Computational Fluid Dynamics Using Finite Volume Method
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Lecture – 31
Finite Volume Method for Convection and Diffusion: Discretization of steady and unsteady convection equation

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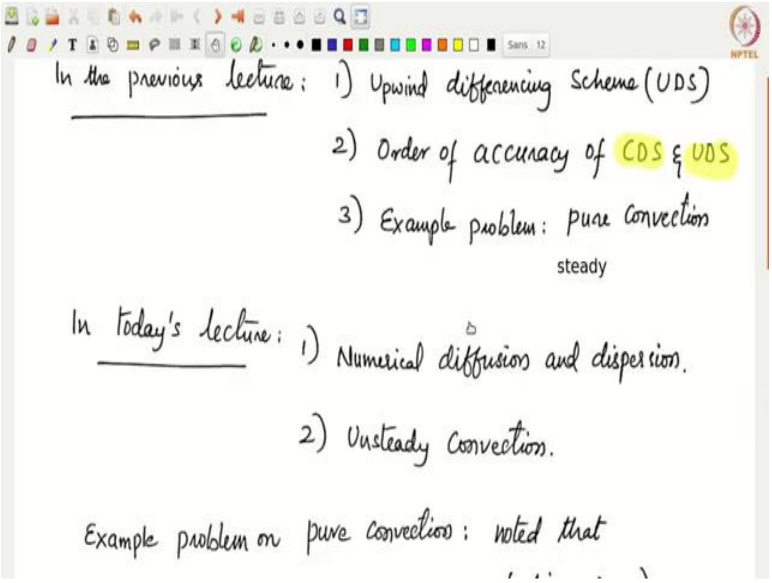

Hello everyone, welcome to another lecture as part of our ME 6151 Computational heat and Fluid flow course. So, in the last lecture what we saw was we discussed about upwind different scheme, which was said to be an alternate for the central difference scheme, right. And, we also looked at the order of accuracy of the two schemes that we have studied, that is the CDS and the UDS and we have also looked at an example problem. So, this was a pure convection problem, but this was a steady pure convection problem, right.

And, then we noted that if we have a pure convection, when we try to solve using central difference method what we got was some kind of wiggles were obtained as compared to the exact or correct solution and we tried to solve with upwind difference scheme, then what we noted was we got dissipation or diffusion in the solution, right. Although, there was no diffusion in the original problem, we saw that we got some diffusion, because of the way the upwind difference scheme kind of works, ok.

So, we noted these two aspects. So, in today's lecture as to continue from where we left off what we are going to look at is, we are going to look at numerical diffusion and dispersion; that is basically this artificial diffusion and dispersion that are obtained, because of UDS and CDS, ok. So, we are going to see this from a numerical perspective, ok.

So, this we are going to analyze from numerical perspective. So, can we see this kind of behavior or can we explain this behavior from the equations that we are solving, ok. So, that is what we are going to analyze today, and depending on the time, we are also going to look at how unsteady convection equation looks and what can be done in order to solve and study convection using again these two schemes that would come in the later lectures, ok.

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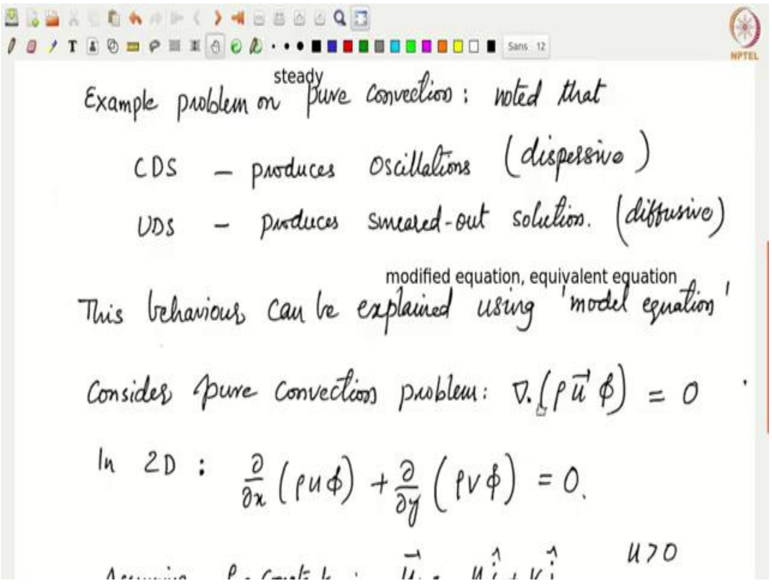

So, then let us go into the lecture essentially; so, the example problem are pure convection, right. So, this was what we studied in the last class example, problem on steady pure convection. We noted that we have the central difference scheme produce or produced oscillations, which we called it as a dispersive scheme and the upwind difference scheme produced a diffusive solution right it kind of smeared out the sharp peaks that we have in the solution, ok. So, that is what we kind of noted.

Now, can we explain this behavior using model equation? So, today we are going to see how to construct this model equation or also known as a modified equation, ok. So, we will explain this behavior of dispersion and diffusion using a modified equation or also known as equivalent equation, ok. So, these are equations which are written from the original governing equation, ok.

So, we are going to see how to construct these modified equations or equivalent equations for a given equation and see if we can explain the behavior, ok. So, that is the agenda for today. Then, let us start off with pure convection problem; again this is a steady pure convection problem, ok. So, we have $\nabla \cdot (\rho\vec{u}\phi) = 0$. Now, if we consider 2-dimensions, then nabla (∇) can be expanded in 2-dimensions, this would be $\frac{\partial}{\partial x}(\rho u\phi) + \frac{\partial}{\partial y}(\rho v\phi) = 0$, ok.

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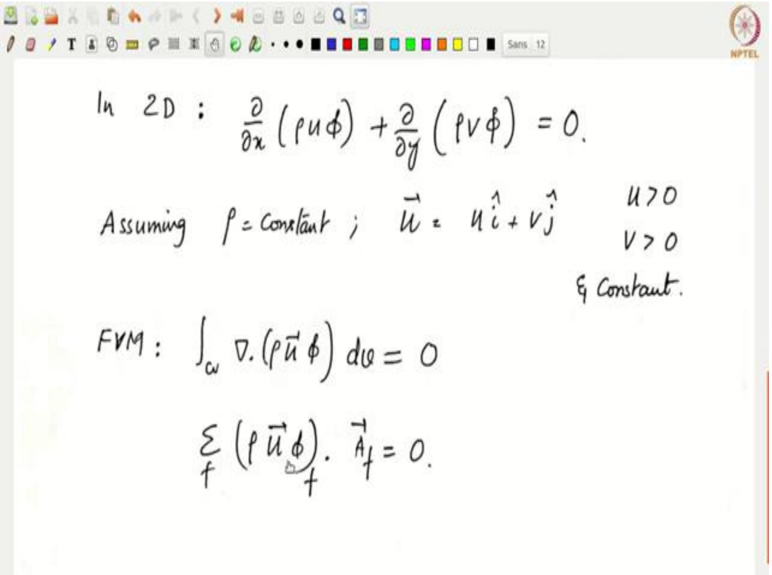

Then, for the sake of simplicity I am also going to consider density equals to constant. So, ρ equals constant and I will also assume that $\vec{u}$ which is $u\hat{\imath} + v\hat{\jmath}$is also a constant vector and the 2 velocities in the x and y directions are also positive, ok. So, we are going to assume that u is a constant and v is a constant and these two are positive constant; that means, u and v are constant and they are also greater than 0, ok.

So, that is what we are going to assume, because then we can simplify this equation very easily, ok. So, with these assumptions, the first step is to infinite volume method is to integrate the governing equation on a control volume's and we can write this as control volume $\nabla \cdot (\rho\vec{u}\phi)\, dV$ equals 0, right.

And, if we apply Gauss divergence method; Gauss divergence theorem and then also convert the resulting surface integral into a summation, then we can convert this equation into sigma f $(\rho\vec{u}\phi)_f \cdot \vec{A}_f$ equal to 0, ok. So, that is the resulting equation we can write alright. Then, we have to choose a particular method here now, right a particular scheme in order to essentially further discretize this problem, ok.

