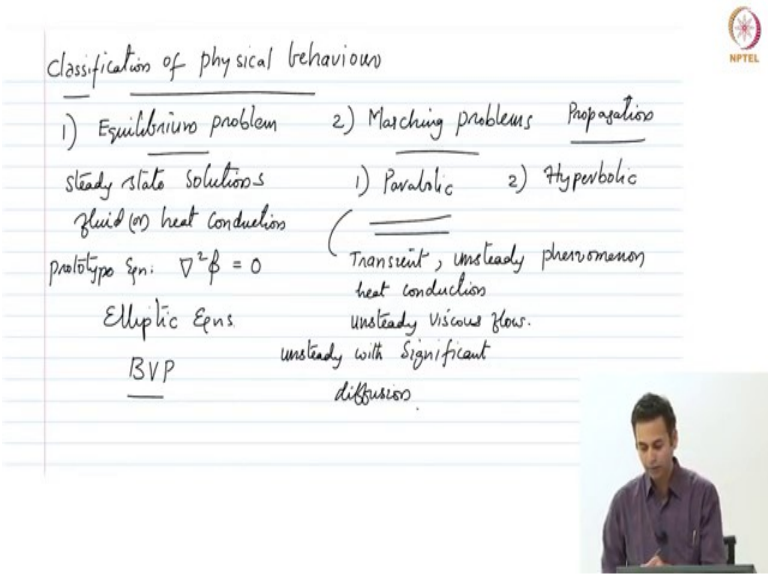

Good morning. Welcome to today's lecture. So, in the last class, we looked at Classification of Physical Behavior. We said there will be two classes of problems that we would like to categorize, one is equilibrium problems, the other are marching problems, right. The equilibrium problems we said these are all the steady state solutions of either fluid flow or heat transfer, right, heat conduction. And we also worked with a prototype equation. What was our prototype equation? It was.

Student: Laplace.

Laplace equation, ok. So, it was $\nabla^2 \phi = 0$del. We also call these type of problems as with a name called elliptic equations. And we noted that these problems require boundary conditions on all the boundaries, so we call these as boundary value problems, ok, alright.

Coming to the marching or propagation type of problems. We have again two types, one is we call it as parabolic and the other one we call it as hyperbolic. First let us look at the parabolic type of equations. These are essentially all the transient phenomena, transient or unsteady phenomena; such as a transient heat conduction or transient or basically unsteady viscous flow to name a few, ok. So, essentially these are unsteady problems that will have a significant amount of diffusion in there, ok. So, these are unsteady problems with significant diffusion, ok.

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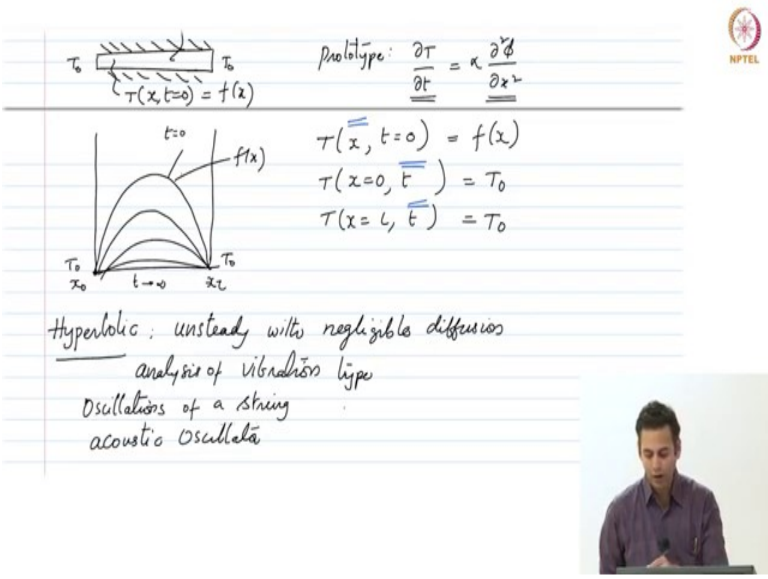

For example, if we consider the if we consider 1D, 1-dimensional let us say we have a solid bar, where we keep; where we keep these two at different temperatures or at the same temperature let us say we keep at T naught which is equal to 0 or something. And then we also insulate the sides of this rod, and we say that the internal heat generation is 0.

So, essentially everything is 0, but then we start off with some initial profile, for this rod. Essentially, we have some initial temperature for the rod which is$T(x, t=0) = f(x)$ T. So, that means, if I draw the initial profile, so this is my x 0, this is my x l and both are maintained at some temperature T naught, ok.

The initial temperature profile in this rod is essentially given by this function, ok. This is at t equals 0 some function of x, ok. Now, what will happen as time progresses? As time progresses there will be diffusion, right, there will be heat transfer through the ends of the rod and the profile will keep getting diffused, and eventually at t equals as t tends to infinity the entire rod will become temperature of t equal t naught, ok. So, essentially as you can see there is a significant amount of diffusion with time, the temperature is diffusing down, ok. So, these kind of problems are something that belong to a parabolic type of problems, ok.

Now, the prototypical a prototype equation for these problems could be the unsteady 1D heat conduction unsteady 1D heat conduction that is $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2}$

So, that is a prototype equation.

Now, from the equation can you tell me what all conditions you would need to solve this problem or what I have already suggested here, right. We would need initial conditions because you have a transient term t here and you have you also need boundary conditions, so I would need boundary conditions on the boundaries, ok.

So, essentially what we need is we need an initial condition that is temperature at for the entire domain that is for all x at t equals 0, right. We would need an initial condition let us call it some f of x and then we would need a boundary condition that is T equals at x equal to 0, we would need a boundary condition, alright.

And do we need this for all the time or only for the initial time? Only initial time. So, that would be t of 0. What do you do at the other times? So, let us say you have a boundary condition of constant temperature, right you maintain this at T equal to t naught.

Do you do that for the entire period of the experiment? You would need it, right. Essentially, if you have a Dirichlet kind of boundary condition then you would need t has to be specified for the temperature on the boundaries has to be specified at particular x for all the times, right. So, essentially you would need $T(x=0, t) = T_0$ Then you would we would call it as some temperature T naught, right you would maintain it as an isothermal condition.

Similarly, $T(x=1, t) = T_0$ for all the time you would need to maintain it as T naught, right according to what we have discussed in the experiment here, ok. So, the initial condition needs to be specified or the all the over the entire domain; whereas, the boundary conditions needs to be specified for the entire time, right. Everybody agrees, ok. So, this is simple, alright, ok. So, if we move on even unsteady viscous flow is a an example of a parabolic type of equations, because of the viscosity you have significant amount of diffusion happening in the flow, alright, ok.

