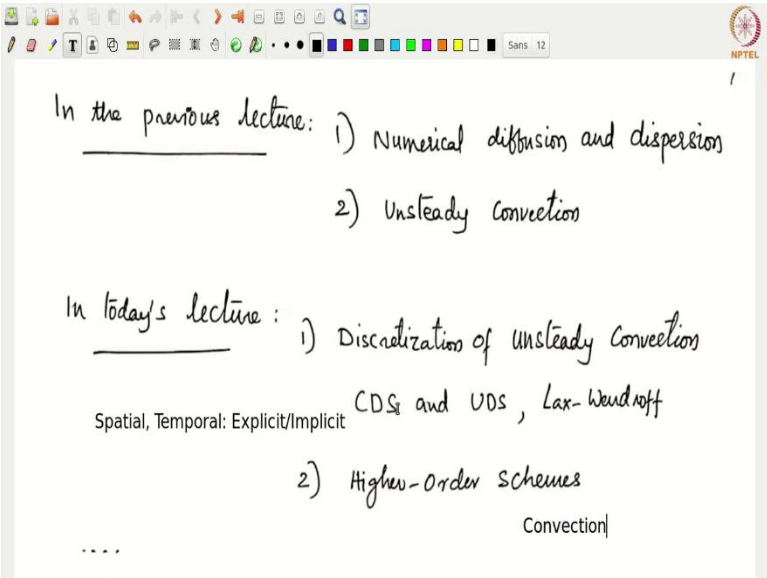
Computational Fluid Dynamics Using Finite Volume Method

Prof. Kameswararao Anupindi
Department of Mechanical Engineering
Indian Institute of Technology, Madras

Lecture – 32
Finite Volume Method for Convection and Diffusion: Discretization of unsteady convection equation

(Refer Slide Time: 00:14)



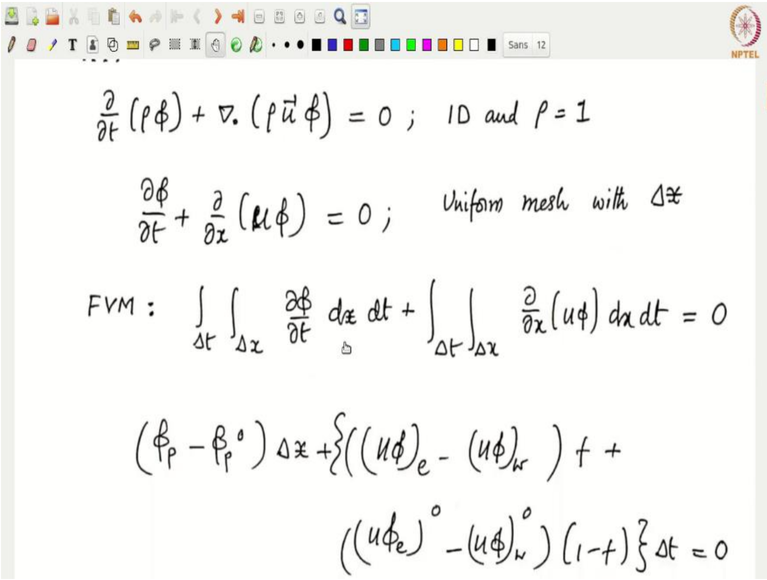
Hello, everyone. Let us get started. So, welcome to another lecture as part of our ME6151 computational heat and fluid flow. So, in the last lecture, we looked at numerical diffusion and dispersion and explained these properties as their type to the upwind difference scheme as well as to the central difference scheme, right, from a numerical perspective.

And, then we also looked at kind of an introduction to unsteady convection and we looked at the exact solution which can be used to compare now when we try to solve this equation using a finite volume method right. So, we can compare it with exact solution and comment on the accuracy of the schemes that we are using ok.

So, in continuation to what where we left off so, in today's lecture what we are going to discretize this unsteady convection equation using central difference scheme or an upwind difference scheme for the spatial derivatives ok. So, for the spatial derivatives we will use either a central difference scheme or upwind difference scheme and we will also use for the temporal derivatives we will either use either explicit schemes or implicit schemes.

And, in addition we will also look at another particular method that is known as Lax – Wendroff discretization and then we will also look at constructing and using some of the higher order schemes for convection. Because till now we have only looked at upwind difference scheme which is first-order accurate in space and central difference scheme which is second-order accurate in space alright.

(Refer Slide Time: 02:06)



Let us move on. So, starting with the unsteady convection equation so, if you set the gamma equal to 0 and source equal to 0, and retain the unsteady part and the advection part, then we get this partial differential equation which is $\frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot(\rho\vec{u}\phi) = 0$, right. In order to simplify the analysis we will make a one-dimensional approximation and also set the density equal to 1, ok.

So, with these two assumptions we can now reduce this equation with rho equal to 1. This term will become $\frac{\partial\phi}{\partial t}$ plus we have $\nabla\cdot(\rho\vec{u}\phi)$, rho equals 1 and this being 1D will only retain one derivative that is $\frac{\partial}{\partial x}(u\phi)$ which is only the x component of this velocity vector equal to 0. To make things further simple we are going to also assume that the mesh is uniform with a cell width of Δx ok, alright.

Now, how do we solve this unsteady convectional equation? Remember, we have done the unsteady diffusion equation a while back in which we had to integrate it not only on the control volume also, but also on the time step, right. As a result, now, we have to integrate both in volume both in space and time for this particular equation so that would read integral Δt that is time going from t to $t + \Delta t$ and integral over the volume.

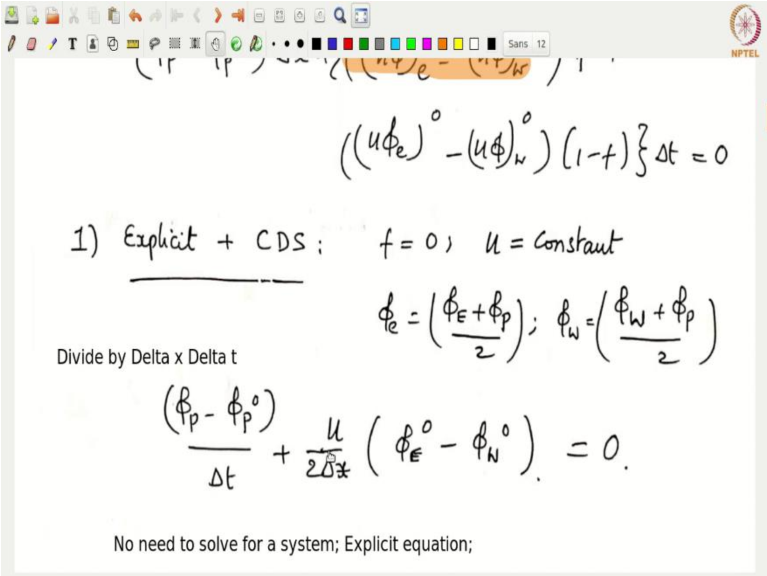
In this particular case, volume happens to delta x. So, this is integral Δt integral Δx, $\frac{\partial \phi}{\partial t}$ and then we have integration on dx dt right, on the volume as well as on the time step plus and then on the second time also we have integral Δt integral Δx, $\frac{\partial}{\partial x}(u\phi)$ dx dt equal to 0, ok.

Now, of course, we can we know how to now integrate these two quantities both on space and time. So, we can write the first quantity as phi p minus phi p 0 times delta x right because this will simplify to partial partial t dt which will give you phi and phi value at the cell centroid can be taken assuming the determines it prevails over the entire cell.

So, this will be $\phi_P - \phi_P^0$, times of course, the volume would be integral times Δx right plus; now, how do we integrate this particular candidate? Essentially we have we apply the Gauss divergence theorem, then $\frac{\partial}{\partial x}(u\phi)$ dx would give you uϕ on the face on both the faces, right. This is uϕ on the east minus uϕ on the west plus we also have a profile assumption for variation of this flux with time right.

That means that we will introduce a factor f times the quantity $(u\phi)_e - (u\phi)_w$ plus $(1 - f)$ times $(u\phi)_e$ at the previous time level $(u\phi)_e^0 - (u\phi)_w^0$ right. Times, of course, this entire thing because being a linear profile assumption as we multiplied with Δt to account for the integration of this quantity ok. So, this is the discretized equation for the volume and time integrated unsteady convection equation here.

(Refer Slide Time: 05:11)



Now, depending on the method that we choose either the value of f will be decided and similarly depending on the spatial method that we choose the approximation for the face values of the dependent variable will be evaluated ok. So, if we use an explicit method for the time step then f would be taken to be 0. So, if you set f equal to 0, the method would be explicit in which case this entire thing would go to would go to 0, right; only the second term remains here, only this term remains.

And, we also say if we use a central difference scheme then ϕ_e and ϕ_w would be taken as arithmetic average of the east value and the P value and the west value and the P value, right. We also assume that u is constant and it is positive in this particular case it does not matter because it is a central difference scheme. So, irrespective of the direction of velocity, the value on the phi on the faces is always computed as the arithmetic average ok, alright.

Then, what we have is $(\phi_P - \phi_P^0)\Delta x$. So, we also want kind of divide throughout divide by $\Delta x \Delta t$ throughout this equation; that means, we will get Δx gets cancelled, we get a Δt in the denominator. So, the first term would be $\phi_P - \phi_P^0$ upon Δt plus this term is 0 right. So, essentially because what we have is an explicit method, right.