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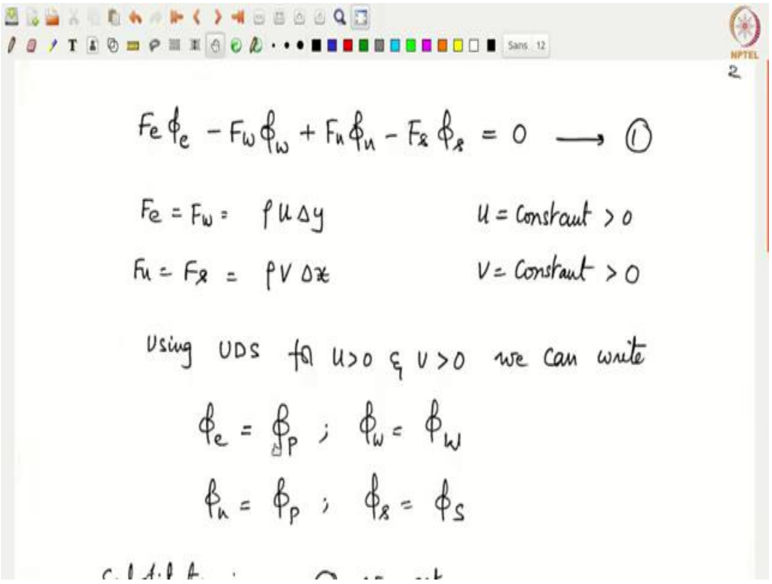

So, then we will use; so, if I go back then, essentially what we have is $(\rho\vec{u}\phi)_f \cdot \vec{A}_f$, then this can be expanded for each of the coefficients right each of the faces that is east, west, north and south, ok. Then, this can be written as $F_e\phi_e - F_w\phi_w + F_n\phi_n - F_s\phi_s = 0$, right. This is already what we saw in the last lectures, because F_e the way F_e and F_w are defined is basically it is ρuΔy, but now, we have assumed that u is constant everywhere and v is constant everywhere.

So, ρu_e equals ρu_w right; as a result both the flow rates are the same ρuΔy right, both are the same and this minus comes in because of the area vector being $-\Delta y\hat{\imath}$ for the west face, ok. So, that is the reason here. Now, what about the F_n and F_s? F_n and F_n are again are both equal to ρvΔx, because v is a constant and it is positive, ok.

So, as a result we can simplify this equation further and also; that means, we can kind of write this as F_e or ρuΔy times ϕ_e minus ϕ_w plus ρvΔx times ϕ_n minus ϕ_s right that can be written. And, if I choose let us say a upwind difference scheme if I choose upwind difference scheme then, what we can do is, we can again write the upwind differencing for the face value of phi; that means, depending on the flow rate we would use upwind for the dependent variable, ok.

And, because we have assumed that u is greater than 0 and v is greater than 0, we can now write phi on the face east face (ϕ_e) equals ϕ_P, right, because u is greater than 0. Similarly, ϕ_w on the west face, because u is greater than 0 will be equal to ϕ_W. Applying the same logic phi on the north face (ϕ_n) can be written as ϕ_P,, because v is greater than 0 and phi on the south face (ϕ_s) is equal to ϕ_S, because v is greater than 0, ok. So, now, this can be written.

Then, we can go and substitute for these ϕ_e, ϕ_w, ϕ_n and ϕ_s into the equation number 1 here, and collect all the terms. Then we can write this as $\rho u \Delta y \phi_P$ right instead of ϕ_e we wrote ϕ_P minus $\rho u \Delta y$ instead of ϕ_w we wrote ϕ_W plus $\rho v \Delta x \phi_n$ instead of ϕ_n we wrote ϕ_P minus $\rho v \Delta x$ ϕ_s and replacing that with ϕ_S equal to 0, ok.

So, this is the equation 1 which is modified with these upwind difference scheme and also substituting for the corresponding flow rates. Alright now, let us divide this entire equation using Δx times Δy, ok.

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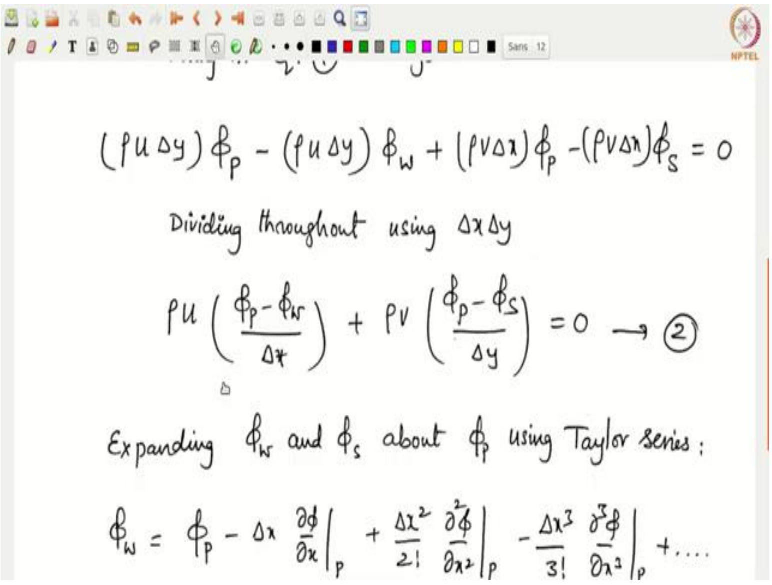

And then, we can write this as $\rho u \Delta y$ divided by $\Delta x \Delta y$. So, 1 Δy gets cancelled and you would get Δx in the denominator and then, what you get is $\phi_P - \phi_W$ by Δx plus from these two terms you would get ρu times $\phi_P - \phi_S$ upon Δy equal to 0 right; that is what we get.

Now, look at this equation. So, let us call this equation as 2. Now, what we do is well up till now, what we have done is you have taken the original governing equation and applied the corresponding discretization scheme. In this particular case it was upwind different scheme and we have just written the finite volume linear algebraic equation right, this is the discrete equation for pure convection using upwind difference scheme that is all we have done.

Now, in order to construct a modified equation, what we have to do is, we have to rewrite this equation by expanding some of these phi values using Taylor series expansion, ok. So, in this what we see is we have 3 quantities ϕ_P, ϕ_W and ϕ_S, ok. So, I am going to expand ϕ_W and ϕ_S about ϕ_P using Taylor series expansion, ok.

So, let me expand ϕ_W and ϕ_S about ϕ_P, then I can write a ϕ_W, as W is to the west of ϕ_P or location x_P by minus Δx right; it is about Δx to the west of P, right. As a result, ϕ_W can be written as $\phi_P - \Delta x \frac{\partial \phi}{\partial x}\Big|_P - \frac{\Delta x^2}{2!}\frac{\partial^2 \phi}{\partial x^2}\Big|_P - \frac{\Delta x^3}{3!}\frac{\partial^3 \phi}{\partial x^3}\Big|_P$ and so on, right. That is expansion for ϕ_W.

Similarly, ϕ_S can be written ϕ_S can be written in terms of ϕ_P being in the Δy direction to the south of location P right. We can write ϕ_S as $\phi_P - \Delta y \frac{\partial \phi}{\partial y}\Big|_P - \frac{\Delta y^2}{2!}\frac{\partial^2 \phi}{\partial y^2}\Big|_P - \frac{\Delta y^3}{3!}\frac{\partial^3 \phi}{\partial y^3}\Big|_P$ and so on, right. So, that is what we have for Taylor series expansion of ϕ_W and ϕ_S quantities, alright.