Now, coming to the hyperbolic problems, these are also unsteady phenomena, these are also unsteady problems, but with negligible amount of diffusion, ok. So, essentially all the analysis or, analysis of vibration problems, vibration type of problems dominate this hyperbolic equations, ok. So, if we have a string that is oscillating or you have a acoustic oscillator, essentially you have a oscillations of a string or we have an acoustic oscillator or even if you have a fluid flow at speeds close to the speed of sound, ok.

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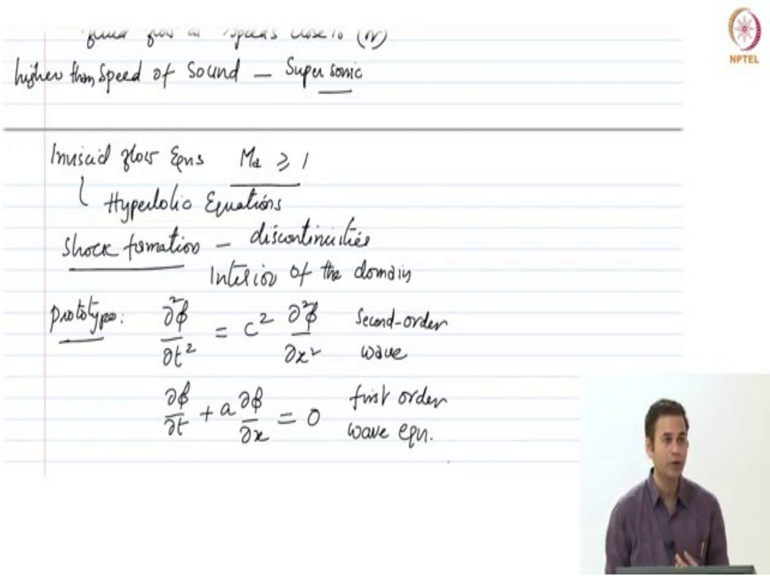

So, these are or close to or more than speed of sound, ok, close to or a higher than speed of sound. These are also these are known as what? Supersonic or sonic, right, essentially sonic or supersonic flows.

These are also hyperbolic in nature, ok. Now, so for example, if you look at the hypersonic flows or supersonic flows they would the viscosity does not play a much role in those flows as a result if you look at the nature of the inviscid flow equations at these speeds that is if we define a Mach number that is greater than or equal to one at these speeds, these inviscid flow equations behave as hyperbolic equations, ok.

So, essentially the viscosity does not play much role in those equations. So, as a result the diffusion is much lesser, ok. And what we also notice is that because of the speeds being very close to the sonic speed or higher, there is a possibility of a shock formation, which is a manifestation of the hyperbolic nature of the problems, ok. Essentially we have because of the shock there could be some discontinuities that get formed in the interior of the domain ok. Now, these discontinuities needs to be also captured by your schemes, in the integral of the domain, ok. As a result it is good to classify the problem before we solve for it, using the particular computational schemes, ok.

Now, what will be a prototype equation for hyperbolic equations? Any guesses here, what will be a prototype equation. A wave equation, right. Essentially, if we have a second order wave equation $\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}$. So, this would be a prototype equation which is a second order wave equation, which describes the oscillations of a string, ok.

Similarly, you can also have of course, a first order wave equation that is $\frac{\partial \phi}{\partial t} + a\frac{\partial \phi}{\partial x} = 0$ So, this is a first order wave equation, ok. Now, the kind of schemes we are going to use to solve hypersonic flows or supersonic flows or for hyperbolic equations have to kind of take into account the consideration that there could be discontinuities in the flow field parameters, ok.

So, that is something what we need to do before we kind of solve these systems. Of course, for the as you know for the purpose of this course we are not going to look at a supersonic or hyperbolic equations, we are only going to look at elliptic and parabolic type of equations, ok.

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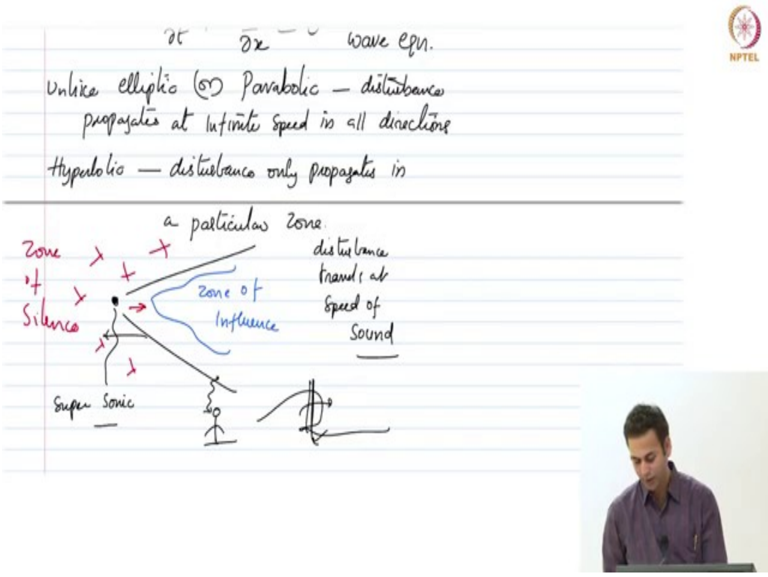

So, unlike elliptic or parabolic type of equations which assume the disturbance to propagate immediately in in all the directions, right, we have discussed is in the context of elliptic equations. If there is a discontinue if there is a jump in the solution somewhere in the domain that is going to propagate through the entire domain, right in all directions. We said in the context of elliptic equations. So, this is true.

So, unlike elliptic and parabolic in which the disturbance propagates at infinite speed theoretically in all directions, ok. Whereas in in hyperbolic equations the disturbance only propagates in a particular zone, right only propagates, so disturbance only propagates in a particular zone, ok. I think this you are aware of for example, if we have a supersonic flow you know that there is a zone of influence and zone of silence, right. Essentially, if you look at, let us say if there is a source of disturbance here this can only influence or send signals which are affected only inside this region which we call it as let us say zone of influence and the pressure fluctuations that are created by this would not be propagating in the other directions, ok.

So, they will not propagate in any of these directions rather they only propagate downstream, right. So, this is a zone of silence when this particular body if it is moving at a sonic or essentially at supersonic speeds, right. So, as a result the kind of schemes you would use I have to take this into consideration, ok. It is not that the disturbance propagates in all directions, but only in a particular direction, ok. So, that needs to be taken care.

What is a good example of this thing? The zone of silence and zone of influence or zone of action. You must have noticed this several times, right. If there is a supersonic flight that is going by the time you hear it for example, let say stand here, right, then by the time you hear it the jet is already ahead of you, right whereas, in a if you have a subsonic flight one of these commercial flights then you would see that you would hear the disturbance before the flight is coming on to you, right before it is approaching, ok. ah

So, as and when this hits you, you are going to get that that boom, right that it is there is a jet and then by the time it is already much ahead of you in its direction, ok. So, that is the that is about a quickly about the hyperbolic type of equations, ok.