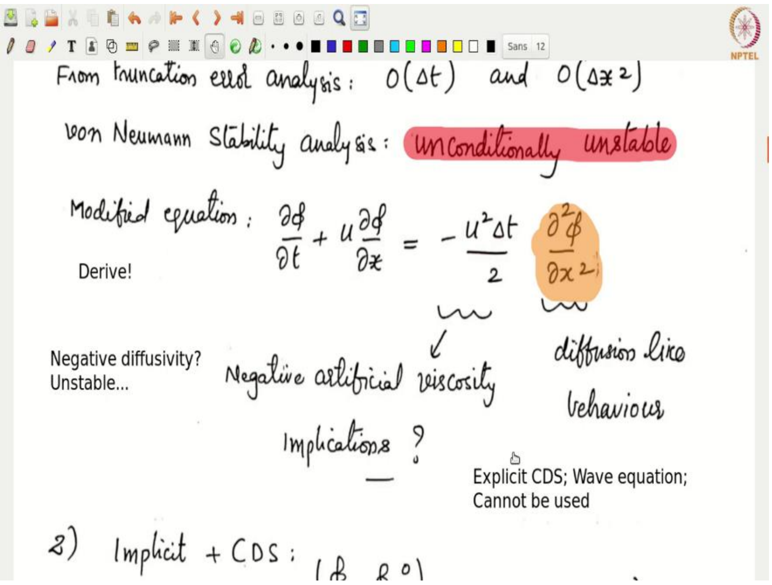
So, f equal to 0 for that, that will give you this quantity is 0 right and only this quantity remains and ϕ_e equals $(\phi_E + \phi_P)/2$. So, as a result what you get is you get a P here and a half phi here and a half phi here, both of which get cancelled, right. And, then remaining terms would be $u/2\Delta x \Delta t$, Δt gets cancelled with this Δt.

So, you get $u/2\Delta x$ and there is a 2 in the denominator. So, you get $u/2\Delta x$ times $\phi_E^0 - \phi_W^0$, both evaluated at the previous time level equal to 0 ok. So, this is the equation when explicit time stepping and central difference scheme is used to discretize this particular one dimensional wave equation ok.

Now, of course, this is an explicit method. So, do we have to solve for a system of linear equations in order to do this? So, no need to solve for a system right because this is a an explicit equation. So, you can directly substitute for the right hand side values and then calculate what is ϕ_P. Of course, we can do a truncation error analysis on this thing right on this particular discretized equation to know what is this spatial and temporal order of accuracy.

And, also we can do a von Neumann stability analysis right and see if that is stable or not similarly we can also do understand what is the modified equation for this to see whether it works or not, ok. So, if you look at the truncation error of course, without doing truncation error we know that the explicit scheme is only first-order accurate in time and the central difference scheme is a second-order accurate in space, ok.

(Refer Slide Time: 08:53)



So, from the truncation error analysis we can write that this is order Δt and order Δx^2, ok. So, which I have not done, but you have to do it and check and again the von Neumann stability analysis we perform oh it shows that essentially this is unconditionally unstable ok. So, this method is a unconditionally unstable.

So, what does that mean? That means, that this cannot be used under any circumstances because whatever no matter what delta t you choose, this is always unstable, but this is a an explicit method. So, what does instability mean? This is an explicit method and this is also for an unsteady equation.

As a result as you go ahead with calculating ϕ_P by substituting ϕ_P^0, ϕ_E^0, ϕ_W^0 from the previous types of values, ϕ_P gets to unbounded values as you go with time as you go with time. So, that is that is the meaning for unconditionally unstable ok; that means, we cannot this method is of cannot be used. So, it cannot it cannot use.

As a result of course, we cannot go ahead with this method. So, we need to change something to get a better method, alright. If you look at the modified equation which you have to derive which I have not done so, you need to derive. What you see is that the modified equation for this particular scheme reads as $\frac{\partial \phi}{\partial t}$ plus $u\,\frac{\partial \phi}{\partial x}$ this is basically the same as the original equation that we started off with assuming u equal to constant.

On the right hand side, what you see is you see $-\frac{u^2 \Delta t}{2} \frac{\partial^2 \phi}{\partial x^2}$ ok. So, the term on the right hand side is has a second derivative. So, this is more like a what is more like a diffusion term, right. This is more like a diffusion like behavior whereas, the diffusion constant itself is actually a function of delta t and not only that there is a minus sign here.

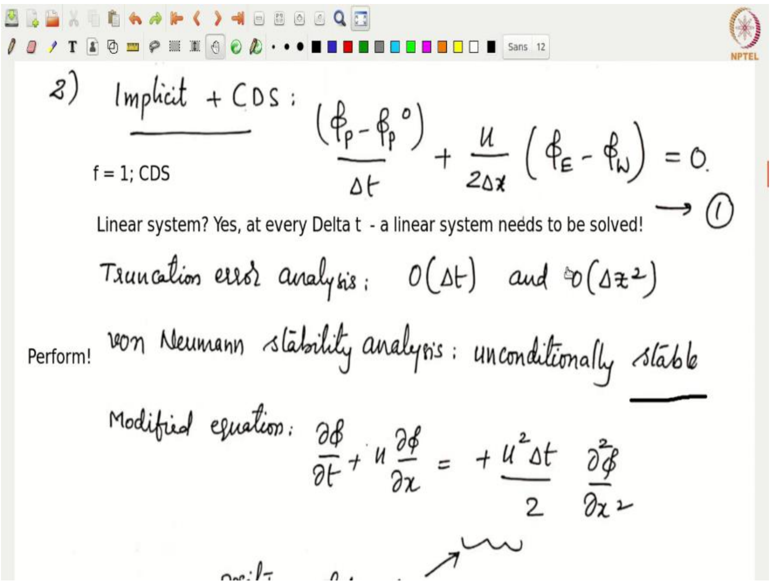
So, the gamma that we would choose till now is now a corresponding value here it has a negative. So, this is more like anti diffusion, right. This is kind of a negative artificial viscosity also we know that if there is a positive viscosity that tends to suppress the instabilities right, essentially it is going to introduce a diffusion like behavior because it is multiplying an even order derivative whereas, if you have a negative value for the viscosity like coefficient.

So, what does that mean; what does that mean if you having a negative diffusivity what will it do to the equation? Essentially it will reduce instead of damping, instead of reducing amplitude it is going to make it unstable, right. So, no doubt why we saw a similar behavior coming from our von Neumann stability analysis which says that it is unconditionally unstable.

And, reason for this instability is basically the negative diffusion that is coming up which can be explained through the modified equation as well, ok. So, as a result the explicit central difference scheme when applied to the wave equation is cannot be used, ok. So, it cannot be used to solve this particular wave equation ok. Then, of course, we can look at the we can look at change in either the (Refer Time: 12:34) scheme or we can look at change in the time stepping scheme.

Let us first look at change in the time stepping scheme instead of having an explicit method, let us have an implicit method for the time derivative and a central difference scheme as usual for the spatial derivative ok.

(Refer Slide Time: 12:42)



Now, if we do this of course, we have the original equation because the spatial scheme is not a different it is basically. So, essentially we are substituting for what here f equals 1 because this is implicit method. And, the central difference approximation is the same ok; that means, we go back here now this term gets retained whereas, this term. So, this term gets retained because f equals to 1 and this term goes to 0.

So, all our values are now evaluated time level 1 right or at delta t. So, that is the only difference; that means, we get rid of these superscripts here. So, as a result our discretized equation reads as $(\phi_P - \phi_P^0)/\Delta t$. Of course, here again we are dividing everything with $\Delta x \Delta t$ plus you have u upon $2\Delta x$ times $\phi_E - \phi_W$, these are evaluated at the current time level.

Now, how do we solve for this equation? Do we need to solve for a system? Do we need to solve for a linear system of equations? Yes, we need to solve for linear system; that means, at every Δt you have to solve for a linear system right because this is an implicit method, right. So, yes, at every Δt a linear system needs to be solved, alright. What about the truncation error? The truncation error remains the same. We know that the implicit method and the explicit method both have order Δt in time and order Δx^2 in space. So, this is the same thing.

What about the von Neumann stability analysis? Von Neumann stability analysis if you perform which I am not doing here. So, you need to kind of perform this step and see if you perform once von Neumann stability analysis, then what you get is, it says that the method is unconditionally stable ok. That is good news because the method is unconditionally stable. This can be used of course, maybe for simulations, but what happens is if you look at the modified equation oops, sorry.

So, if you look at the modified equation what we get is basically $\frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x}$ equals you get a plus instead of minus right in this case you get a minus and here in this case you get a plus, is that easy to see? Is that straight forward to see? Because we have just changed the time stepping here maybe if you perform or equation you will be able to see that ok.