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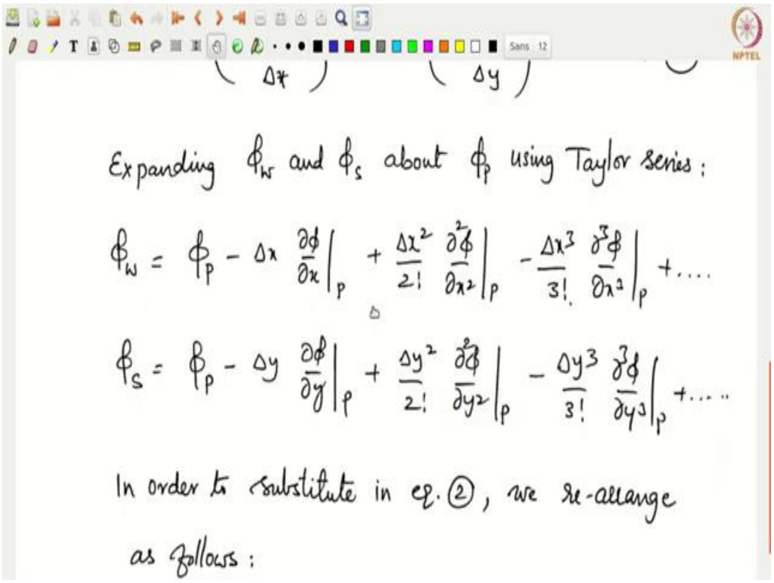

Now, in order to, we essentially want to replace this quantity as well as this quantity using the Taylor series expansion we have. So, if you want to write $\phi_P - \phi_W$. So, we divided by Δx. So, we can write this from here. Essentially, $\phi_P - \phi_W$; that would give you this entire term sent to the left hand side, but then we want to divide by Δx, right. So, what would remain for $\phi_P - \phi_W$ divided by delta x that would be essentially, if I go back and make it continuous, ok.

So, essentially $\phi_P - \phi_W$ by Δx would be $\left.\frac{\partial\phi}{\partial x}\right|_P$ right Δx gets divided. So, this will minus $\frac{\Delta x}{2!}\left.\frac{\partial^2\phi}{\partial x^2}\right|_P$ then, this will be plus $\frac{\Delta x^2}{3!}\left.\frac{\partial^3\phi}{\partial x^3}\right|_P$, right.

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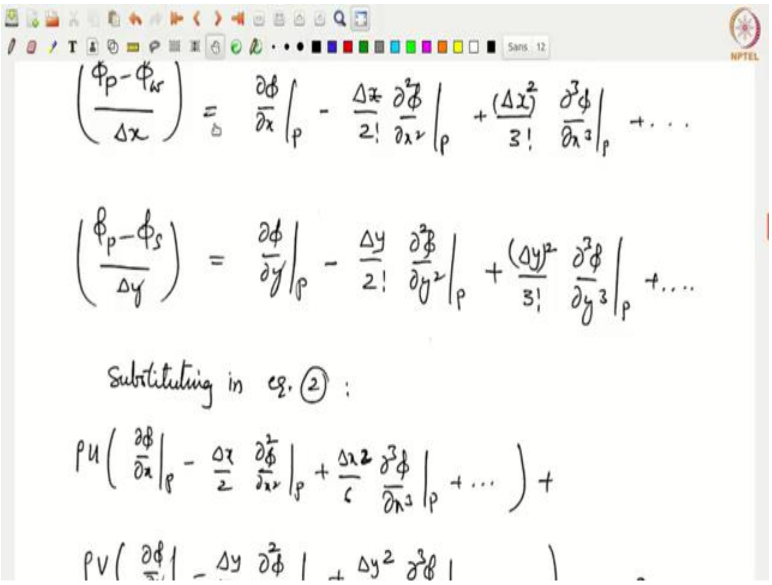

So, that is what we have; that means, $(\phi_P - \phi_W)/\Delta x$ equals $\left.\frac{\partial\phi}{\partial x}\right|_P$ minus $\frac{\Delta x}{2!}\left.\frac{\partial^2\phi}{\partial x^2}\right|_P$ plus $\frac{\Delta x^2}{3!}\left.\frac{\partial^3\phi}{\partial x^3}\right|_P$, right. Essentially, we just rewritten what is this quantity from the 1st equation that we have here, alright. Similarly, can we write what is $(\phi_P - \phi_S)/\Delta y$ from this equation? Yes, we can write that; that means, send ϕ_S to this side and send all these guys to the left hand side and divide by Δy right what may remain is partial $\left.\frac{\partial\phi}{\partial y}\right|_P$.

So, that would be $\left.\frac{\partial\phi}{\partial y}\right|_P$ similar to here we will get $\frac{\Delta y}{2!}\left.\frac{\partial^2\phi}{\partial y^2}\right|_P$ plus $\frac{\Delta y^2}{3!}\left.\frac{\partial^3\phi}{\partial y^3}\right|_P$ and so on, right. So, essentially, we got expressions for these two quantities right from Taylor series expansion, ok. So, we have these two quantities. Now, what we do is, we just substitute for these two quantities in that equation, right.

So, essentially the equation is, what was the equation? Equation was ρu times this quantity plus ρv times this quantity so; that means, ρu times $\left.\frac{\partial \phi}{\partial x}\right|_P$ minus this $\frac{\Delta x}{2!}\left.\frac{\partial^2 \phi}{\partial x^2}\right|_P$ plus $\frac{\Delta x^2}{3!}\left.\frac{\partial^3 \phi}{\partial x^3}\right|_P$ plus we have ρu times $\left.\frac{\partial \phi}{\partial y}\right|_P$ minus $\frac{\Delta y}{2!}\left.\frac{\partial^2 \phi}{\partial y^2}\right|_P$ right, this should be plus $\frac{\Delta y^2}{3!}\left.\frac{\partial^3 \phi}{\partial y^3}\right|_P$ and so on equal to 0, right. So, essentially, we got now one equation, which is again a continuous equation, right.

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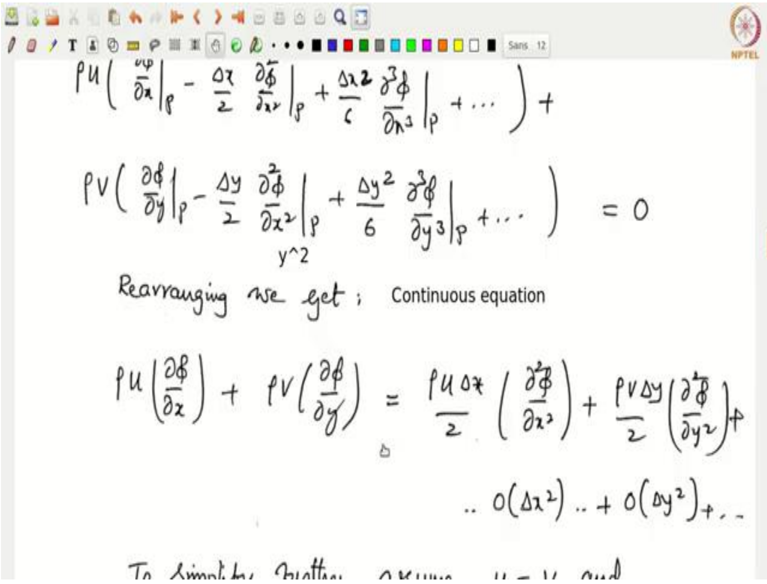

So, this is a continuous equation right. This is essentially it is taking care of the terms which we have; which we would probably neglect, because of the discretization, right. Those terms are now substituted back. So, we got some continuous equation, from the discrete equation that we have constructed, alright. Then, if you look at what is if I rearrange these terms, then I can write ρu times $\frac{\partial \phi}{\partial x}$ plus ρv times $\frac{\partial \phi}{\partial y}$; we have these two terms.

And, I would like to send the remaining quantities to the right hand side that would be ρuΔx by 2 times $\frac{\partial^2 \phi}{\partial x^2}$ plus ρvΔy by 2 times $\frac{\partial^2 \phi}{\partial y^2}$, right plus order Δx^2 plus order Δy^2 and so on, right. So, we get so many of these terms, right.