Now, we are going to see from a mathematical perspective I will just come to you we will see from a mathematical perspective how to kind of classify these equations from a particular classification method which is going to come next, ok. Yes.

At infinite speed whereas, in the hyperbolic or equations the disturbance only travels at the speed of sound, ok. It only travels at speed of sound. So, this is disturbance travels at speed of sound. Well, it is it travels at speed of sound also in elliptic equation theoretically, but sorry in practice whereas, in in theory they can propagate at infinite speed, ok.

Student: (Refer Time: 15:30).

Yes.

Student: (Refer Time: 15:34).

Discontinuities is not aware of in solid mechanics or I mean they could happen in solid mechanics as well, if you have a material discontinuities and things like that which may kind of give rise to strings I am not aware of as such. But in heat transfer I do not think you would get those, it kind of depends on the you know you have to kind of make the disturbance or the object that creates disturbance or travel higher than the local speed of sound, alright. So, when whenever that happens he will get a kind of shock, right.

But, if you look at well it kind of comes back to fluid mechanics again, if you look at the ocean waves, right you have a shock that kind of gets formed, right. If you go to a beach if you see that there is a toppling of the waves that reaches the shore, right. Essentially, the bottom of the wave kind of catches up with the with the with the bottom and then essentially you get a toppling over, essentially at some point it will be like a shock wave, right. So, these happen, but again it comes to fluid mechanics, ok. But this is in in water and as supposed to air we are talking about, ok. Other questions, ok.

Then let us move on to; now, we have seen that how does how does the physical behavior happened will also look at a classification method of these partial differential equations, ok.

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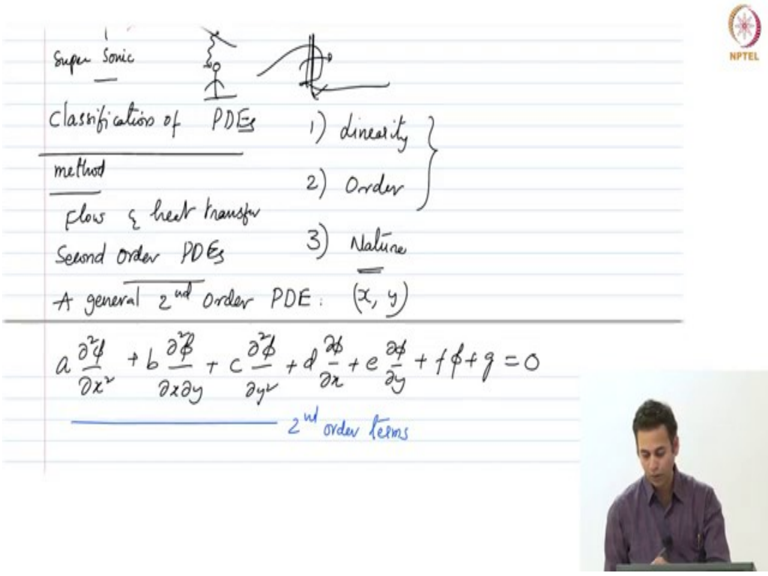

Now, when we say a classification you can of course, so essentially we are looking at classification of a partial differential equations. Of course, this is two generic terms, so we are looking at a particular method of classifying them. So, of course, you can classify in multiple in different ways, right, one classification that you know could be based on linearity, right you could say a particular PDE is a linear or non-linear, ok. So, you can look at linearity and say that I will going to classify equations that I get based on this. What are the other classifications that you know off?

Student: (Refer Time: 17:46).

Based on the order, right. You can say first order equation, second order, third order based on the order you can classify the equations, right if you want to just segregate them. Anything else? Of course, what we have just seen, right, essentially into parabolic elliptic and hyperbolic depending on the nature of the equations, ok. So, we are not going to worry about the first two; because we somehow already know that. So, we are going to look at the nature that is based on whether it is how the disturbance propagates and things like that.

Now, all the equations that we get in fluid flow and heat transfer or second order or second order PDEs, right. So, there are only second order PDEs. So, we are only going to look at a particular method that kind of classifies a general second order PDE, ok. So, we are looking at this particular method of classification which says we look at a general second order partial differential equation and say we would classify them based on certain criteria, ok. Because this is these equations are the ones we are going to encounter in the entire course, ok.

Now, of course, this is not the only method, there as suggested there are several different methods of classifications. This is one particular method for classifying second order PDE or a set of PDEs, that are second order, alright.

Now, when I say a general PDE we are going to consider a general partial differential equation in x and y that is in two dimensions, so which I would try to like write it as

a∂2ϕ∂x2+b∂2ϕ∂x∂y+c∂2ϕ∂y2+d∂p∂x+e∂ϕa $\frac{\partial^2 \phi}{\partial x^2} + b\frac{\partial^2 \phi}{\partial x \partial y} + c\frac{\partial^2 \phi}{\partial y^2} + d\frac{\partial p}{\partial x} + e\frac{\partial \phi}{\partial y} + f\phi + g = 0$

So, let us say this is a general second order PDE that we kind of come across.

Of course, the classification method or the type of the equation depends on the highest derivatives of a PDE, ok. So, as a result as a result we are going to look at the highest derivatives that is the second order derivatives that we have. So, we are going to look at these second order terms in order to classify this particular equation, ok. Then what we kind of do here is we are going to look for simple wave like solutions, ok.

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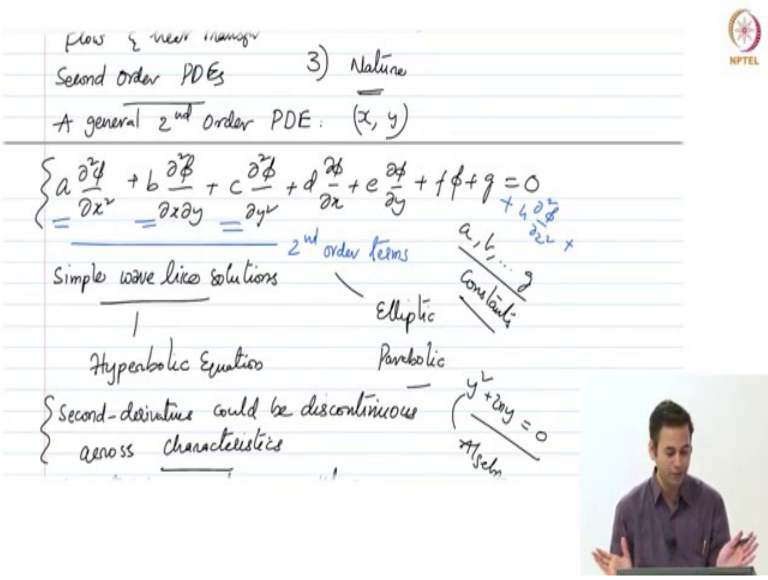

So, essentially if there are simple wave like solutions that exist for a given equation then we are going to classify this as a hyperbolic problem, alright, ok. So, essentially it will be like a hyperbolic equation and if we are not going to get simple wave like solutions then we are going to classify them based on something else as either elliptic or a parabolic, ok.

