Computational Fluid Dynamics Using Finite Volume Method
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Lecture - 23
Finite Volume Method for Diffusion Equation: Stability analysis and steady diffusion in unstructured meshes

Good morning, let us get started. So, in the last class we were discussing about the Von Neumann Stability Analysis right. That is what we did not finish it let us kind of continue with that.

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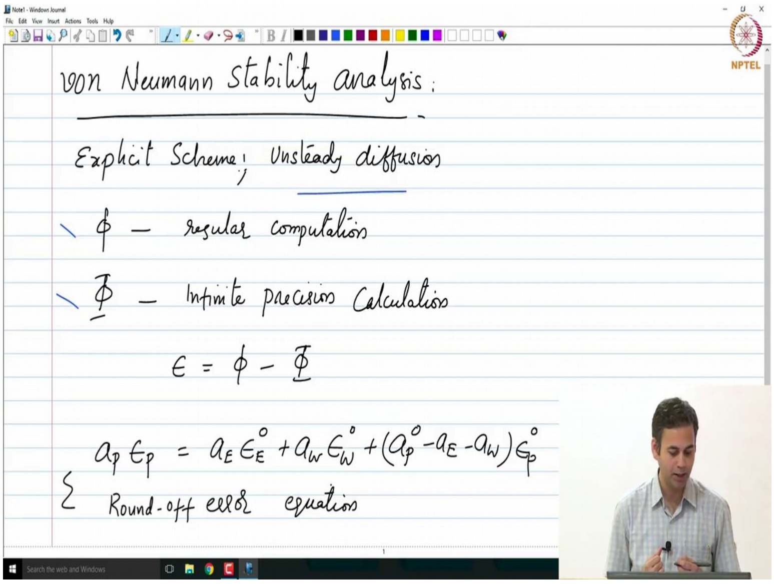

So, this is Von Neumann Stability Analysis. Essentially, we would like to know if the time steps that we are using the time stepping scheme that we are using would keep round off errors bounded or not right. To the round off errors grow from time step to time step or are they going to be not you know are they going to be bounded.

So, that is where we kind of looked at this formal analysis in which we started off with the explicit time stepping scheme. So, this is the explicit scheme applied to unsteady diffusion, right. And then we have we have said that there is a solution ϕ that we obtain on a regular computer right.

Essentially, this is the regular computation and then we also said there is another solution Φ which is obtained on an infinite precision computer right which does not have any round off error associated with it ok. So, this is basically coming from an infinite precision calculation and we equated the round off error ϵ as the difference between the regular computation and the one without any roundoff error.

So, we said the ϵ = ϕ - Φ and we substituted ϕ and Φ into the unsteady diffusion equation the discretized equation; and subtracted one from the other and then found an equation for the error itself right. Found an equation for the roundoff error itself and what was that? That was $a_p \epsilon_{\mathrm{p}} = a_E \epsilon_{\mathrm{E}}^o + a_W \epsilon_{\mathrm{W}}^o + (a_P^o - a_E - a_W)\epsilon_P^o$ right that is what we have.

Student: Epsilon (Refer time: 02:41).

Sorry this is ϵ_P^o ok. So, this is ϵ_P^o good. So, we have an equation for round off error itself.

So, this round off error equation right which is basically the same equation as the governing equation for the dependent variable right fine.

Now, we said we have to kind of introduce a model for the round off error and one good model is basically it is composed of several spatial and temporal components in an exponential sense. And because this governing equation is linear we do not have to look at each and every component of the error that we have model rather we can just look at one component of it and analyze the solution.