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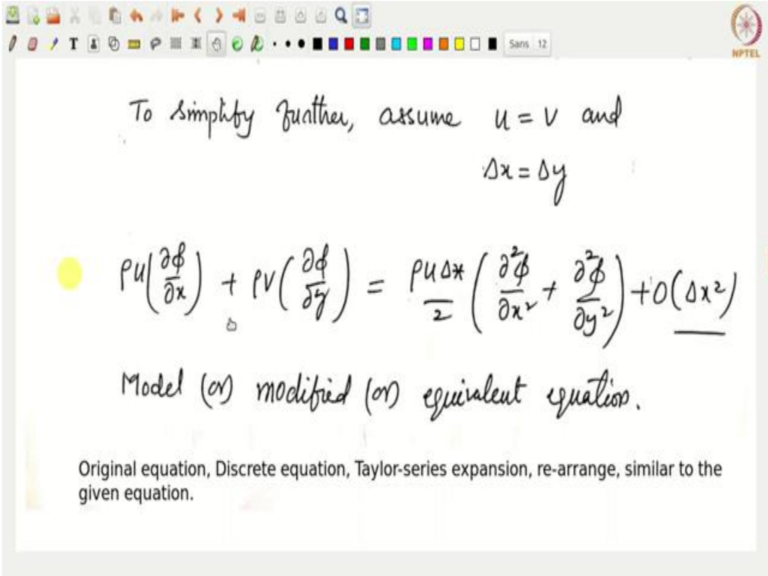

So, we got now an equation, which is actually the modified equation, but we would like to do some more simplification before we call it a modified equation, ok. So, in order to make things little more simple what we do is, let us assume that u and v are not only constant, but they are also equal, ok. They are positive and they are also equal, ok. And, then let us also assume that we have a uniform mesh in both the directions that is Δx equal's Δy, ok.

These are basically couple of simplifications to make the analysis easy, alright. Then, we can write this equation as $\rho u \frac{\partial \phi}{\partial x}$ plus $\rho v \frac{\partial \phi}{\partial y}$ equals, because these two $\frac{\rho u \Delta x}{2}$ and $\frac{\rho v \Delta y}{2}$ are now one and the same, because u equals y and Δx equal Δy. I can write this as combine these two quantities.

We can write this as $\frac{\rho u \Delta x}{2}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right)$ plus order Δx^2, alright. So, this equation is what we call as the modified equation, ok. So, we call this as the modified equation or model equation or the equivalent equation, right. Now, how did we construct this?

Essentially, we started off with the original equation right, start with the original equation, then write the corresponding discrete equation right for the finite volume method, then in the discrete equation use Taylor-series expansion and then rearrange, right. So, that it looks similar to the given equation right, ok. So, now, why do we say this is similar? It is similar, because we were started off to solve a pure convection equation in 2-dimensions, right.

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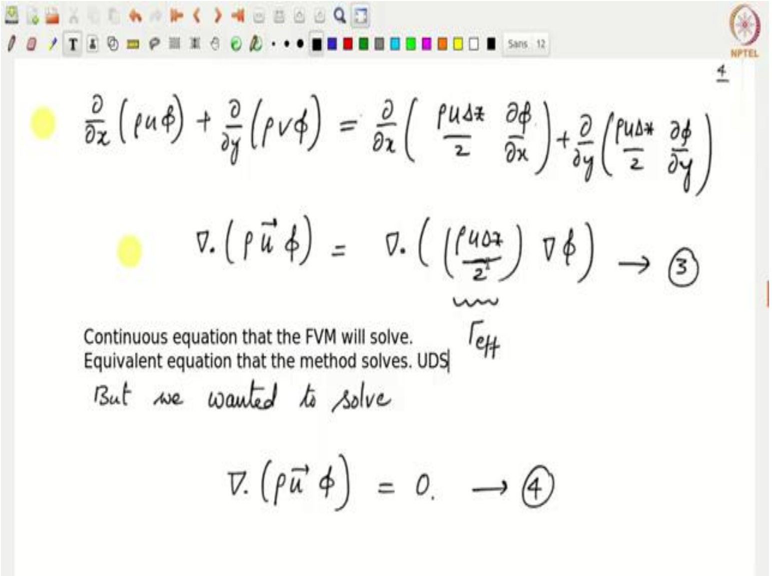

Now, that means, I can, because ρ, u, v are all constants I can take these guys inside the derivative. I can write this as $\frac{\partial}{\partial x}(\rho u\phi)$ plus $\frac{\partial}{\partial y}(\rho v\phi)$ on the right hand side. Again, this is all these are all constants. I can now write this as instead of $\frac{\rho u\Delta x}{2}\left(\frac{\partial^2\phi}{\partial x^2}\right)$, we can write this as $\frac{\partial}{\partial x}\left(\frac{\rho u\Delta x}{2}\frac{\partial\phi}{\partial x}\right)$ plus $\frac{\partial}{\partial y}\left(\frac{\rho u\Delta x}{2}\frac{\partial\phi}{\partial y}\right)$ plus of course, we have order Δx^2 terms here, ok.

Now, we can of course, rearrange this as $\nabla\cdot(\rho\vec{u}\phi) = \nabla\cdot\left(\left(\frac{\rho u\Delta x}{2}\right)\nabla\phi\right)$, right. We essentially converted this equation into a vector equation that looks like this, right. Now, ok; so, this is basically something similar to what we have learned this is basically the convection term, this is the diffusion term, but this is not something that we started off to solve, right.

We started off or we wanted to solve pure convection equation, where del dot rho u bar phi equals 0 right; whereas, what we ended up with is this equation 3, which is not the same equation rather this is the equation right or the continuous equation or the continuous equation that the finite volume method or the method that we have employed will solve, right.

So, this is the equivalent equation that the method solves, right which is not the same as what we wanted to solve, ok. Now, why is this coming? This is happening, because of the discretization scheme that we have chosen which was the upwind difference scheme, right.

So, what does this look like? So, essentially there is a diffusion part in there. So, although we wanted to solve pure convection, we have some diffusion coming into play.

That is why, when we looked at in the last lecture, there is some diffusion of the spearing out of the sharp discontinuities was occurring, right. Now, what will be the effective diffusion coefficient? The affected diffusion coefficient is equal to $\frac{\rho u \Delta x}{2}$. This is more like gamma right this is more like gamma; so, that means, there is some diffusion which we call it as numerical diffusion, ok; which is coming into play, because of the upwind difference scheme that we have chosen.

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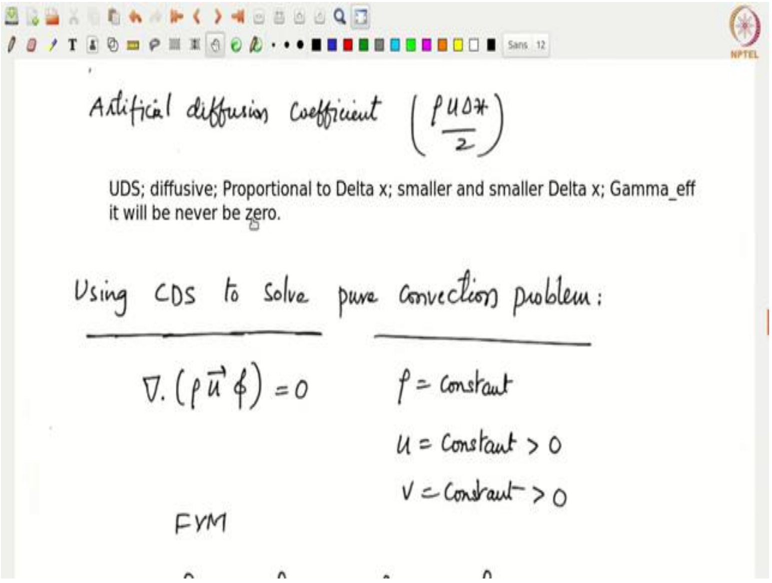

And, the artificial diffusion coefficient is not a constant, rather it is equal to $\frac{\rho u \Delta x}{2}$ right; that means, if we try to solve pure convection using UDS it will be diffusive. However, the diffusion coefficient is proportional right to Δx; that means, if you take smaller and smaller Δx, then the gamma effective can be taken to go to 0, right, but it will never be zero it can be made to go to as small as possible, right.