Now, what has gone in behind this thing which we are not discussing is basically whenever you have a hyperbolic equation then there could be discontinuities across the let us say for example, shockwaves, right. So, essentially these shockwaves if we consider them as characteristics there could be discontinuities across these characteristics, ok. Now, that is what we are invoking, ok, assuming that the there could be, so essentially there could be second derivatives could be discontinuous across something that is known as characteristics, ok. Now, these characteristics could be thought of as the shock waves, ok.

Now, we kind of invoke this as well as the continuity of the first derivatives and we come up with a characteristic equation, ok. So, if we have a characteristic equation that I can construct from this second order PDE and if that characteristic equation has two real roots then I can say that I have wave like solutions coming up from my PDE, ok. That is something which I am not discussing here, but you would take it granted here in terms of the equations.

So, essentially if we look at a characteristic equation for the second order PDE that we wrote above that would be $a\left(\frac{dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c = 0$ ok, so where dy dx is the kind of gives you the slope of the characteristics, ok. Now, this is a characteristic equation. If this equation has real roots then we are going to say that what I am going to get is a hyperbolic equation; because that will have two real characteristics through which the information propagates, ok.

Now, all this is kind of has started the classification of these things have started because you if you can find these characteristics you can actually reduce your PDE to an ODE along these characteristic lines, ok. And then being an ODE you can solve it in a much simpler way than a PDE that is how all these methods have kind of originated, ok. That is why we are giving so much emphasis on the characteristics, and the wave like solutions and so on, ok. But we are not discussing that as part of this course, ok, as part of the CFD course.

But if you have taken some course in gas dynamics or something you would be taught some of these characteristics and reduce in the equations along the characteristics and solving for them, ok, so which we are not doing it here. But this is known as method of characteristics which you can look up and which reduces the PDE to ODEs along these characteristics, ok, alright.

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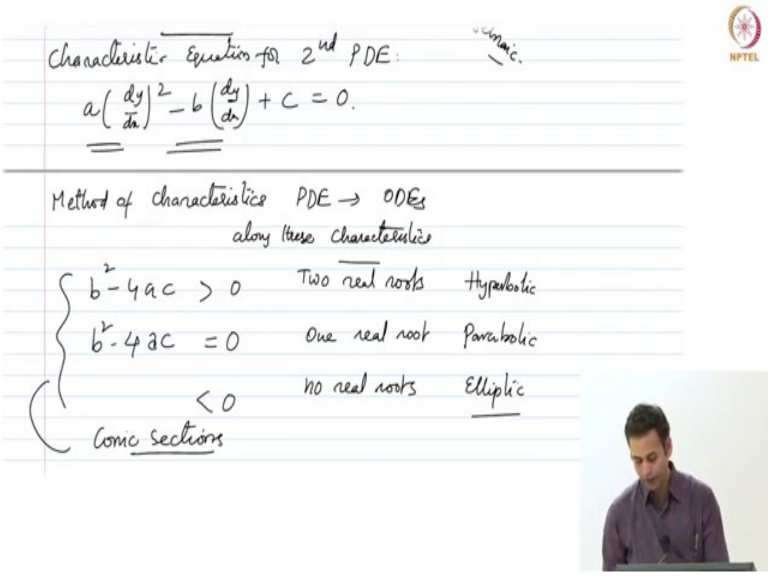

Now, we take it for granted that there is a characteristic equation that we can write for the given PDE and that is$a\left(\frac{dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c$ Where a b c are what? They are coming from the, they are the same values as the coefficients that we have for here, ok. Those are the same values. Now, I here what I assume is that I assume that for this PDE, a, b all the way to g are all constants, ok. I am assuming that these are all constant.

As a result what will be the type of this equation, is it linear or non-linear linear? Because they are all constants, this will be a linear second order PDE. Now, what would make it a non-linear PDE? What should I set a to z as or at least a to f as?

Student: (Refer Time: 25:03).

Either function of?

Student: (Refer Time: 25:05).

No, if they are function of x y is it non-linear? It is still non, it is still linear, right because x y are known, right that is still the.

Student: (Refer Time: 25:15) already.

It should be either function of phi or its derivatives, right. It should be either function of phi or its derivatives then it will be a non-linear PDE, right. Of course, you are right in some in some sense if there were functions of x y, if my equation was not a PDE if it was an algebraic equation it will be a non-linear algebraic equation, right. If I have something like y square plus 2 x y equals 0 or something then because there is a y multiplying y that is a non-linear equation, but this is a non-linear algebraic equation, right.

But in the context of ODEs or PDEs, the coefficients have to be either functions of phi essentially they should be functions of the dependent variable or its derivatives not the functions of independent variables, right. If they are in functions of independent variables that is still a linear PDE or ODE, right, ok, alright. So, we initially assume that these are a to z are all constants. However, this method works even if they are not constants, ok, this method of characteristics and things like that which we will kind of briefly discuss, ok.

Now, for again now this equation to have two real characteristics, the essentially theb^2-4ac b is what I have to look for, right, has to be if it is when will this equation have two real roots. When this is greater than 0 you would have two real roots, ok, which would make the equation as what?

Student: (Refer Time: 26:42).

Hyperbolic, ok. Now, if you have$b^2-4ac=0$ how many roots we would have? We have only one real root and that would make this equations as sorry, we should make it as parabolic, ok. Similarly, if we have a, it is less than 0 you would not have any no real roots; that means, we do not have real characteristics, you only have like imaginary characteristics. So, this we call this as elliptic type of problem, ok.

Now, well I have been discussing this parabolic hyperbolic elliptic all these things for now couple of lectures. How does these come about? Why do we call them hyperbolic, why not some other name? It is essentially because of this b square minus 4 ac is what we are using to kind of classify them, ok.

This kind of resembles to the conic sections that you might have learnt, right where if you have these greater than 0 you are going to get a hyperbola, effort is equal to 0 you are going to get an ellipse and so on. So, this essentially stems from the conic sections theory and other than that there is no other resemblance between the hyperbolic and all these things, ok. So, that is one thing which you have to keep in mind, ok. So far so good.

So, now we have kind of a found an equation using which we can classify the second order PDE that we have, ok.