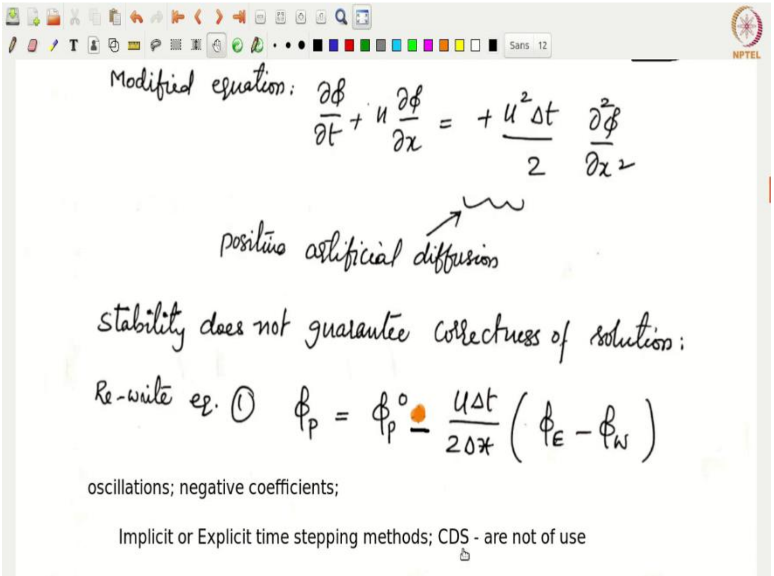
So, essentially $\frac{u^2\Delta t}{2}\frac{\partial^2\phi}{\partial x^2}$. So, that means, again we got a diffusion like term alright and then there is a positive value for the diffusion coefficient. So, that means, this also acts like most alike an upwind scheme right, essentially it is going to introduce some kind of diffusion. Now, that is what we think.

But, however, if you look at if you actually try to solve this system although this is unconditionally stable what you see is that your solution would kind of give you a stable solution, but the results will not be accurate for any delta t you choose. That is because the von Neumann stability analysis only tells you that it is unconditionally stable; that means, it is not going to be unbounded it is not going to be infinite values for any of these solution variables. But, it does not guarantee that it will give you the correct answer ok.

So, this stability analysis only tells you whether it is gives you boundedness or not or it gives you stable solutions or not, but it does not tell you whether it is accurate or not ok. So, that this does not mean that it will be accurate or correct or deflect physically whether it is possible or not. That can be that behavior can be explained by looking at the coefficients right in our regular analysis we always look at the coefficients.

So, if I rewrite this equation by sending everything to the right hand side we know that there are the coefficients here one has a positive sign and one has a negative sign right. Now, does not matter what our Δt we take, the a east and a west will always have opposite signs, right.

(Refer Slide Time: 17:02)



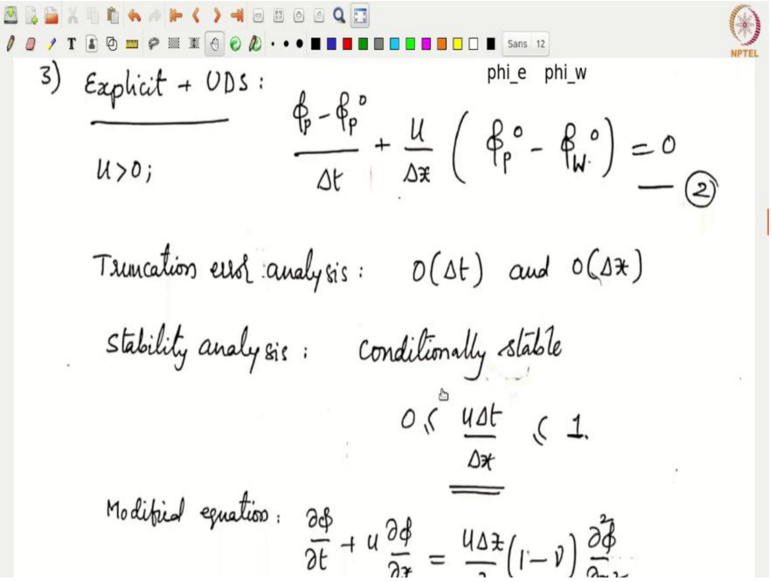
That means, I can rewrite this as $\phi_P = \phi_P^0 - \frac{u\Delta t}{2\Delta x}(\phi_E - \phi_W)$, right. So, we can write that here. This should be a minus here ok. So, this should be a minus because I have send it to the right hand side ok. So, this is not a plus this is a minus, fine, alright.

So, that means; that means, what we have here is that, we have the coefficients being opposite signs this E is would always produce oscillations, right. This will produce oscillations as a result of the negative coefficients that we have and does not matter whatever Δt we take, this will always produce oscillations ok. So, that is the reason why the solution will not be physically correct. However, the results will be bounded.

So, you will get still some numbers which will not grow to infinite or something, but this method would not be of use because we cannot really use the apprehend solutions to compare with anything ok. So, essentially both the method that we have seen till now either the implicit or the explicit time stepping methods together with this central difference scheme are not of any use alright. So, we could not solve for it.

Now, let us explore the other two methods which are either the implicit or explicit, but used with the in the context of the upwind difference schemes ok, alright.

(Refer Slide Time: 18:49)



Let us look at upwind difference schemes. So, let us get started again with explicit method for time stepping and upwind difference schemes for the spatial derivatives ok. So, because it is explicit, we expect that everything to be evaluated at the previous time level; that means, everything will have a superscript of 0. The unsteady term remains as it is this is $(\phi_P - \phi_P^0)/\Delta t$.

Now, we make an assumption that the velocity is positive because upwind scheme requires you to have a kind of depends on the direction. So, as a result I make u is greater than 0. So, basically once u is greater than 0, we can write for phi remember this term is basically

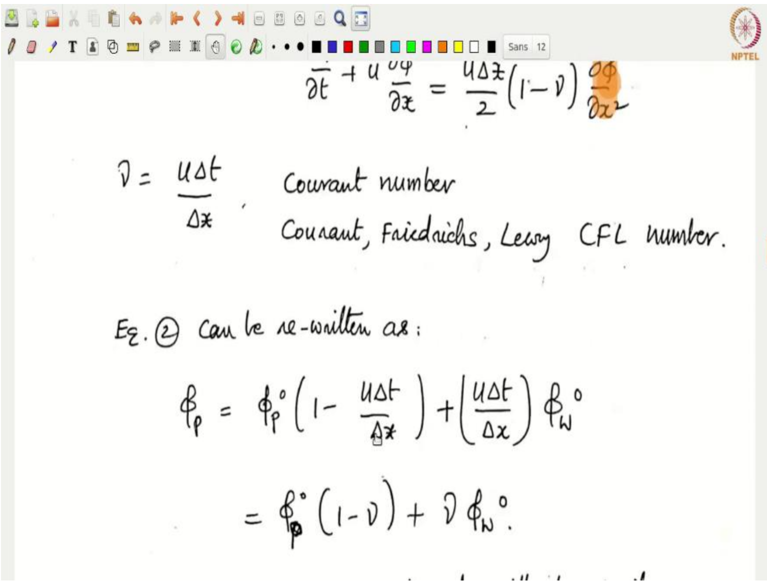
ϕ_e and this term is our ϕ_w right because this is basically you have a phi is u is greater than 0.

So, ϕ_e will be equal to ϕ_P and this quantity would be equal to ϕ_W, right. This is on the west face on the west face if u west is also positive, then this will be phi W, right. You need to verify this once again alright. So, we have this particular equation which is discretization with explicit time stepping and upwind difference scheme ok.

Now, of course, if you perform truncation error analysis what you get is you get a first-order in space and first-order in time because upwind difference scheme is first-order in space as well, ok. Now, if you perform von Neumann stability analysis what you see is that the method says this is conditionally stable.

So, conditionally stable which means that essentially if you choose the so, that means, the this method can be used depending on under certain conditions of your Δt ok. So, which is basically which tells you and the and the stability condition you would get from this stability analysis is basically $\frac{u\Delta t}{\Delta x}$ which is a particular ratio should be between 0 and 1. So, if it is between 0 and 1, then this method can be used to get correct results, ok. So, that means, this method can be used, now let us look at the modified equation to kind of describe its behavior.

(Refer Slide Time: 21:18)

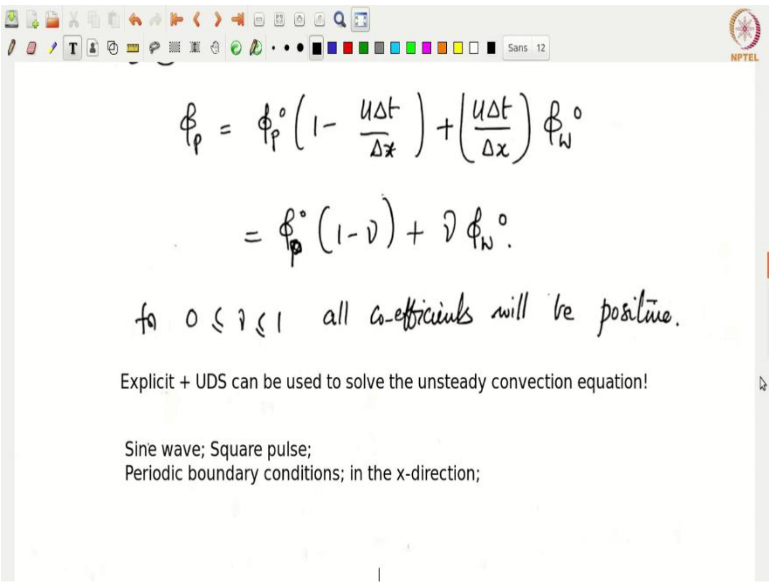


So, the modified equation looks like $\frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x}$ equals; now, on the right hand side again we have a second-order derivative which is similar to the diffusion like behavior ok. Not only that we have we have also the diffusion coefficient is now $\frac{u\Delta x}{2}(1-\nu)$, ok. This nu ($\nu$) is nothing, but the $u\Delta t/\Delta x$ ok. So, this is basically the same now value here we have now called it as nu ok.