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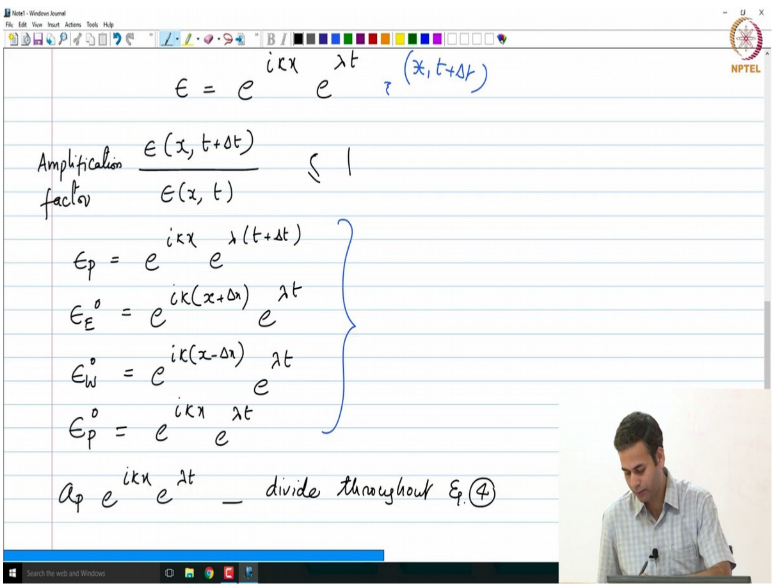

That is where we wrote this as $\epsilon = e^{ikx}e^{\lambda t}$ and we of course, want kind of or interested in the growth or decay of this roundoff error from time step to time step. So, what we are interested in we are interested in is the $\frac{\epsilon(x,t+\Delta t)}{\epsilon(x,t)}$ all right. This is we have termed this as what?

Student: (Refer time: 04:01).

Amplification factor and we would we would like to have this ratio as what?

Student: Less than (Refer time: 04:11).

Less than or equal to 1 such that our roundoff error either remains the same or it does not grow right that is what we would like. Now, if we plug in the spatial and temporal values of p, e and 0 superscripts right into the definition for error, we will obtain all these quantities right.

So, we said if you plug in the x and t values for the corresponding items here. So, that will be ϵ_{P} would be what? Would be if you plug in the values into this p would be coming at x and at a time of t+Δt right. So, that would give you a $\epsilon_{\text{P}} = e^{ikx}e^{\lambda(t+\Delta t)}$ right.

And then similarly we have ϵ_{E}^{o} this would be $\epsilon_{\text{E}}^{o} = e^{ik(x+\Delta x)}e^{\lambda t}$

Student: x plus (Refer time: 05:03).

Student: t.

t because we are looking at the time level t 0. So, this is t similarly we can plug in other quantities which is ϵ_{W}^{o} this is $e^{ik(x-\Delta x)}e^{\lambda t}$.

And we have one more quantity that is ϵ_{P} at the previous time level that is ϵ_{P}^{o} right this is $e^{ikx}e^{\lambda t}$ that is all now what we say is we kind of want to substitute all these values into the governing equation that is this one all right we plug in all of these things. And we of course, also would like to divide the entire equation with $e^{ikx}e^{\lambda t}$ after substitution and also with a_p right.

So, essentially divide throughout. What is the equation?

$a_p\epsilon_p = a_E\epsilon_E^o + a_W\epsilon_W^o + (a_P^o - a_E - a_W)\epsilon_P^o$. We call this as 4 or something. Yesterday we call this as equation 4 I guess. So, this is equation 4 essentially plug in these values and then divide everything with $a_p\, e^{ikx}e^{\lambda t}$ in equation 4 alright. Now, once you do that what is the final equation you are going to get?