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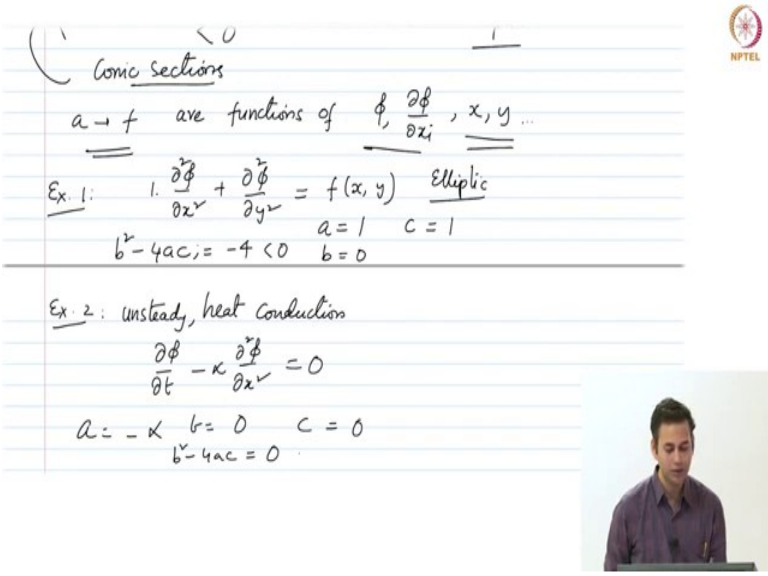

Now, we say that this could be we say that this method still applies if you have a if you have a to f functions of either phi or its derivatives, ok. Now, what does that mean? The moment you have these coefficients functions of ϕ and $\frac{\partial \phi}{\partial x}$ or if they are functions of x y and so on.

What will happen to the type of the equation? The type of the equation may change from point to point, right in the domain because now these are functions of x y, the relative values of a b c could change as a result the value of b square minus 4 ac could change from location to location because of this thing, right. So that means, when we talk about a particular PDE depending on the coefficients the type of the PDE could change, ok.

So, that is what I kind of want to emphasize here. For example, let us look at let us look at an example problem. Essentially, example 1, we take the equation which is Laplace equation,

$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=f(x,y)$. Can you tell me what is b square minus 4 ac here? Or to begin with what is a? a would be 1 because that is the coefficient here. What would be b?

Student: (Refer Time: 29:49).

b is 0 because you do not have a term of a mixed derivative, right partial square phi by partial x partial x is not there, so b equals 0. What would we c?

Student: 1.

c is 1. So, what is b square minus 4 ac minus?

Student: Minus 4.

$b^2-4ac=-4<0$ This is less than 0, so that is why this equation is elliptic, ok. That kind of satisfies what we earlier discussed in the context of elliptic equations, ok.

Let us look at another example. If we take the unsteady heat conduction, ok, so that is a

$$\frac{\partial \phi}{\partial t}-\alpha \frac{\partial^2 \phi}{\partial x^2}=0$$

. Now, what is a b c in this? a is minus.

Student: Alpha.

Minus alpha; b.

Student: 0.

b is 0 because you do not have mixed derivative, right. We do not $\frac{\partial^2 \phi}{\partial x \partial y} \vee \frac{\partial^2 \phi}{\partial x \partial t}$ none of them are there. What about c?

Student: (Refer Time: 30:53).

c is also 0. So, what is the b square minus 4 ac here?

Student: 0.

0. So, is this does it qualify as what type?

Student: Parabolic.

Parabolic like what we have written, right. So, this is 0. So, this would come out to be a parabolic type, parabolic, right.

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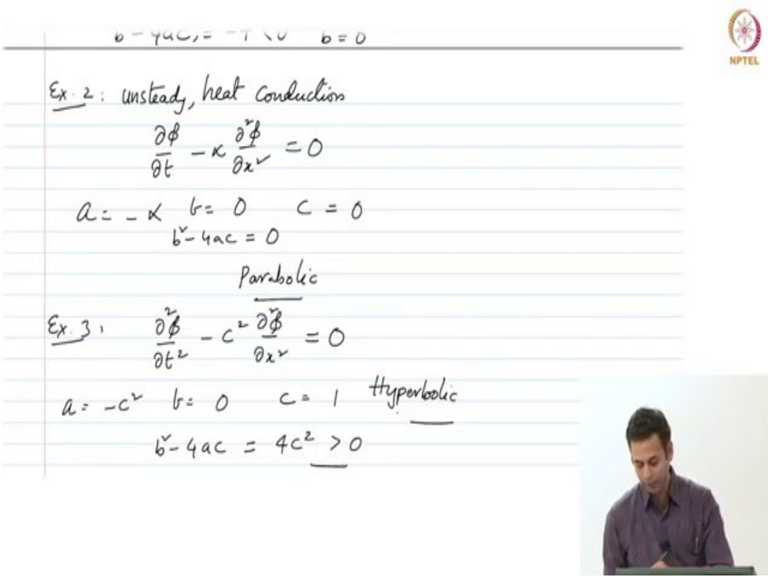

So, this is a parabolic equation like what we discussed before. All the transient equations with negligible with significant amounts of diffusion, ok. Now, let us look at another example. Of course, the only thing remaining is the hyperbolic. So, what is equation we have for hyperbolic? This was second order if it was $\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} = 0$ right that is what I wrote.

Now, what is a b c here? What are the independent parameters that were independent quantities here? X, not x and y like what we had before here it is x and t, ok. So, what is a b c? a is minus c square, ok. b?

Student: 0.

0, because we do not have mixed derivative. c is 1. So, what would be $b^2 - 4ac$

Student: 4 c square.

$4c^2 > 0$ So, this is a hyperbolic, ok. We can of course, keep doing this for several problems fine, ok. Now, let us look at; so, all these till now all the 3 equations we have seen have a b c as constants, right.

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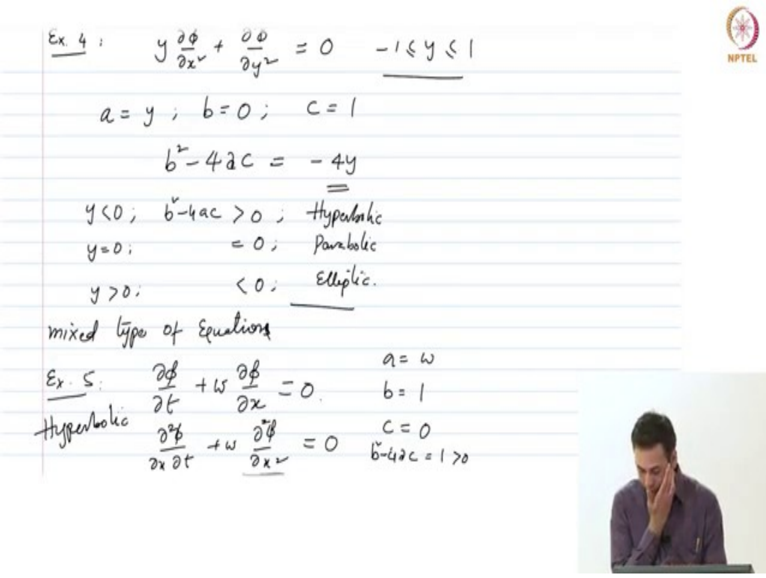

Now, let us have another problem where the coefficient is not a constant, but it is dependent on; example 4, but it is dependent on one of the coordinate directions. For example, I will take y dou square phi by dou x square plus dou square phi by dou y square equals 0, ok.