So, if I say ν is $\frac{u\Delta t}{\Delta x}$ which is in the literature referred to as either Courant number or it is actually referred to as CFL number which is named after Courant, Friedrich's and Lewy in the after the scientist who first proposed this particular who first performed the stability analysis and discovered this thing ok. So, as a result if we can choose Δt such that it is less $\Delta x/u$, then we can certainly work with the equation in integrating this thing ok.

So, now let us come back to our heuristic analysis where we can rewrite this equation as ϕ_P equals ϕ_P^0 by Δt, right. This is ϕ_P^0 times $1-\frac{u\Delta t}{\Delta x}$. So, I am sending everything to the right hand side and then this becomes $\frac{u\Delta t}{\Delta x}$ times ϕ_W^0 ok. So, this is basically what we get ok. So, if I do this, this is basically what we get. $\phi_P = \phi_P^0\left(1-\frac{u\Delta t}{\Delta x}\right) + \left(\frac{u\Delta t}{\Delta x}\right)\phi_W^0$

(Refer Slide Time: 22:47)



Now, this equation is nothing but if I replace $\frac{u\Delta t}{\Delta x}$ with ν what we get is ϕ_P^0 times $1-\nu$ plus ν times ϕ_W^0, ok. Now, what we see is if because we said from the stability analysis if nu can be taken to be between 0 and 1, then it turns out then both coefficients are always positive, is it not? Which was not the case when we had x when we had implicit CDS, right.

No matter what delta do you take, the coefficients were always coming out to be some coefficients coming out to be negative because there the coefficients were in terms of phi p, phi east and phi w whereas, here we have only two in terms of phi p and phi w as a result they are the term coming from the unsteady can be clubbed with the one of the spatial values ok. So, that means, if nu can be taken to be between 0 and 1; that means, this will be positive and of course, this will be positive.

As a result both the coefficients will be positive and this equation would give you stable solutions and physically possible solution without oscillations right because the coefficients will be both coefficients will be positive, alright. As a result the explicit plus upwind difference scheme can be used to solve the unsteady convection equation ok. So, it can be used ok. We now found a one method to solve our equation, alright.

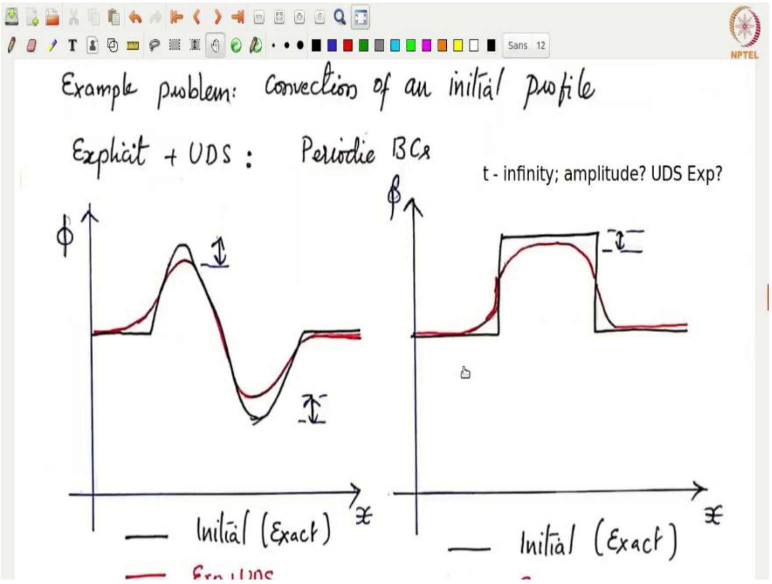
Now, do we need to solve for a system here, a system of linear equations? No, we do not have to because this is an explicit method, everything is known on the right hand side and it can be just substitute and we can get the value of ϕ_P at every Δt. Only condition is that your Δt cannot be any random number rather it has to be between to satisfy this CFL number criteria, right. So, it has between 0 and 1, $\frac{u\Delta t}{\Delta x}$ ok. So, that is the only condition that is put on the method alright.

Now, let us see if we can use this explicit upwind difference scheme to solve let us say our wave equation with two initial conditions; one set of problem would be with an initial condition of a sine wave the other one would be with a. So, one would be with a sine wave; the other one would be with a square pulse, ok. Now, we assume that we have let us say periodic boundary conditions what does that mean?

That means, that in the x-direction we have periodic boundary condition. So, if something is leaving through the domain through one direction it will come back through the through the left hand side ok; if it leaves the right hand side, it will come back through the left hand side that is a periodic boundary condition that is what we implemented in the code

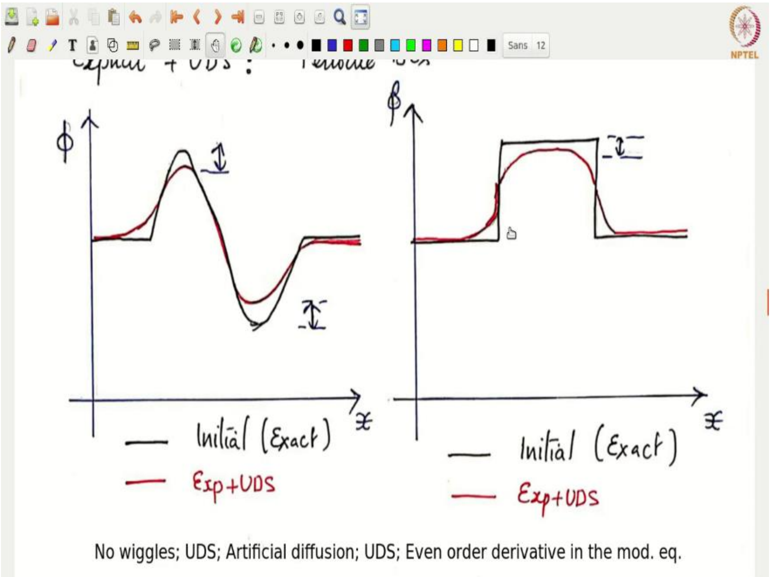
So, what does it mean? How do you implement a periodic boundary condition? We implement a periodic condition by saying that your east neighbor for the cell that is near the boundary would be same as this guy, right, on the left hand side here. Let us see ok.

(Refer Slide Time: 25:55)



Let us look at this example problem convection of an initial profile. We know that the exact solution is basically the same profile shifted by u times time right velocity times the time is what the shift is. Now, we will use a explicit and upwind difference scheme for time stepping and spatial derivative and of course, also use a periodic boundary conditions in the x-direction.

(Refer Slide Time: 26:20)



So, here on the left hand side you see a initial sine wave on the right hand side you see the problem of the square pulse. So, let us see if you have a so, I have periodic boundary conditions here and here; that means, whatever goes out of here will come back from here. Now, the initial profile is shown here in black this is basically approximately a sine wave and this sine wave the exact solution also after one period that means, it will travel it will travel it will come back and it will come back to the same location.

So, we have solved the problem let us say for one time period one period of this travel. So, it has come back to the same location. So, the exact solution also coincides with the initial profile that is with the black one. Now, if you have solved it using explicit and upwind difference scheme, the result you would obtain is basically shown in red color here where you can see that there is a decrease in amplitude. So, we start off with this black profile and solve the system using explicit upwind difference scheme.

Then, the solution after one period of revolution or one period of movement would be equal to the line shown by red right. So, you can see that there is a diffusion that is coming into here. This diffusion is basically dictated or controlled because of the term we saw here right based on $\frac{u\Delta x}{2}$ times $1-\nu$ this is the diffusion coefficient depending on how large Δx , Δxt we take. The amount of diffusion we see here will be controlled ok.

But, one thing we see is that there are no wiggles in the solution right that is because we have not used the CDS scheme rather we have used an upwind difference scheme. So, we see some artificial diffusion right. We see artificial diffusion which is a characteristic of the upwind difference scheme right because of the even order derivative in the modified equation.

Now, what about on the right hand side? So, on the right hand side we again start off with a square pulse that is the black solution that is our initial profile and again we apply periodic boundary conditions. So, the square pulse essentially moves ok. The square pulse moves like this to the right hand side once it reaches the boundary it will start coming back from the left hand side ok.

So, again have periodic boundary condition and we let it come back to the same position to the same location that is one cycle of revolution and we again start off. So, that is the exact solution dictated by $\phi(x - ut)$ the exact solution for the pure convection equation.

Now, if we solve again this initial profile that is shown in the black with explicit and upwind difference scheme what you see is that, you see again a profile that does not have any wiggles, but it has a decrease in amplitude, right. You can see a diffusion like behavior that is because of the even order derivative that is coming up in the modified equation ok.

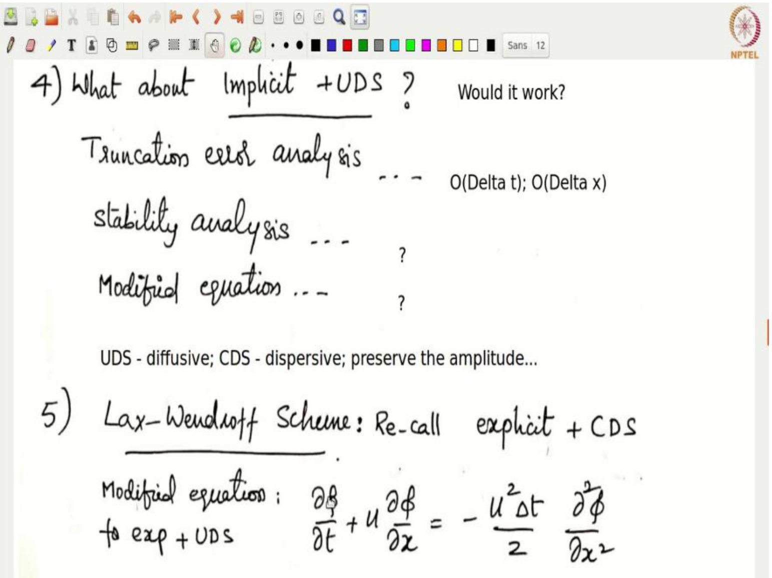
So, this is a solution of course, it is not exact rather it gives you a solution which is a kind of diffusive in nature, but without any wiggles ok. So, explicit upwind difference scheme can be used with a proper delta x delta t such that we can get somewhat results with by using the method ok.