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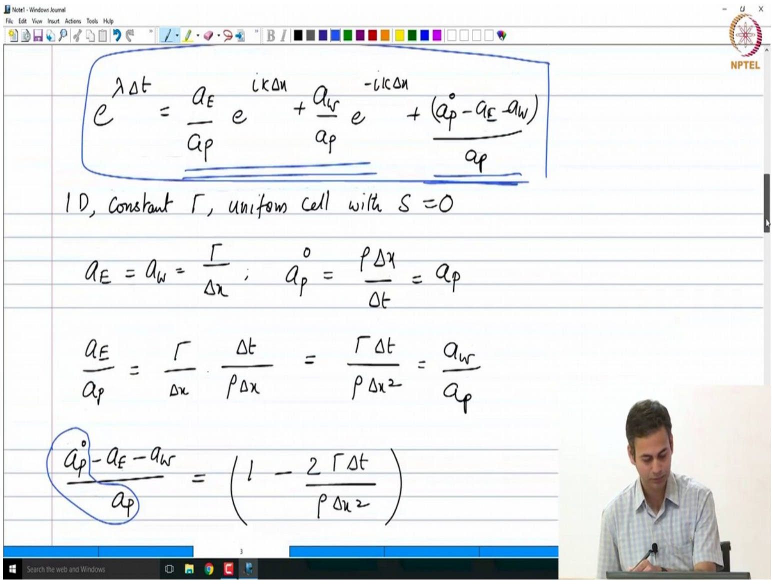

$$e^{\lambda\Delta t} = \frac{a_E}{a_p}e^{ik\Delta x} + \frac{a_W}{a_p}e^{-ik\Delta x} + \frac{(a_P^o - a_E - a_W)}{a_p}$$

Student: t.

Student: a (Refer time: 06:34).

Student: a p and (Refer time: 06:36).

Student: a p and (Refer time: 06:38).

Student: e power.

Student: e power.

Student: i k (Refer time: 06:44).

$e^{\lambda \Delta t} = \frac{a_E}{a_p} e^{ik\Delta x} + \frac{a_W}{a_p} e^{-ik\Delta x} + \frac{(a_P^o - a_E - a_W)}{a_p}$ that is all this is what we got ok. Now we what we would like to do is, we also realize that what is our this is basically a one-dimensional constant Γ and uniform cells right with source term equal to 0 right. Those are the assumptions we have made in order to analyze this thing.

So, in that context what is your a_E? a_E and a_W are equal to $\frac{\Gamma}{\Delta x}$ and a_p is how much? $\frac{\rho \Delta v}{\Delta t}$.

So, this will come out to be $\frac{\rho \Delta x}{\Delta t}$. So, how much would be a_E. sorry this is a_P^o which is also same as a_p. So, what will be these coefficients $\frac{a_E}{a_p}$ how much would this be?

Student: Gamma.

$\frac{\Gamma}{\Delta x} \frac{\Delta t}{\rho \Delta x}$ right. So, this is how much?

Student: Rho delta.

$\frac{\Gamma \Delta t}{\rho \Delta x^2}$ ok. So, this is same as $\frac{a_W}{a_p}$ right because a_E equals a_W both are the same. Then what about this last term that we have here this guy? How much will this be? $\frac{(a_P^o - a_E - a_W)}{a_p}$. This will be how much?

Student: 1.

1 minus you have a_E equals a_W. This is 2 times $\frac{a_E}{a_P}$. This is $2\frac{\Gamma\Delta t}{\rho\Delta x^2}$ is it alright? a_P^o equals a_P. So, as a result the first terms is 1 this is 1 right and the remaining parts is $2\frac{\Gamma\Delta t}{\rho\Delta x^2}$ fine ok. Now, we will kind of substitute all these things back into the equation. So, that will give us

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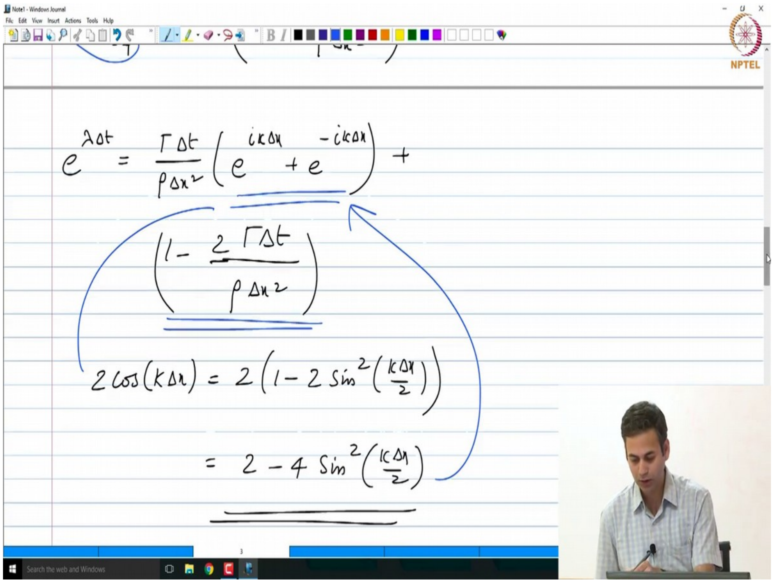