So, as a result this can be made small, but it will never be zero, right; it will never be zero, because it is always proportional to Δx fine so, but that kind of gives us some insight on why the upwind difference scheme is diffusive, right. Like what we have observed yesterday.

Now, in general not only upwind difference scheme, but any method that we use if it gives in the modified equation if it gives this kind of even order derivatives; if it gives even order derivatives, then that method can be said to be; that means, if you have in the equivalent equation the even order derivatives such as, these guys $\frac{\partial^2 \phi}{\partial x^2}$ 2nd order derivative, then we can call this equation or that particular equation as always diffusive, right.

So, in the modified equation if the if the derivatives are of the 2nd order, 4th order, 6th order and so on, then that scheme can be called as a diffusive scheme right, it produces diffusion. Now, what will be the order of accuracy of this method? Order of accuracy is of course, 1st order, because this is the truncation error right; that means, this is order delta x.

So, the method is 1st order accurate, but it is diffusive, alright. So, then let us look at now the other scheme that we have tried solving the pure convection equation that is the Central Difference Scheme, right. We would use the CDS scheme to solve for pure convection problem, ok.

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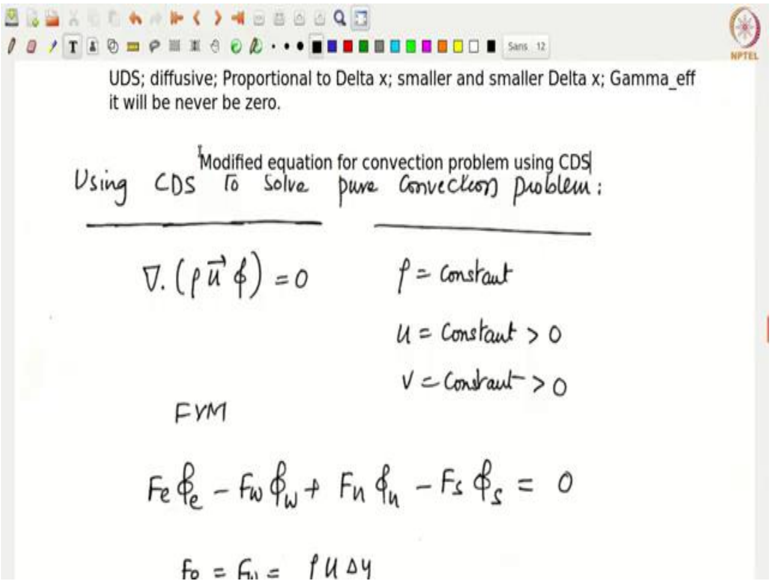

And essentially, we want to also construct what is the; what is the modified equation for a pure convection problem, for convection problem using the central difference scheme right. So, that is what we want to do alright. Then, let us again get started with the pure convection that is $\nabla \cdot (\rho \vec{u} \phi) = 0$, right.

The right hand side $\nabla \cdot (\Gamma \nabla \phi) + S_\phi$ is equal to 0 anyway and then, you are we again assume that density is constant so, ρ is constant. When the velocities u and v the two components of velocity are also constants and they are greater than 0 that is what we assume. And, we apply finite volume method and then transform this equation into a discrete equation, right.

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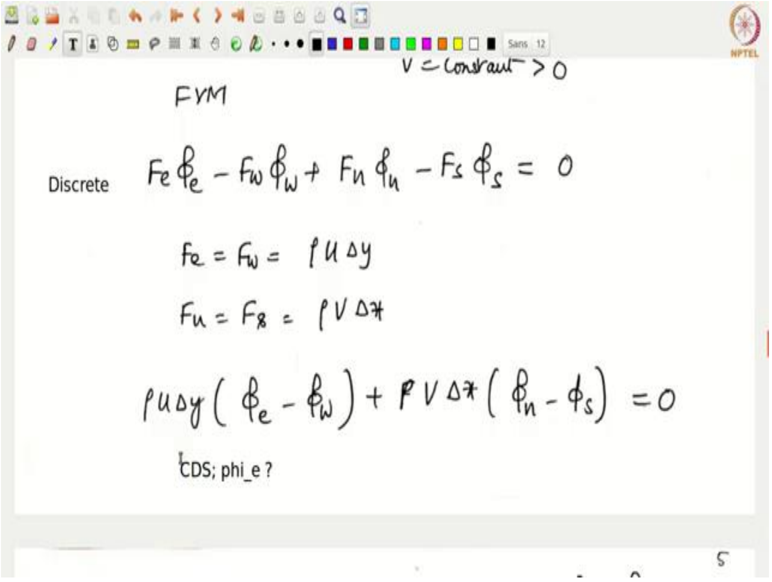

So, the discrete equation reads like this. $F_e\phi_e - F_w\phi_w + F_n\phi_n - F_s\phi_s = 0$, ok. Now, in this one again F_e and F_w are both equal to ρuΔy and F_n and F_s both are equal to ρvΔx right, because we assume that u and v are constants, their value on the east face is the same as west face and everywhere else. Same for the phi the value is same on the north face and the south face and everywhere else, ok.

Then, what we get is we get ρuΔy times $\phi_e - \phi_w$ plus ρvΔx times $\phi_n - \phi_s$ equals 0, ok. So, this is our equation by substituting for the flow rates in terms of the ρuΔx will be ρuΔy and ρvΔx, ok.

Now, we have to introduce the discretization for the central difference scheme right that is basically, how do we say ϕ_e is what how do we describe the value of phi on the east face, right. This is using central difference scheme we write it as the linear average or arithmetic average of the two cell values right.

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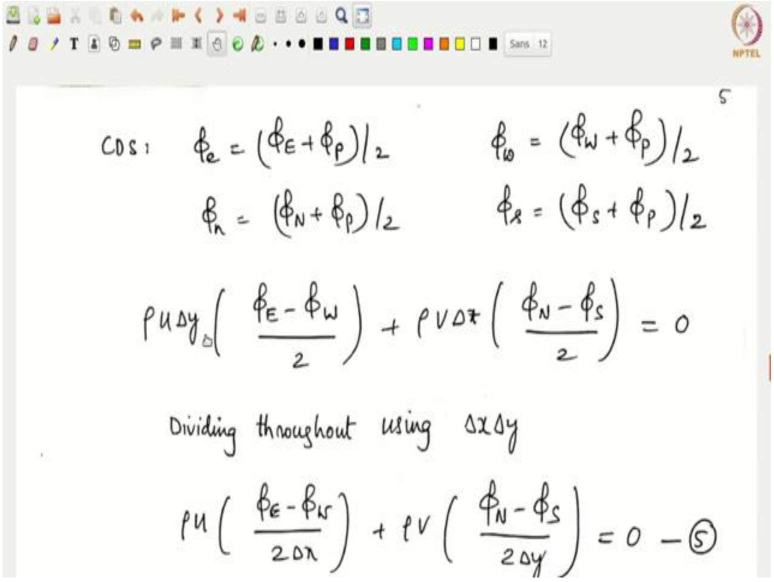

So, in the CDS what we said is, we say ϕ_e equals $(\phi_E + \phi_P)/2$. And similarly, ϕ_w equals $(\phi_W + \phi_P)/2$ and ϕ_n equals $(\phi_N + \phi_P)/2$ and ϕ_s equals $(\phi_S + \phi_P)/2$, right. So, we have the central difference scheme definitions then we will plug it into this equation.

What we see is that, ϕ_e as E and P and ϕ_e which is coming with a minus has W and P as well; that means, the P gets cancelled and what we get is instead of $\phi_e - \phi_w$ we get $(\phi_E - \phi_W)/2$, right. So, this equation is nothing, but ρuΔy times $(\phi_E - \phi_W)/2$ plus ρvΔx times $(\phi_N - \phi_S)/2$ equals 0, ok.

So, that is what we have. Then, let us again divide throughout by ΔxΔy, then this Δy gets cancelled and then we get a Δx in the denominator. And, we also get a Δy in the denominator, because this Δx gets cancelled, alright. Then, what we can do is we can write the final equation this as $\rho u \left(\frac{\phi_E - \phi_W}{2\Delta x}\right)$ plus $\rho v \left(\frac{\phi_N - \phi_S}{2\Delta y}\right)$ equals 0, ok.