So, we want to know what is the type of this equation in the over the range minus 1 less than or equals 1, y less than or equals 1, in the range minus 1 to 1 we want to know what is the type of this equation. Because earlier the type of the equation is the same, everywhere in the domain that you considered irrespective of the domain extent, ok. Now, what is what is a here?

Student: y.

y. b is 0. c is 1. So, what would be b square minus 4 ac? Minus.

Student: 4 y.

$b^2 - 4ac = -4\,y$ Now, is this greater than 0 or less than 0?

Student: (Refer Time: 33:24).

That depends on the value of the y of course, ok. So, if $y<0$ then this would be.

Student: b square (Refer Time: 33:33).

$b^2-4ac>0$ So, this will be what?

Student: Hyperbolic.

Hyperbolic. If $y=0$ of course, $b^2-4ac=0$ this would be parabolic. And if $y>0$ the discriminant will be less than 0 and this will be elliptic, ok, fine.

So, essentially the equation now we have written here is a mixed type of equation, right, whose is a mixed type of equation which whose type depends on the region that you are looking at fine, ok. Now, you may have a question here, you just told us that depending on the method we would choose a particular depending on the type of the equation we would choose a particular scheme or a type of method, like what do you do now if you have an equation like that. These are very tough to solve, ok.

If you have an equation where you are going from let us say parabolic to elliptic to hyperbolic in one solution domain, these are very difficult to solve numerically, ok. That means, what we are looking at? We are looking at a simple domain where you have a subsonic flow followed by some kind of transonic flow and then supersonic flow all happening in like one region those are really tough problems, probably you cannot do it with one solver. So, you have to make some compromises and things like that, ok. Any other questions? Yes.

First order wave equation, very good, ok. I will come to that now. So, another example is a

$$\frac{\partial \phi}{\partial t}+c\frac{\partial \phi}{\partial x}=0$$

right. So, this is a wave equation of a first order. I said this is hyperbolic. Now, how do we prove that this is a hyperbolic? Ok. Of course, one thing one caveat I said before was we are looking at a classification of second order PDEs only. So, I did not look at the first order PDEs as such.

But of course, you can convert this to a second order PDE by just differentiating with respect to x or t, right. I would just differentiate with respect to x. So, this would be partial phi by partial x partial t plus c assuming that c is constant this will be $\frac{\partial \phi}{\partial x^2}=0$ Now, what is a b c here? a equals, a equals 0. b equals?

Student: Equals c.

b equals 1, right.

Student: a equals c equals c.

a equal to c, ok, fine. Let us call with a different name I think we have two cs here conflicting. Let us call this as plus some velocity, let us call it w. So, what would be a? Ok. a you want to call it as w, right because we said this term is the first term, ok. Is that correct? And then b equals?

Student: 1 (Refer Time: 36:26).

1. c equals?

Student: (Refer Time: 36:27).

0. What would be b square minus 4 ac?

Student: (Refer Time: 36:31).

4 ac is.

Student: 1.

1, which is greater than 0. Is this what type of this equation is?

Student: Hyperbolic.

Hyperbolic, ok. Now, of course, you do not have to stop here. A follow up question would be now, how do I convert this equation to a second order wave equation, right.

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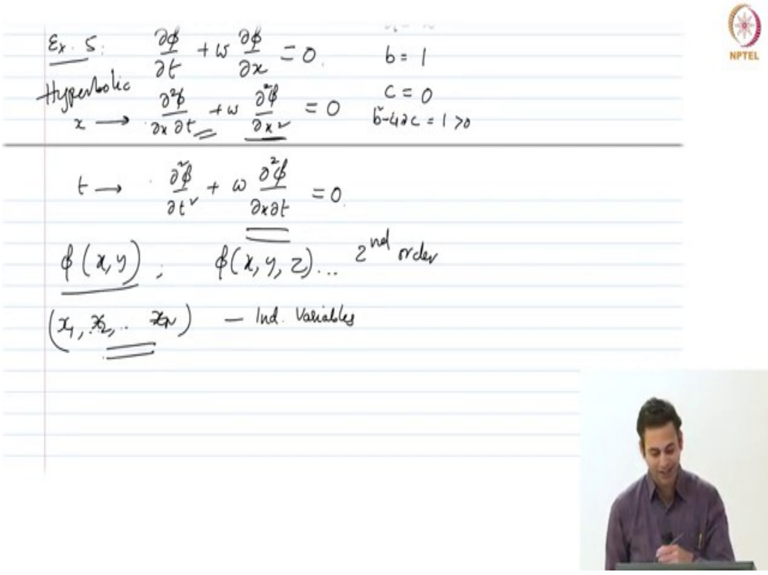

So, here I have differentiated with respect to x, you differentiate with respect to time what are you going to get is $\frac{\partial^2 \phi}{\partial t^2} + w \frac{\partial^2 \phi}{\partial x \partial t} = 0$ Now, you could form a second order equation from these two terms, right which is same as before, ok. So, you kind of get rid of the mixed derivative from these equations and you can end up with an equation which is in terms of a second order wave equation, ok.

Now, in general this method will not apply if you have anything other than second order PDEs, ok. Other questions. I am expecting lot of questions here, ok. If you do not have questions, I will ask your questions and answer them.

Student: (Refer Time: 37:35).

Yes.

Student: Below characteristics you said PDEs reduce to the ODE.

Yeah.

Student: Did you solve another?

Along the characteristic lines, yes, yeah.

Student: Where it is a (Refer Time: 37:46).

Yeah.

Student: (Refer Time: 37:49) greater than (Refer Time: 37:50) either two slopes.

There will be two characteristics.

Student: Two characteristics.

Yes.

Student: Then which type (Refer Time: 37:56)

You would solve along both of them.

Student: (Refer Time: 37:59).

Right. Essentially you would solve along both the characteristics that is a one on the top, one on the bottom.

Student: (Refer Time: 38:06) there is a pattern to all.

Yes.

Student: (Refer Time: 38:08).

Which goes as a plus and a minus and then you would draw along both of them, would kind of solve along both of them, ok.