How much is this thing? What do we have here $e^{\lambda\Delta t}$. $e^{\lambda\Delta t}$ equals $\frac{a_E}{a_P}, \frac{a_W}{a_P}$. So, both are the same. So, we can take that out that would be $\frac{\Gamma\Delta t}{\rho\Delta x^2}(e^{ik\Delta x} + e^{-ik\Delta x})$ right coming from these two terms right and then we have plus we have this guy. Plus we have $(1 - \frac{2\Gamma\Delta t}{\rho\Delta x^2})$ alright that is the total value. Now how much is $e^{ik\Delta x} + e^{-ik\Delta x}$? How much is this?

Student: (Refer time: 10:15).

Student: k delta (Refer time: 10:21).

This is $2cos(k\Delta x)$. So, if I were to write in a square terms this would be cos 2^{θ} would be how much?

Student: (Refer time: 10:28).

Or

Student: (Refer time: 10:31)

cos θ = 1-$2sin^2(\frac{\theta}{2})$. So, this would be 2[1-$2sin^2(\frac{k\Delta x}{2})$] right that is fine. So, then this would be essentially 2-$4sin^2(\frac{k\Delta x}{2})$ right. So, we will plug this back into this term here right and see what happens. So, that is how much? $\frac{\Gamma\Delta t}{\rho\Delta x^2}$.

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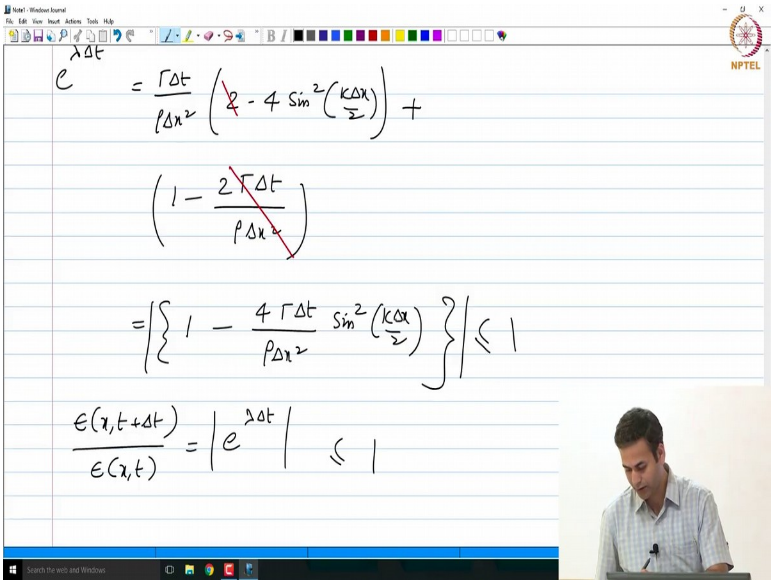

So, equals $\frac{\Gamma\Delta t}{\rho\Delta x^2}$ [2-$4sin^2(\frac{k\Delta x}{2})$] plus we have this guy that is your 1 minus 2 gamma delta t by rho delta x square. That is basically $(1-\frac{2\Gamma\Delta t}{\rho\Delta x^2})$. All these things are equal to $e^{\lambda\Delta t}$ right fine ok. Mistakes is it correct all right. So, do we have any terms that that may get cancelled.

Student: (Refer time: 11:53).