Now, one question here is that we have solved it let us say for one period of revolution, now let us say if I if we keep doing this with explicit UDS for let us say few hundreds of cycles, what will happen to this peak? Would it be the same or would it reduce? Or in the let us say in the limit of t tends to infinity, what would be the amplitude that you see here predicted using the upwind difference and the explicit method?

What would that be? What will happen with this amplitude? Would it remain the same as you go with time or would it? Of course, it will shift right it will shift it will come back, but the next time it comes back here would that be smaller or would that be the same? That should be smaller, right? Because continuously you are as you are integrating the equation, your diffusion is taking away the energy from the solution right.

So, as a result it will be smaller and smaller and what will happen in the limit of infinite amplitude? The exact solution will remain the same because it dictates that it has to preserve the amplitude and it only shifts in location, whereas if you solve it with one of these methods eventually the amplitude will go to 0, right. Everything will disappear alright.

(Refer Slide Time: 31:07)



Now, that means, we found one method of doing or solving this unsteady convection equation that is the explicit upwind difference scheme, now what about the implicit upwind difference scheme would that work or would it would it not work? Will an implicit UDS work that is something you have to find out and you can again perform truncation error analysis, of course, if you do this thing you know that this is order Δt and order Δx, right. This you already know.

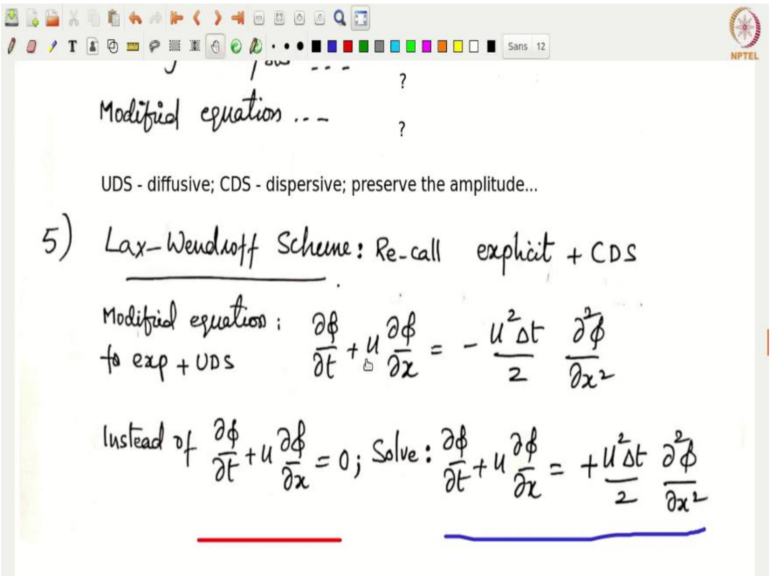
If you perform stability analysis you need to see what you would get would it be unconditionally stable or unconditionally unstable or whatever it is. So, you need to perform a stability analysis and also you need to look at modified equation for this and comment on the on the solution ok; that is something I leave it for you to do, ok. Of course, you can draw similar conclusions by looking at how this term will be right, we have done it for the implicit series. So, you can see how the modified equation looks like and so on by looking at the explicit method as well, alright.

So, we have now out of these four methods we have looked at one of the methods. Now, the of course, the upwind difference schemes are viable; they can be used for the solution, but the issue is that the upwind difference schemes are also diffusive right. When we looked at the central difference schemes they cannot be directly used either with explicit or implicit methods, but one property that the central difference schemes has is they are dispersive in nature.

However, they also have this artificial viscosity, the negative viscosity that is coming into play; however, they are not dissipative. So, they kind of you know preserve the amplitude right which is a good thing. So, several attempts have been made in the literature and several modifications have been have been done and one of such schemes is basically a Lax – Wendroff scheme which is basically tries to address kind of tries to fix the problems associated with the explicit CDS method ok.

So, it will take basically the explicit central difference scheme which will not work because of the negative viscosity that we have in the modified equation and it will try to fix that such that it works ok.

(Refer Slide Time: 33:45)



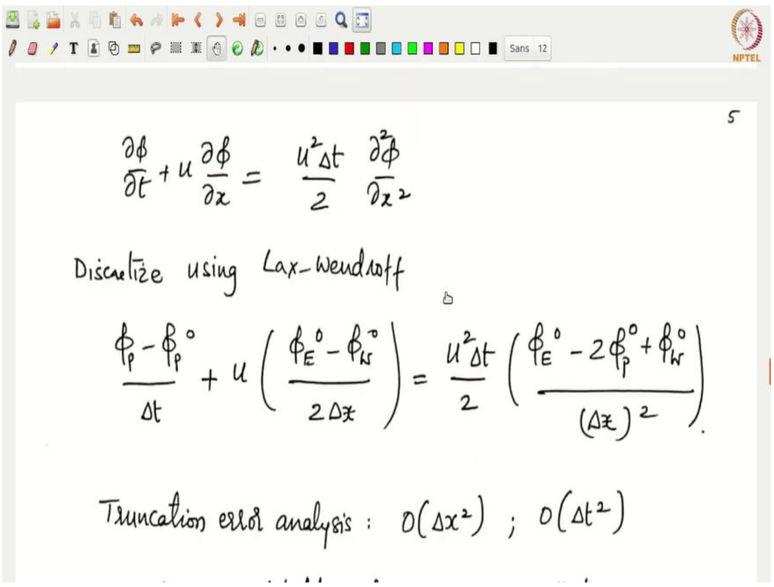
So, essentially you if you would recall the explicit CDS method the modified equation for explicit CDS method what we got was $\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = -\frac{u^2 \Delta t}{2}\frac{\partial^2 \phi}{\partial x^2}$. We said this is more like a diffusion like a term and then the negative viscosity, negative diffusion coefficient is what is causing the problem in terms of instability ok.

So, the idea is can we get rid of this coefficient or this particular term from the from this modified equation? Ok. So, that means, yes, we can. Essentially, we should not solve for $\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0$ in the first place, rather if you add a positive value of the same quantity to the equation; that means, instead of solving for the originally wave equation if you would solve for you add a plus here right to the original equation such that in the modified equation when you get this minus this gets cancelled with this plus ok.

So, if you start off with this problem instead of the original problem, then you will not have the negative viscosity coming into play; in the modified equation as a result it will work fine. So, that is the idea with which it was started off with. So, instead of instead of solving with instead of solving with this, you solve you start off with the essentially a wave equation with the diffusion like term on the right hand side where the diffusion coefficient is basically your negative of what you get in the modified equation; that means, if it is a positive $\frac{u^2 \Delta t}{2}\frac{\partial^2 \phi}{\partial x^2}$ ok.

So, this is basically changing the problem. So, you are asked to solve for a 1D wave equation and instead you will not solve for it because you want to work with explicit central difference scheme, you will solve a different equation with something else on the right hand side which would have a better behavior ok. So, that is what Lax – Wendroff proposed and that is what Lax – Wendroff scheme is applied to 1D wave equation ok, alright.

(Refer Slide Time: 35:50)



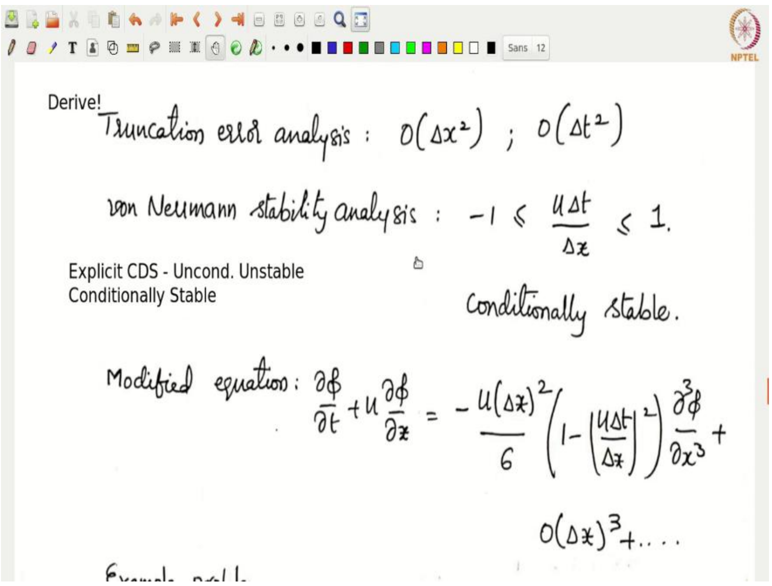
That means, now, how do we discretize this equation? So, we actually made it now second-order derivative here. So, what now starting point is this equation that is $\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = \frac{u^2 \Delta t}{2}\frac{\partial^2 \phi}{\partial x^2}$ that is our starting point.