Yes.

Student: (Refer Time: 38:16).

Exactly.

Student: What will happen to this (Refer Time: 38:19).

Of 3 variables, yes. So, essentially the question is we assumed of course, that phi is a function of only x y, what will happen if phi is a function of x, y, z or something else, right. So, but still the PDE we are looking at is a second order PDE, ok. We are not going more than that, ok. That is fine because let us say you want to solve in a 3-dimensional domain in the Navier-Stokes equations which are second order, but the domain is now 3D, right I have x, y, z. Of course, what we discussed does not work, ok. But a logical extension of this will work and that would be our next discussion that is essentially if we have let us say x 1, x 2 and so on x n, these are the independent variables that we have. And we are looking at a general second order PDE that is in n dimensions, ok.

Then what you are going to do is the treatment is similar, but then the way we classify them is slightly different, but which brings in the same kind of concepts, ok. So, I am going to kind of not derive it rather write the solution here. So, if we have an equation like this, then what would happen? If you go back what will happen to our general PDE, it is here.

So, you are going to get more terms, right. What terms do you get more? You of course, get some h times dou square phi by dou z square, alright plus you would get mixed terms in terms of x z and so on, right. So, it will be gigantic we are not going to write it, ok. So, what we write it is, we would write it in a nice compact form, ok.

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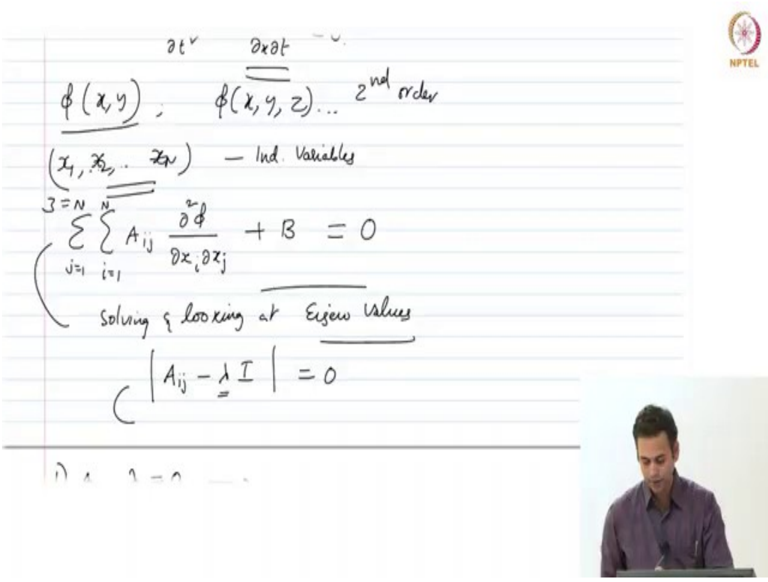

So, we have we can write that as a essentially we are going to get terms which are multiplied with $A_{ij}\frac{\partial^2\phi}{\partial x_i \partial x_j}$ where x i, x j could be either 1,1, 1,2 and so on, so this could be xx, so xy, yz and so on. Each of them will have some coefficient. All those coefficients we are going to write it as sum A ij, ok. That is kind of a matrix, ok. We have all these terms in there plus we have lot of terms which are not second order derivatives, ok. Let us call it some B, ok. These are all the first order terms and the constants and so on, ok.

And all these have to be summed up on what? i equals 1 to capital N, j equals 1 to capital N, right equals 0. So, this is our now second order PDE in n dimensions, right. If you plug in N equals 3 you are going to get for x, y, z, ok. Now, what do we do for this? Of course, it is kind of a logical extension of what we discussed. The answer to this lies in solving for or in solving and looking at the eigenvalues, ok. The eigenvalues of this particular coefficient matrix A ij which is similar to what we have done in the context of just two coordinates, ok.

So, we have to construct $[A_{ij} - \lambda I] = 0$ I construct a characteristic Look at the eigenvalues that you get here, these lambdas. Now, depending on the value of lambdas you are going to classify the equation, ok. Now, this is something I am going to give you as a recipe without proof.

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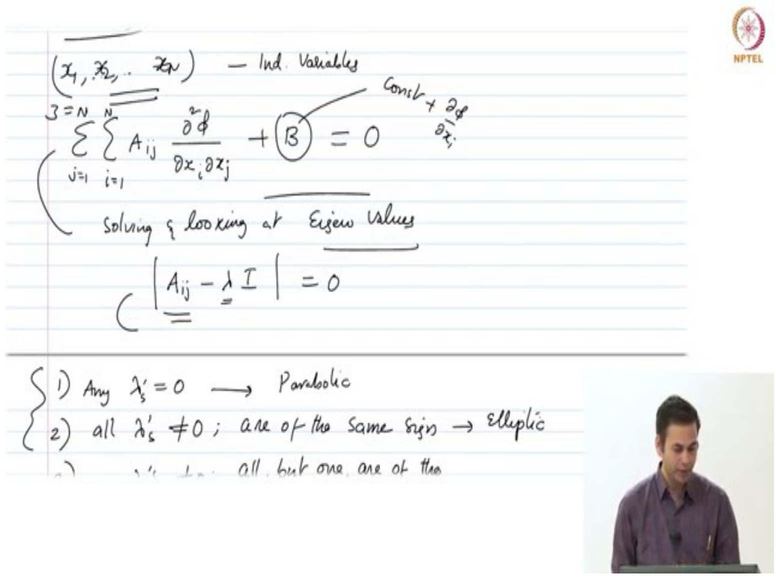

So, if you have, if there is any lambda that is equal to 0 then we are going to call this system of equations as elliptic, ok. What do we have here? We had similar one, equal to 0 was parabolic, right, this is parabolic, ok. So, if you have any lambda equal to 0 we are going to call it as a parabolic type of equation. If all the lambdas are not equal to 0, but are of the same sign, then we are going to call it as a elliptic type of problem. Again all the lambdas are not equal to 0, but all, but one, are of the same sign, then we are going to call it as hyperbolic type of equation. Now, this is you would need this, right, if you want to classify the Navier-Stokes equations in 3-dimensions, ok.

Now, of course, again depending on these A ij values the classification we are looking at could depend on region to region, that could also be possible. Other questions. Yes.

So, say it again, sorry.

Student: (Refer Time: 43:25).

Yeah all, but one of the same thing. For example, if you have looking at 3 dimensional. So, the question is all, but one or the same sign.

Student: (Refer Time: 43:34).

In 3 dimensions. So, if we have 3 for example, the lambdas could be two of them positive one of them negative and things like that. So, one eigenvalue has to be a of different sign than the other two.

Student: (Refer Time: 43:47).