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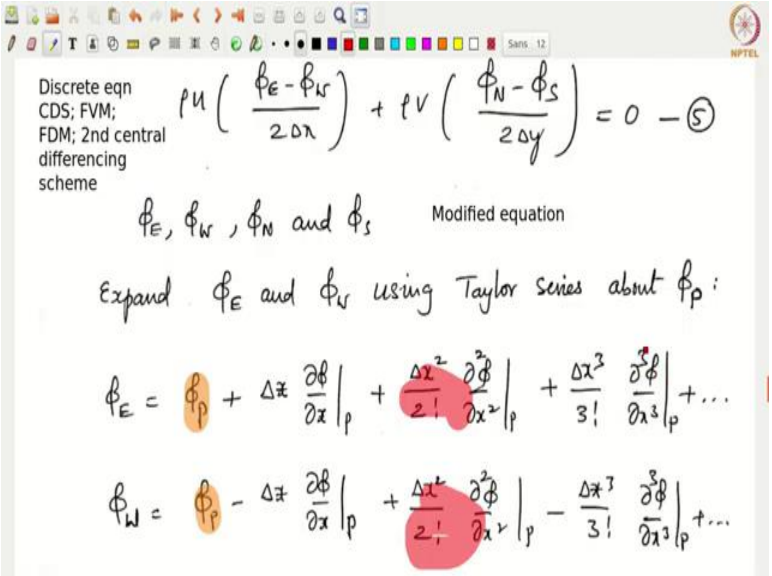

So, this is the; this is the discrete equation right using a central difference scheme, ok. Now, what you see is that of course, this was derived using finite volume method, but what you see is that this would be the same as the equation that you would if you had used a finite difference method right and substitute it for directly for partial phi partial x in terms of a 2nd order central difference scheme as well, right that would be the same alright.

So, that is what you would note from here. This equation e would be the same even if you have started off with the finite difference method, because right now, we have assumed lot of things to be constant and stuff like that ok; that is why we could write and eventually it looks similar to the central differencing scheme coming from the finite difference method right, ok.

So, equation 5 is our final discrete equation. Now, what do we do? In order to construct a modified or modified equation what we have to do is; what we have to do is we have to plug in for ϕ_E, ϕ_W, ϕ_N and ϕ_S in terms of ϕ_P, right. That is what we have to do, in order to construct the modified equation. So, therefore, let me expand ϕ_E and ϕ_W using Taylor-series about ϕ_P.

So, what is ϕ_E? ϕ_E is nothing, but ϕ_E is at delta x from ϕ_P. So, $\phi_E = \phi_P + \Delta x \frac{\partial \phi}{\partial x}\Big|_P + \frac{\Delta x^2}{2!}\frac{\partial^2 \phi}{\partial x^2}\Big|_P + \frac{\Delta x^3}{3!}\frac{\partial^3 \phi}{\partial x^3}\Big|_P$ and so on.

Similarly, $\phi_W = \phi_P - \Delta x \frac{\partial \phi}{\partial x}\Big|_P - \frac{\Delta x^2}{2!}\frac{\partial^2 \phi}{\partial x^2}\Big|_P - \frac{\Delta x^3}{3!}\frac{\partial^3 \phi}{\partial x^3}\Big|_P$ and so on, ok. So, that is what we have written for this quantity.

So, can we write what is $\frac{\phi_E - \phi_W}{2\Delta x}$ from here? Yes, we can write that will be this minus this divided by 2Δx that would give you ϕ_P gets cancelled, right. So, essentially this guy gets cancelled right and then, these two would ϕ_E minus ϕ_E this would add up to 2Δx, but that is what we are dividing with.

So, what remains is partial phi partial x remains. Then, what about this guy? This guy also this is a plus. So, this this guy also gets cancelled right, this guy gets cancelled these two gets cancelled. Then, this guy will become Δx^3 by 6, but you are adding them up. So, this will become Δx^2 right divided by we are also dividing by 2Δx. So, this would become 12 right 6 and 6. So, this would be divided by 2Δx. So, you would get something of the order of Δx^2 times $\frac{\partial^3 \phi}{\partial x^3}\Big|_P$, right. So, that is what we would get.

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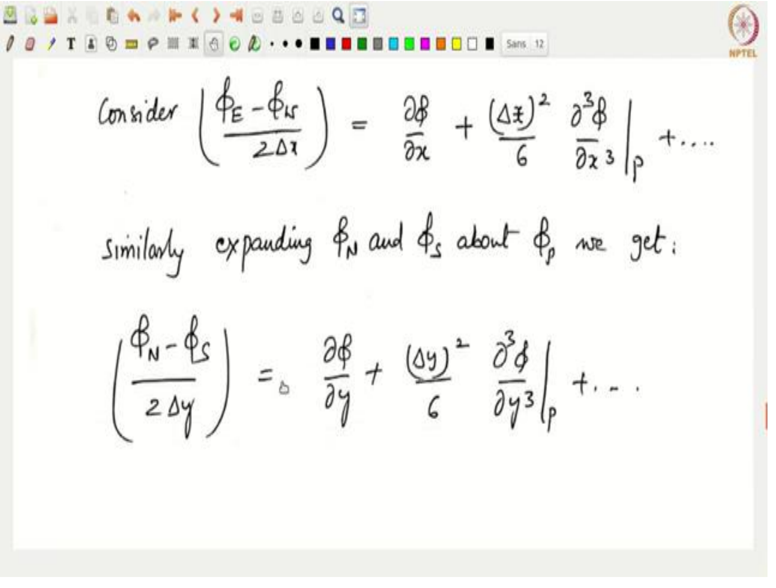

That means, if I write this equation what you get is $\frac{\partial \phi}{\partial x}$ which survives from here, plus $\Delta x^2/6\, \frac{\partial^3 \phi}{\partial x^3}$; that is what we would get for this 1st quantity, ok. Now, what we have to do is, We have to also expand the ϕ_N and ϕ_S in terms of ϕ_P and then find out a value for $\frac{\phi_N - \phi_S}{2\Delta y}$ that is what we would do which I have not done, but which you can on a very similar lines by replacing Δx with Δy and $\frac{\partial \phi}{\partial x}$ with $\frac{\partial \phi}{\partial x}$ and so on.

You can get you can show that $\frac{\phi_N - \phi_S}{2\Delta y}$ equals $\frac{\partial \phi}{\partial y}$ plus $\Delta x^2/6\ \frac{\partial^3 \phi}{\partial y^3}$ and so on. So, let us plug these two quantities back into the equation number 5 ok; that means, rho u times this quantity plus ρv times this quantity, ok.

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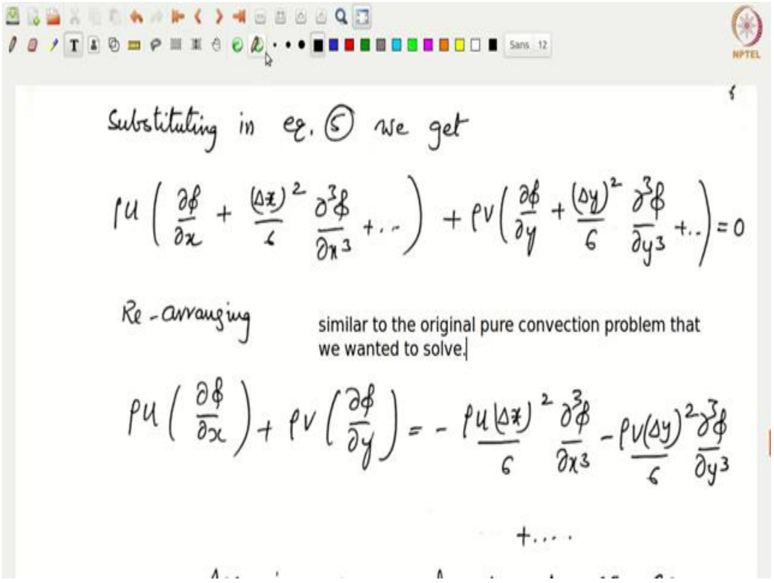