Now, in the Lax – Wendroff scheme we again use explicit method; that means, and central difference scheme for these derivatives and also we need to now come up with central difference scheme for the second derivative, that is, the $\frac{\partial^2 \phi}{\partial x^2}$, ok. So, what we have is, for the first term we have $\frac{\phi_P - \phi_P^0}{\Delta t}$ plus the second term is u times this is basically when you integrate and substitute for ϕ_e and ϕ_w, you get $\frac{\phi_E^0 - \phi_W^0}{2\Delta x}$ that is what you get.

And, on the right hand side what you get is $\frac{u^2 \Delta t}{2}$ and the second derivative here it appears as if we have substituted using a finite difference formula, but this is done with finite volume you get a pretty much the same result here. So, you get basically $\left.\frac{\partial \phi}{\partial \mathrm{x}}\right|_e$ minus $\left.\frac{\partial \phi}{\partial \mathrm{x}}\right|_w$ and each of them would have $\phi_E^0 - \phi_P^0$ minus $\phi_P^0 - \phi_W^0$ that will give rise to minus $2\phi_P^0$ and all other terms.

And, when you had divided with delta x delta t there is already a Δx here and this becomes Δx^2 and there is a Δt that comes up in the a numerator as well this is basically coming because of the extra term we started off with it ok. So, this is the discretization for the Lax – Wendroff method alright

(Refer Slide Time: 37:44)



Now, we can of course, do a truncation error analysis and von Neumann stability analysis and we can look at the modified equation for this for this Lax-Wendroff scheme; that means, we are looking at a modified equation for the modified wave equation ok.

So, that means, if you do a truncation error analysis which I have not done here which you know how to do it. So, if you kind of derive this part. This is basically gives you second-order accuracy in space that is because of the central difference schemes we have used and it also gives you second-order accuracy in time that is because of these terms that are coming up here, ok. So, this you need to verify.

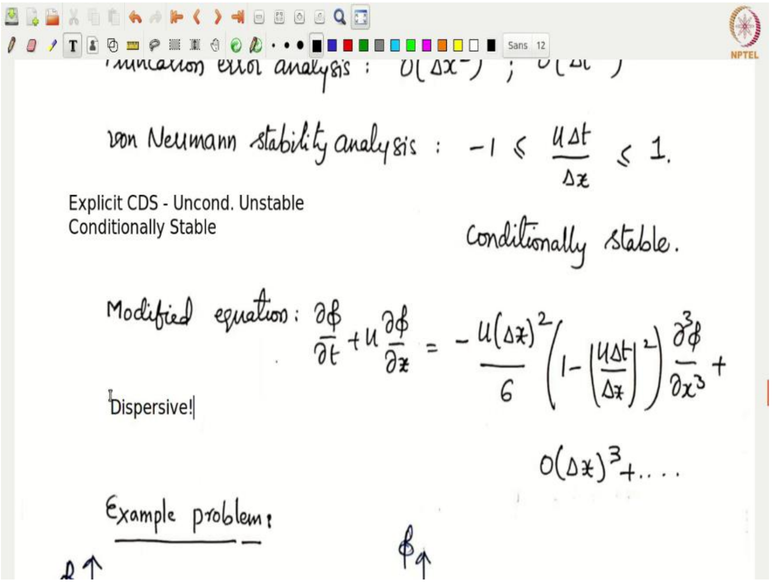
Now, if you perform von Neumann stability analysis for this Lax-Wendroff scheme what you get is essentially your $\frac{u\Delta t}{\Delta x}$ the CFL number has to be between minus 1 to 1. So, it only gives you a conditional stability; that means, you have to choose your Δt such that it is kind of satisfies this condition and if you can choose it that way then the method would be stable, ok.

So, we have changed the unconditionally stable method right which was originally the you remember the explicit CDS was unconditionally unstable right. So, that we have changed it and now, conditionally stable method by incorporating this extra second-order derivative on the right hand side, ok. That is what Lax-Wendroff scheme has done, alright.

Now, we can look at the modified equation for the Lax-Wendroff method; that means, we are looking at modified equation for this guy right. So, which of course, will have will not have this particular term because this gets cancelled with whatever term that usually you get and you get now a third-order derivative. So, the modified equation would read $\frac{\partial \Phi}{\partial t} + u\frac{\partial \Phi}{\partial x}$ equals you get some number times $\frac{\partial^3 \Phi}{\partial x^3}$.

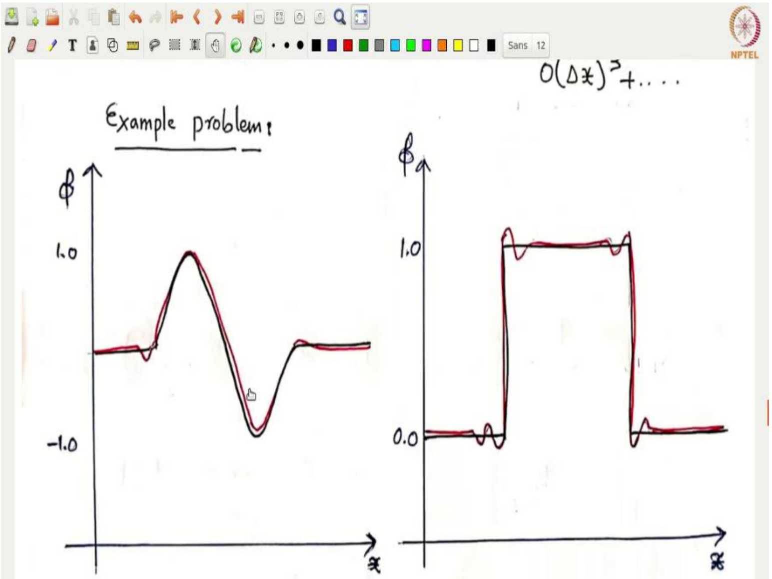
So, you get a third derivative with something multiplying it which is basically in terms of your CFL number and order delta x square. So, this is basically telling you your method is second-order accurate and this being a third derivative this is basically what kind of a term is this?

(Refer Slide Time: 40:03)



This is dispersive, right. So, this is not a dissipative it produces dispersion. So, depending on $\frac{u\Delta t}{\Delta x}$ value, this we can solve for the method right, alright.

(Refer Slide Time: 40:18)



So, that means, a let us look at an example problem example problem is again same sine wave and the step function. So, on the left hand side here we have the sine wave where again the black denotes the initial condition or the exact solution after one cycle of evolution and the red denotes the solution obtained using the Lax-Wendroff scheme ok.

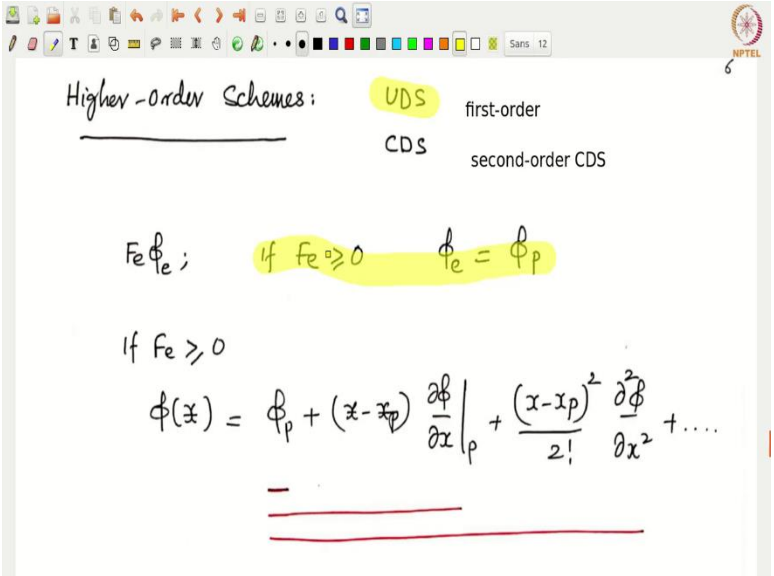
So, what we see is that because there is no dissipation like term, unlike the upwind difference scheme we see that the peaks are now very much preserved, ok. I have drawn this red to be somewhat closer to this, but it is exactly over lying on top of it, I just drew it very close such that you can perceive it ok. So, there is no diffusion. However, you see that where there are discontinuity you see again some kind of wiggles that show up, ok. This is because of the dispersive nature, ok.

Again if you look at the sine wave what you have is the black one is the initial condition as well as the exact solution after one cycle of revolution; that means, having periodic boundary conditions on the left and right. The red one as you can see preserves the amplitude, but again at the locations where there as there is a jump in the solution you get some kind of oscillations that come up this is again because of the dispersive nature of the of the method ok.

However, the Lax-Wendroff seems to be much better than upwind difference scheme which produced a differences in the amplitudes ok. So, as a result there are several other methods which are proposed which will try to fix these oscillations as well, and particularly, some of them are in the class where we look at we kind of construct a higher order schemes for upwind differencing.

Because upwind difference schemes would not produce this oscillations as a result the upwind difference schemes are somewhat preferred because you do not get nu u maxima and minima, rather what you get is your amplitude decrease in amplitude can be fixed by going to higher order ok.