Yeah, it is kind of. So, it will be the same conditions even if we have n dimensions. So, the question is then what will have what will be the type of the equation if this is not satisfied, is it? So, if, but then you would end up with some 0s in the lambdas. You would not have a condition where more than one lambda is of the same sign in the classification.

One thing is it is very difficult to look at more than 3-dimensions in this. Of course, you can include involved time as another dimension, as a 4th dimension, but we do not have another classification method here. So, I guess if you end up with something probably that will again fall into a hyperbolic type of equation depending on the method, ok. Other questions see.

Student: (Refer Time: 44:33).

Yeah, I mean there are several things that may come up, but I think they are all belong to the hyperbolic thing, ok. You do not have the conditions on the parabolic and elliptic are clear in terms of, if you have a 0 then that is parabolic you could have multiple 0s also that will be in the parabolic, but if they are all the same sign that will be elliptic and if you have all let us say mixed type of lambdas then you end up with hyperbolic. But this will be very difficult to get in practice. We probably may have like a toy problem that may show these kind of things, but in practice very difficult to have a physical problem that will give anything else than what is kind of displayed here. Yeah.

Student: (Refer Time: 45:07).

In general equation.

Student: (Refer Time: 45:10).

Oh, in the in the in this equation itself.

Student: (Refer Time: 45:13).

In this equation itself, yeah. So, there will be of course, there will be terms with. So, all those terms which are. So, the question is will there be a more terms which are partial phi, partial x, right are they are all observable in to b, because we are only looking at the highest derivatives to classify this equation, ok. So, b is not a constant, but contains the constants plus all the first order derivatives partial phi partial x i, ok. Other questions.

If you have a non-linear equation, so A ij is a phi or partial phi, partial x, right. So, then the current iterate value of that phi or partial phi partial x has to be used.

For example, you are solving for it, you you have to give give some value for the phi. So, essentially the nature of the equations will change as you progress your solution, right because it is non-linear.

So, it is like the type of the equation will not be known until you get a solution or it will keep changing as you progress the solution, right because we are we are trying to solve for phi and if phi comes as a coefficient here we would not be able to tell the type, ok. But those are also very hard to get unless you have. Well, some equations are there which we kind of belong to the hyperbolic family for example, Fergus equation and things like that we will where you have the coefficient is multiplied with a u, where the unknown is also u, ok. Other questions.

Now, ok, one important question is what is the type of the Navier-Stokes equations? Let us say if you have, right that would be your next question. So, Navier-Stokes equations you can show that the incompressible flow equations are of parabolic elliptic type, ok. So, it is kind of a mixed thing.

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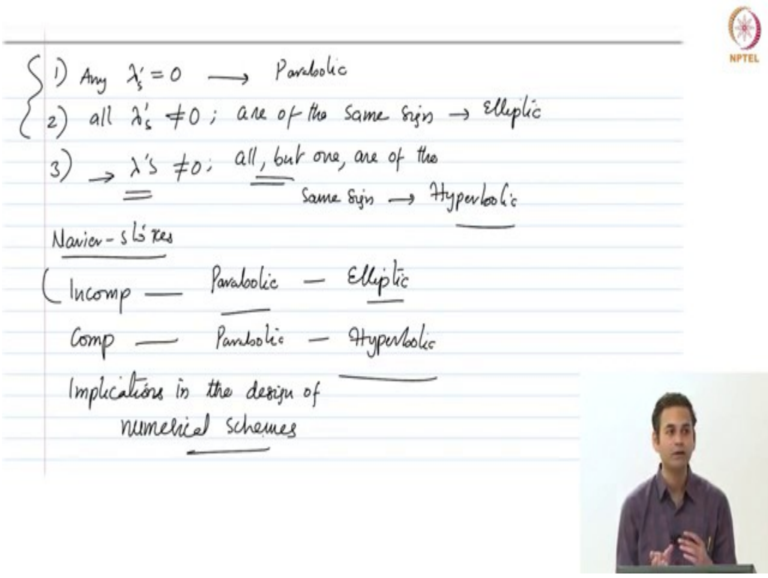

So, we can show that in Navier-Stokes the incompressible equations are of a parabolic elliptic type. So, I think. So, what I mean is elliptic parabolic is basically the pressure terms behave like elliptic equations and the velocity terms behave like parabolic, because now you have extra variable pressure, right. You have velocity as well as pressure, unlike these equations we have written, right here the only variable is phi, right. But in Navier-Stokes you have pressure and this one. And if you go to compressible Navier-Stokes equations this behaves more like parabolic hyperbolic, ok.

So, this has lot of consequences in designing the numerical schemes for the solution of them, ok. So, this has implications in the design of numerical schemes for solving these type of problems, ok.

I think the takeaway message is if you have a elliptic kind of problem, the schemes you would use would be kind of sending the information in all directions. Whereas, if you have hyperbolic then you have to be careful in terms of how you gather the information, ok. Especially, if you have a shock and things like that you have the numerical scan schemes have to be kind of careful enough that they do not pick up the data from all the sides, ok. So, these are very important. Other questions. Ok.

I have one question. So, why not just write compressible flow equations and develop a solver for that and then run it for incompressible flow, right. Why do we care? We will just write for a compressible flow solver and make it work for incompressible also, would that be, would that be fine? Because I just said that, it will be very difficult to solve for incompressible compressible or a range of Mach numbers and things like that. Can we do that?

No, no we cannot do. So, we cannot just work with a compressible flow solver at incompressible speeds, right in as an incompressible situation. Why?

Student: (Refer Time: 49:21).

Sorry, essentially the nature of the equations would change as a result the things you are trying to will kind of trying to solve would become with kind of degenerate and you would end up with kind of a stiff equations, ok. So, the same problems that you would encounter in directly working with incompressible flow equations is what you would encounter if you try to solve from a compressible perspective as well, ok.

So, in the limit of let us say if we have a compressible solver you try to run it at a speeds at a much lower speeds. Let us say for example, Mach numbers less than 0.3 and 0.2, then you would encounter the same kind of problems as you would encounter in if you start off from an incompressible flow equations, for which we have some remedies, ok. So.

Student: (Refer Time: 50:07).

Right.

Student: And when do you.

You usually will have lot of difficulties in converging the solver. So, there could be some special treatments which are which you have to borrow from the concepts of incompressible flow solutions to be put into the compressible thing, ok. In which case you can actually probably just be well off solving the incompressible flow equations rather than the compressible ones. Because the speeds are so small that there are no density variations, the density variations less than you know 5 percent or 3 percent, ok, alright.

Then, I am going to stop here in the next class. So, this kind of finishes the chapter 1, that is the review of the governing equations looking at the general scalar transport equation. So, we have chapter 2 coming up in the next class, that is overview of numerical methods that is also a kind of very short chapter. We will try to finish it sometime over the next week, ok.

Thank you.