So, that is basically, if you substitute back you get ρu times $\frac{\partial \phi}{\partial x}$ plus $\Delta x^2/6$ times $\frac{\partial^3 \phi}{\partial x^3}$ plus ρv times $\frac{\partial \phi}{\partial y}$ plus $\Delta y^2/6$ times $\frac{\partial^3 \phi}{\partial y^3}$ equals 0 plus and so on, there are several terms here. Now, of course, we can again rearrange this; such that it looks somewhat similar to the original pure convection problem that we want to or wanted to solve, right

So, let us kind of do the rearrangement. If we do the rearrangement, then it is ρu $\left(\frac{\partial \phi}{\partial x}\right)$ plus ρv $\left(\frac{\partial \phi}{\partial y}\right)$ equals the remaining terms go to the right hand side; that is, $-\frac{\rho u(\Delta x^2)}{6}\frac{\partial^3 \phi}{\partial x^3}$ minus $\frac{\rho v(\Delta y^2)}{6}\frac{\partial^3 \phi}{\partial y^3}$ plus and so on, right alright.

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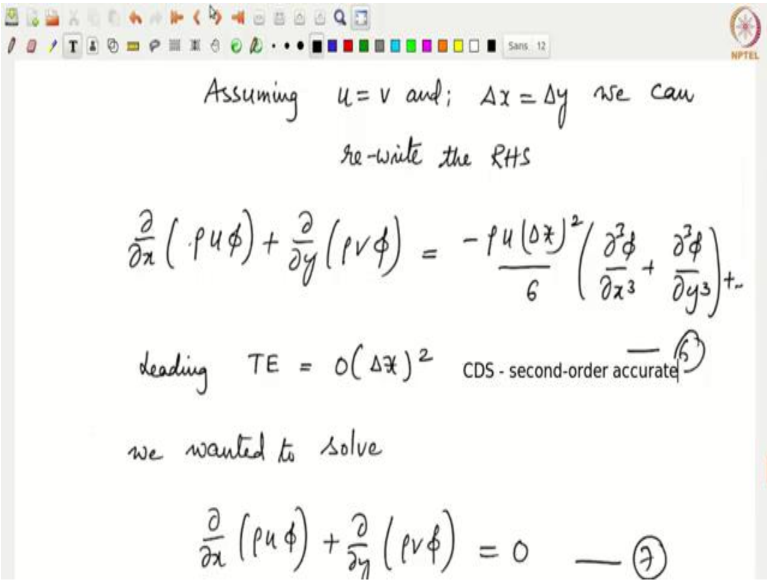

Let me also assume that again, for the sake of simplicity I am assuming u equals v and Δx equals Δy. So, that we can rewrite the right hand side and of course, the left hand side I am writing it as, because rho u and v are constants we can write it as $\frac{\partial}{\partial x}(\rho u \phi)$ plus $\frac{\partial}{\partial y}(\rho v \phi)$ equals this would be these two are the same one and the same. So, this will be $\frac{-\rho u(\Delta x^2)}{6}$ times $\frac{\partial^3 \phi}{\partial x^3} + \frac{\partial^3 \phi}{\partial y^3}$ cube, ok. So, that is what we have here.

Now, what is the leading truncation error here? It is 2nd order accurate. This we already know, right. We know that the central difference scheme is a 2nd order accurate right that we already know. And, but we wanted to solve $\frac{\partial}{\partial x}(\rho u \phi) + \frac{\partial}{\partial y}(\rho v \phi) = 0$, but what we ended up getting is equation 6, which has some term here, this is not we cannot call this diffusion, because this does not have even order derivatives, right.

This we would call in as dispersion all the this term and which is basically responsible for the oscillatory behavior of the central difference schemes, ok. So, in general, any derivative on the right hand side or in the modified equation that comes with odd derivatives, ok.

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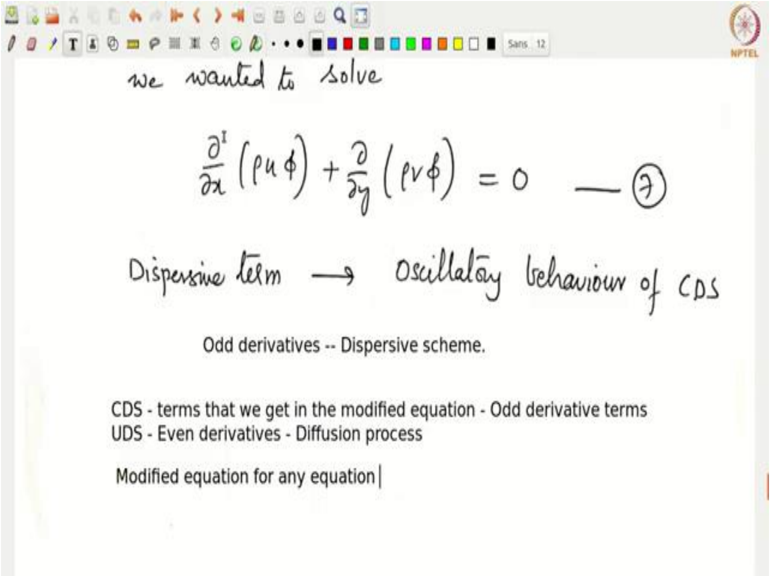

So, let me write it here. If it comes with the odd derivatives, then we can call it as a dispersive scheme, ok. So, if there are odd derivatives appearing in the modified equation, then we can call this as a dispersive term. Again, we know that the dispersive term is basically, of the order of delta x square it is kind of proportional to the grid size.

So, as we refine the mesh the dispersive term goes smaller and smaller; that means, we can expect the wiggles to become lesser and lesser, but nonetheless they will be there, because the mesh is always something that is finite, but small, ok. So, this is how we can now, describe the behavior of the central differencing and the upwind difference scheme based on the terms that we get in the modified equation, ok.

Essentially, for this we got odd derivative terms which we are responsible for something known as dispersion and for the UDS, we got even derivatives which are similar to diffusion process and as a result we can call UDS as diffusive process diffusive scheme and CDS as a dispersive scheme, ok.

So, now of course, you are also in a position to write a modified equation for any equation that will be given to you right for an equation. I mean essentially the process is construct the finite volume scheme that is basically, the construct the discrete equation then use Taylor-series expansion and substitute back for all the phi values at all the neighboring points in terms of phi P.

And then, rearrange it to look like the original equation right and then, that is your modified equation and then you can analyze the modified equation and comment on what would be the solution be depending on the terms that you get in the modified equation, ok. So, that is an explanation from the numerical point of view for the artificial or numerical or false dispersion and diffusion that we have seen when using CDS and UDS schemes in the solution of pure convection equation, alright.

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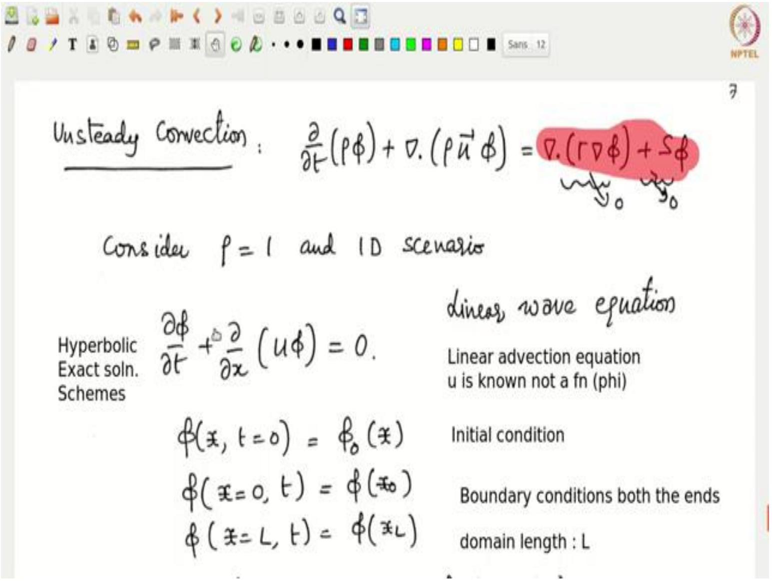

So, now, let us move on to the unsteady convection equation; that means, we are introducing the unsteady component. So, that means, if we go back to the general scalar transport equation which is $\frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot(\rho\vec{u}\phi) = \nabla\cdot(\Gamma\nabla\phi) + S_\phi$, because we are only again interested in the pure convection, but unsteady convection we are setting these two terms equal to 0, ok.