(Refer Slide Time: 42:23)

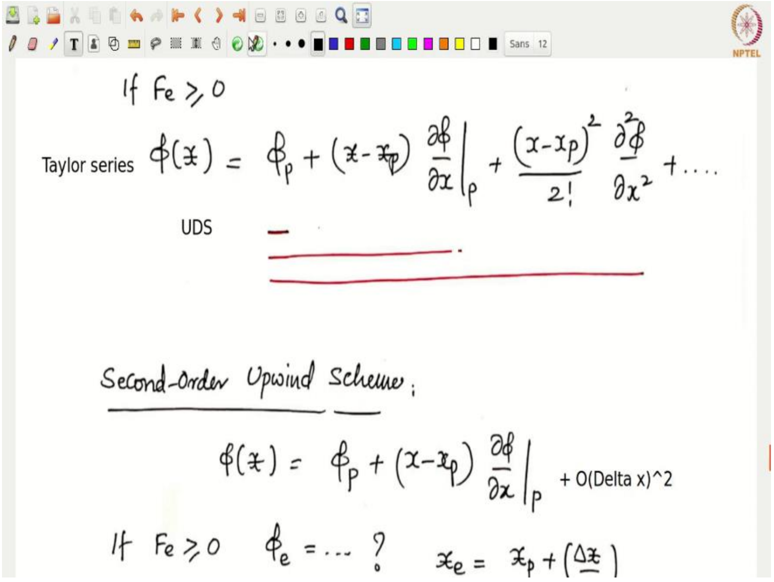


So, the next topic is basically construction and use of higher order schemes those are based on upwind difference method ok. So, up till now we have looked at the first-order upwind difference scheme and the second-order central difference schemes, but these per se as such are not very usable because upwind difference schemes are more diffusive in nature. And, the central difference schemes are although they are dispersive in nature, there cannot be used because of the problems with unconditional unstability and these things.

So, as a result the higher order schemes are proposed which are based on which are based on the higher order which are based on the upwind difference schemes but constructed to the high order alright. So, now, we will see how to construct the higher order schemes. So, the convection term we get is basically in the form of $F_e \phi_e$ and if F_e is positive; that means, the velocity on the face is positive, then ϕ_e equals ϕ_P, that is what we have.

So, we construct everything for this particular condition ok. So, I use this condition and construct all the schemes and later on you will you would realize how to change this how to modify this for the other conditions that exist ok.

(Refer Slide Time: 43:53)



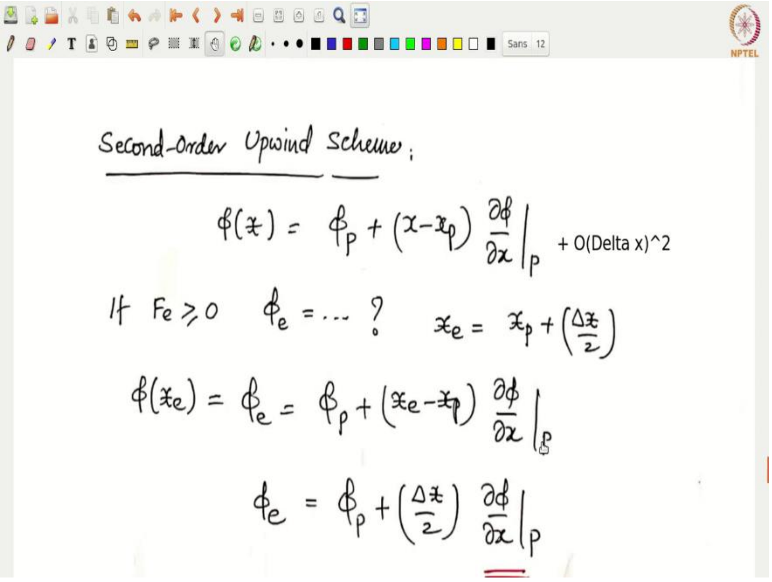
So, means if F_e is greater than or equal to 0, of course, this can be written always. I can write $\phi(x)$ that is phi in the neighborhood of x_P can be written in terms of the ϕ_P right the value of phi at the cell centroid P. This is basically using Taylor series expansion. So, we can expand using Taylor series this is $\phi(x) = \phi_P + (x - x_P)\frac{\partial \phi}{\partial x}\Big|_P$ plus you get the second derivative that is $\frac{(x-x_P)^2}{2!}\frac{\partial^2 \phi}{\partial x^2}\Big|_P$ and so on, ok.

Now, up till now for the first-order UDS, this is our first-order upwind difference scheme right. Essentially, if you only consider this first term here then this gave you your first-order UDS right the upwind difference scheme. Now, if you include not only the first term if you also include the gradient term; that means, the first two terms in the Taylor series, then you what you going to get is basically a second-order accurate scheme.

Of course, now we need to introduce an approximation not only for the face value, but also for the gradient ok, just like the way we have done it for the diffusion terms. Now, if you include first three terms then they other term this would be of the order Δx^3. So, you get a third-order accurate scheme and so on. So, depending on the number of terms you retain in the Taylor series expansion for a $\phi(x)$ we can get more and more accurate schemes for the upwind difference scheme of the quantity dependent quantity phi ok.

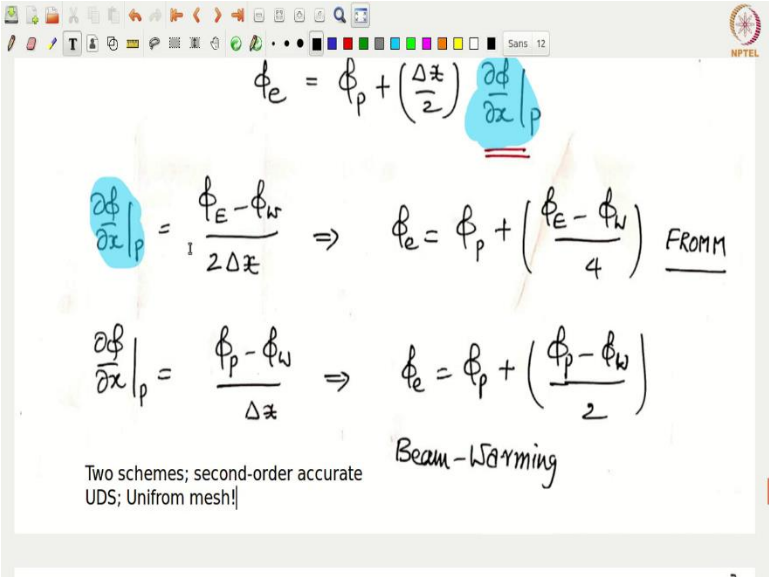
So, let us look at by including the first two terms let us construct something known as a second-order upwind second-order accurate upwind scheme by including the first two terms; that means, a $\phi(x)$equals ϕ_P plus $(x - x_P)\left.\frac{\partial\phi}{\partial x}\right|_P$ plus this is basically plus this is order Δx^2, right. This is basically order in the x square second-order accurate in space.

(Refer Slide Time: 45:58)



So, again I should that the mass flow rate the F_e is greater than or equal to 0 then we need to see how to calculate how do I approximate phi sub e, ok. We know that now the x location is basically x_e which is that means, x_e minus x_P would be equal to how much? $\Delta x/2$, right. The distance between the east face and the cell centroid p would be $\Delta x/2$ ok. That means, I want to calculate what is $\phi(x_e)$ which is nothing, but ϕ_e equals ϕ_P plus $(x_e - x_P)\left.\frac{\partial\phi}{\partial x}\right|_P$.

(Refer Slide Time: 46:35)



So, this is nothing, but ϕ_e is ϕ_P plus $(x_e - x_P)$ is $\Delta x/2$ times $\frac{\partial \phi}{\partial x}\Big|_P$. Now, how do we evaluate this particular derivative that again defines a particular second-order upwind difference scheme ok.

So, we can of course, write this in two ways: one is using a central difference formula that is $\frac{\partial \phi}{\partial x}\Big|_P$ can be written as in central difference terms this is $\frac{\phi_E - \phi_W}{2\Delta x}$ that is the distance between east and west and which will give you ϕ_e equals ϕ_P plus $\frac{\phi_E - \phi_W}{4}$, right because you have a 2 here and you have a 2 here and Δx gets cancelled, right.

We just substituting for $\frac{\partial \phi}{\partial x}\Big|_P$ from basically we are substituting $\frac{\partial \phi}{\partial x}\Big|_P$ from here into this term here, right. So, if we substitute for this, then we get for $\phi_e = \phi_P + \left(\frac{\phi_E - \phi_W}{4}\right)$. So, this particular scheme of evaluating the gradient the first derivative using central difference formula gives you something known as FROMM scheme ok. So, this is what is referred to it in the literature as.

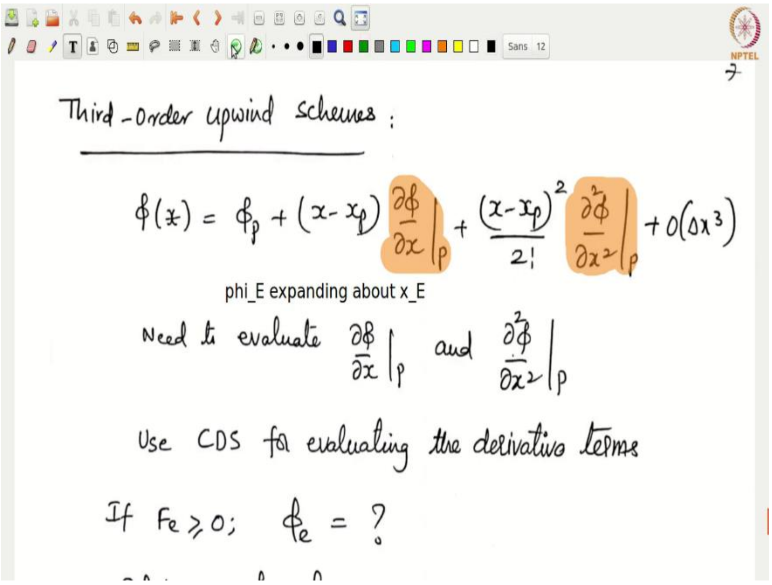
Now, similarly the first derivative can also be approximated using a backward difference formula right because you have ϕ_P it can be$\frac{\phi_P - \phi_W}{\Delta x}$. Now, why cannot we use a forward difference formula here? That is because we have assumed F_e to be greater than 0, right. So, either you can use central difference or a backward difference because flow is going from left to right.

So, you cannot use the forward difference here right $\phi_E - \phi_P$ would try to you can you can you can actually use it, but it will instabilities you can actually try that ok. So, that is why this is $\frac{\phi_P - \phi_W}{\Delta x}$ using a backward difference formula one sided formula this will give you phi little e equals phi p plus there is no 2 here there is a 2 here. So, this will be $\frac{\phi_P - \phi_W}{2}$, right; the Δx gets cancelled. So, this particular scheme is known as Beam-Warming scheme.

Now, this of course, this becomes $\phi_E - \phi_P$ if F_e is negative if F_e is less than 0, ok. So, this basically now gave you two schemes which are both second-order accurate and both are second-order accurate upwind difference schemes.

Now, of course, remember that all these things are derived for F_e greater than equal to 0. So, you need to modify them accordingly for other faces and for other values of F_e; if F_e is less than 0 and so on. Another thing is these are all derived for assuming a uniform mesh. So, you need to fix things if mesh is not uniform ok, alright.

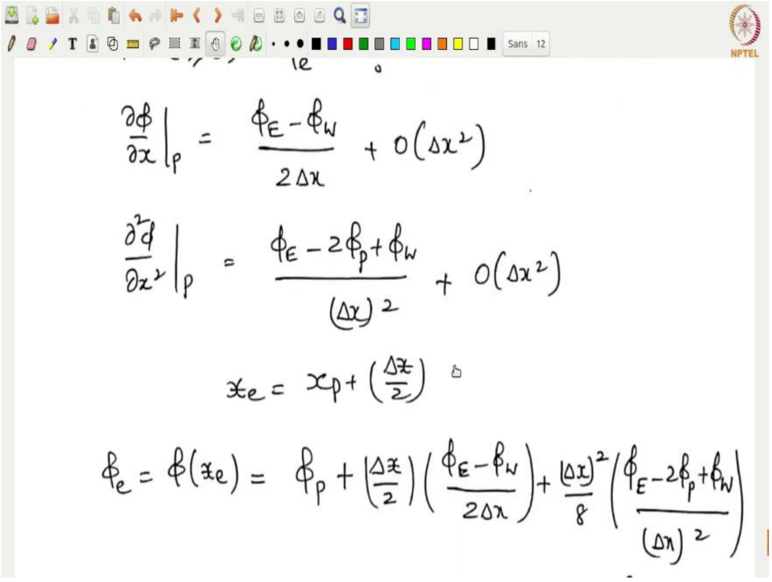
(Refer Slide Time: 49:39)



Then if you include let us say the first two terms then what you get is basically a third-order upwind scheme that would read you that would read as $(x) = \phi_P + (x - x_P)\frac{\partial \phi}{\partial x}\Big|_P + \frac{(x-x_P)^2}{2!}\frac{\partial^2 \phi}{\partial x^2}\Big|_P$ plus order Δx^3 ok. So, this is basically a third-order accurate scheme.

Of course, just like what we have done for the first derivative, now we have to also model the we have to also introduce something some kind of model for the second derivative as well, right. We have done this for the first derivative, now we have to do something for the second derivative as well. So, if you can do this then scheme is complete right.

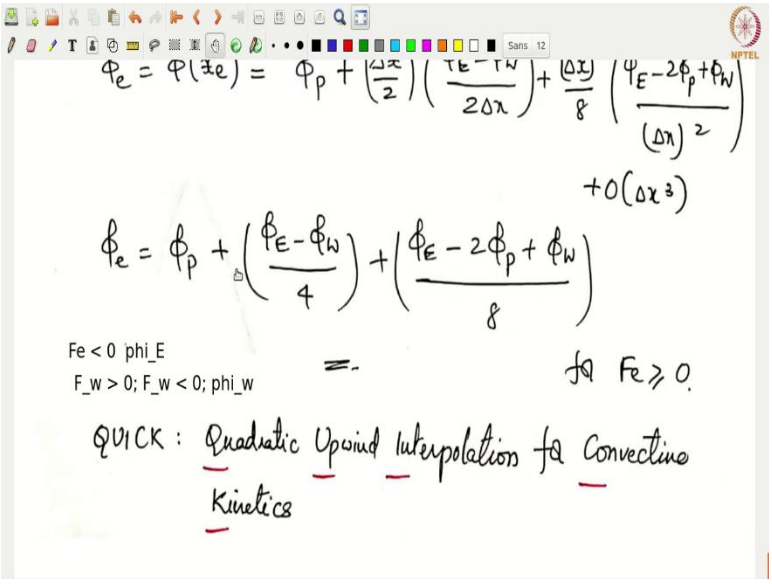
(Refer Slide Time: 50:24)



So, we need to evaluate $\frac{\partial\phi}{\partial x}\Big|_P$ and partial square phi by partial x square at p. Now, again depending on what schemes you use you get a particular final scheme ok. So, if you use a central difference scheme for evaluating derivatives, then you get a particular scheme, ok. Now, again I am assuming that e is positive and we want to calculate what is phi on the face east, ok.

So, if you use central difference scheme we can write the first-order derivative as $\frac{\phi_E-\phi_W}{2\Delta x}$ and the second derivative as $\frac{\phi_E-2\phi_P+\phi_W}{(\Delta x)^2}$ ok. So, if I plug in these two back into the original equation here and here and substitute for $x_e - x_P$ as $\frac{\Delta x}{2}$ and $x_e - x_P$ as $\left(\frac{\Delta x}{2}\right)$ whole square, then what you get is you get an equation for ϕ_e.

(Refer Slide Time: 51:16)



As ϕ_P plus $\left(\frac{\Delta x}{2}\right)$ times this quantity $\left(\frac{\phi_E-\phi_W}{2\Delta x}\right)$ plus $\left(\frac{\Delta x^2}{8}\right)$, right. We have a 2 coming from here and as it there is already 2 in the denominator times the second derivative plus order Δx^3. So, if we simplify this, what you get is $\phi_e = \phi_P + \left(\frac{\phi_E-\phi_W}{4}\right) + \left(\frac{\phi_E-2\phi_P+\phi_W}{8}\right)$ ok.

Remember, this entire thing only works for F_e greater than or equal to 0 so, if Fe is not greater than or equal to 0, if F_e is negative what would you do? Essentially, if F_e is negative where do you start off with? F_e is negative, you start off with do you write ϕ_e equals ϕ_P? No. We start off with if F_e is negative we start off with ϕ_E, right.

So, this would we start off with ϕ_E and try expanding about expanding about the east x_E, right that is where you start off and then you have to substitute the central difference formula for these in terms of the centered around E cell ok. That is what you have to do and so on. Similarly, you want to do all these things for the other faces F_w and F_w when it is positive the phi w when F_w is positive and negative ok. So, we have only done for one case ok.

So, this particular scheme that we got by using the central difference schemes for the first and second derivatives in the Taylor series expansion for ϕ_e is known as in the literature known as QUICK scheme which is third-order accurate which stands for Quadratic Upwind Interpolation for Convective Kinetics ok. This is basically quadratic is basically second-order accurate second-order interpolation that is what we have used. That is why it is called quadratic upwind interpolation.

Now, this is upwind is because all this we will work for F_e greater than or equal to 0, we have developed in terms of phi p that is the upwind part here, but not the these parts these are always central differences ok. So, accordingly that means, you get another formula for ϕ_e if F_e is negative. Similarly, you get one formula when F_w is positive for ϕ_w and one formula for ϕ_w when F_w is negative ok, fine.

So, that may not be in terms of let us say for example, if let us say for example, if your F_e is negative, then what do you expect? What would be your ϕ_P? ϕ_P would be shifted to ϕ_E right, this would be ϕ_E and what would be the central differencing for F_e less than 0? If F_e is less than 0 your central cell is basically ϕ_E, then the central difference formula would read this as east east would be your east and then this would be your west will be p cells.

So, this would be east east minus p and similarly, this would be east east and this will be east and this will be p right and so on. So, accordingly you have to develop for the west face you want to type for essentially when F_w is greater than 0 and F_w is less than 0 you need to come up with formulae for ϕ_W or ϕ_w, alright.

So, that, kind of finishes the two higher order schemes or the three higher order schemes that is basically FROMM scheme, Beam-Warming scheme and then the QUICK scheme – the Quadratic Upwind Interpolation for Convective Kinetics, alright. Now, we going to stop here.

So, in the next lecture next lecture we will pickup from these higher order schemes how do we implement them actually in the code and then we move on to the discussions on the unstructured meshes right because till now we have not looked at unstructured meshes for convection ok.

So, that will be kind of the agenda for the next lecture, ok. I am going to stop here. If you have any questions write back to me, we will discuss them alright.

Thank you. Talk to you in the next class.