So, essentially, we are only just unsteady convection. So, let us see we are also want to kind of make a simplification. So, we will consider only a 1 D scenario or a 1 D situation and then, we also want to set the density equal to 1 everywhere. Then, this equation can be written as $\frac{\partial}{\partial t}(\rho\phi)$ plus, because it is 1 D you can write this as $\frac{\partial}{\partial x}(u\phi)$ right, ρ is 1 this is uϕ equals 0.

So, this equation is known as in the literature as the linear wave equation or it is also known as what linear advection equation. Now, why is it linear? It is linear, because u is known and is not a function of ϕ right; that is why, it is linear, alright. Then, what do we need to solve for this equation. This equation is basically a 1st order equation and it requires it has one time derivative one space derivative. So, it would require one initial condition, right.

So, this would require an initial condition that is basically, it requires a specification of phi 0 or the entire domain that is initial condition and then it requires boundary conditions right at the, at both the ends of the problem right, if you have a domain of length L; domain length is L then, we would need boundary conditions at x 0 and x L, ok. Now, what kind of an equation is this? This is a 1 D linear advection equation.

What is the type of this equation? Is it elliptic, parabolic or hyperbolic? We have done this in the past. So, this is basically hyperbolic equation right, alright. So, then for this equation fortunately exact solutions exist. So, that is why we have chosen this equation, because this can help us understand how does the different schemes the discretization schemes that we would choose behave when compared to the exact solution that we know, ok.

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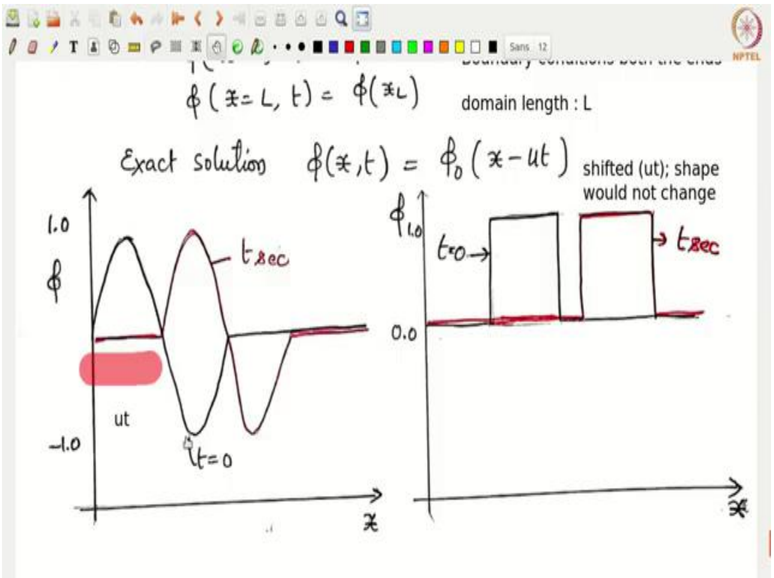

So; that means, the exact solution to this linear advection equation is basically $\phi(x,t)$ equals $\phi_0(x-ut)$. It is basically, whatever is the initial condition ϕ_0 of x that basically gets shifted by u times t right; it basically gets shifted by a minus u t right or u t.

Essentially and the shape as such of the initial profile would be would not change right that is preserved that is the property of 1 D linear wave equation. So, whatever is the initial profile given, that initial profile will remain the same, but only thing is that there will be a shift of this initial profile to a different location depending on what is u and what is time by a distance u times t right, ok.

That is what this says. Phi of x t at any time is nothing, but phi naught of x instead of x you have x minus u t. So, the shape would not change, but it only gets shifted, alright. Let us consider two initial conditions which are shown in black here, ok. So, one initial condition is basically using a sine wave. So, this is at t equal to 0. Now, at a later time if you solve using 1-D wave equation, this solution would come out to be something that is shown in red.

Essentially it will get shifted, but without a change in shape it gets shifted by some distance. Now, how much is this distance? This distance is nothing, but u times t right; this is your u times t, ok. After t seconds it would have traveled u times t, but the shape as such remains the same, ok. Similarly, if you take a square pulse that is basically again shown using black at t equal to 0, this is the square pulse, ok. Now, this also gets shifted by a distance u t in after a time of t seconds, ok.

So, the red one is basically it goes this way and the peak gets shifted, then the initial condition gets shifted, but there is no change in the shape or size of the initial pulse, right. So, the amplitude remains at 1.0 same as the here, 1.0 and minus 1.0. It is only a rigid body translation of the initial pulse that is the exact solution that is given.

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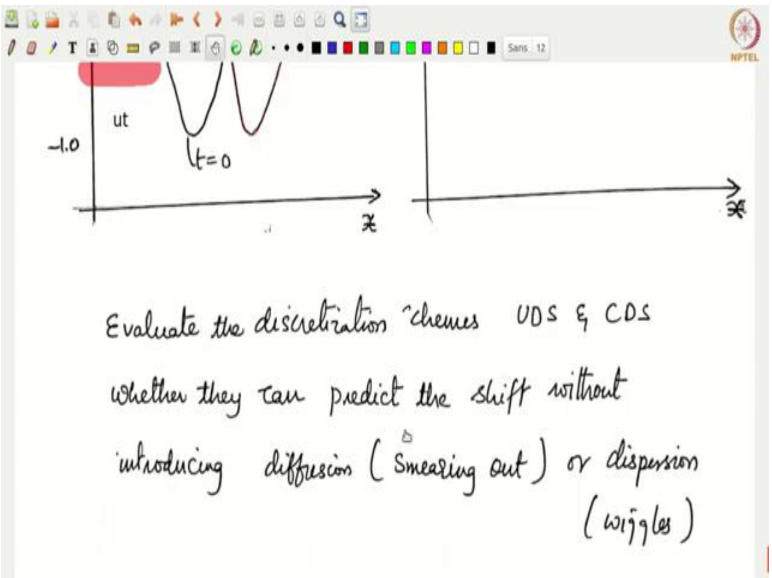

Now, this is very useful, because then, we can now apply the different schemes that we were to work with which are basically the upwind difference scheme and the central difference scheme; of course, together with the some kind of time stepping scheme for the unsteady term. And, we can evaluate; how does these schemes behave in this context, because we know the exact solution.

We can compare it and see well, do you see what it is supposed to be like from the numerical solution or not, ok. So, that kind of comparison can be done because the exact solution is known, ok. So, essentially the motivation is to see if the schemes that we would use that are constructed based on the upwind differencing and central differencing, can they construct they predict the shift without introducing diffusion that is the smearing out or the dispersion, right. These oscillations or wiggles. So, can we solve for these or not? So, that is the question we can pose.

Now, that would be our discussion for the next lecture. Essentially, using central differencing and upwind differencing schemes and some form of time stepping scheme, we would like to see what will be the solution that is obtained from these and compare it with the exact solution.

Now, this may look very simple, because it is a 1-D wave equation. It is linear. Well, will that be useful? Yes, because of its simplicity it is very extensively used in the literature to compare any new scheme that is introduced. However, the insights that we gain from working with this 1-D wave equation can be useful in the more complex situations where we have non-linear and coupled equations such as in the context of Navier-Stokes equations, ok.

So, what we learn from here can be directly useful there as well. So, that is the motivation to study such a simple problem using the different schemes alright. So, I am going to stop here. And, we will continue with the discretizing the unsteady convection equation using finite volume method and with the different schemes in the next class alright.

Thank you.