$\frac{2\Gamma\Delta t}{\rho\Delta x^2}$ gets cancelled. So, essentially what we will be left with we will be left with the.

Student: 4.

$1 - \frac{4\Gamma\Delta t}{\rho\Delta x^2} sin^2 (\frac{k\Delta x}{2})$ right. That is all remains because this term gets cancelled with the 2 $\frac{\Gamma\Delta t}{\rho\Delta x^2}$ fine. What is what is the amplification factor in the current context?

Student: (Refer time: 12:35).

Amplification factor is nothing, but $e^{\lambda\Delta t}$ right. So, essentially $\frac{\epsilon(x,t+\Delta t)}{\epsilon(x,t)}$ is nothing but $e^{\lambda\Delta t}$ right. If you plug in $e^{ikx}e^{\lambda t}$ this is what remains.

So, we want this guy to be less than or equal to 1 right; that means, we want modulus of this guy to be less than or equal to 1 right. That is the condition we are imposing in order to get a relation if there if there any exists between Δt, Δx and other parameters right. So, essentially ok. So, what will be this value now? Essentially, modulus of this guy. So, what is the maximum value that $sin^2 (\frac{k\Delta x}{2})$ can take?

Student: 1.

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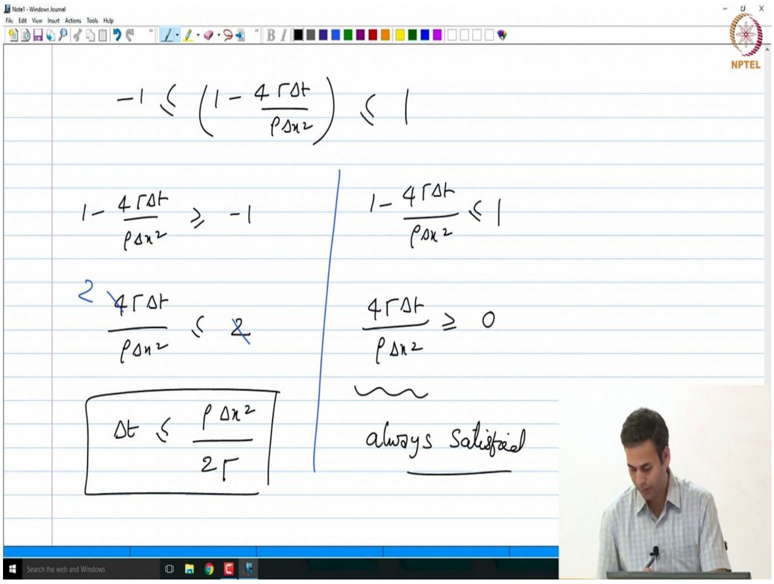

1 ok. So, essentially we are we have to look at in the worst scenario this will be $-1 \leq [1 - \frac{4\Gamma\Delta t}{\rho\Delta x^2}] \leq 1$. Alright that is what we have. So, we have two conditions the first one is for the minus.

So, this is $[1 - \frac{4\Gamma\Delta t}{\rho\Delta x^2}] \geq -1$. This would give us what? $\frac{4\Gamma\Delta t}{\rho\Delta x^2} \leq 2$ right that is what it gives. So, this will be how much? $\Delta t \leq \frac{\rho\Delta x^2}{2\Gamma}$ right. It gets cancelled ok. That is from the first condition.

What about from the second condition? That is $[1 - \frac{4\Gamma\Delta t}{\rho\Delta x^2}] \leq 1$. This would tell us $\frac{4\Gamma\Delta t}{\rho\Delta x^2}$ is greater than or equal to how much? 0, that means, delta t should always be positive right. This will be the case always right because your Δx is positive, Γ is positive.

So, this is always satisfied. What about this one? This is the condition that let us you choose delta t according to the Δx and the density and the diffusion coefficient that you have in the problem right. So, this is the condition which we have also written down I think right while back for the explicit scheme. So, this is the condition for stability such that the errors do not grow right.

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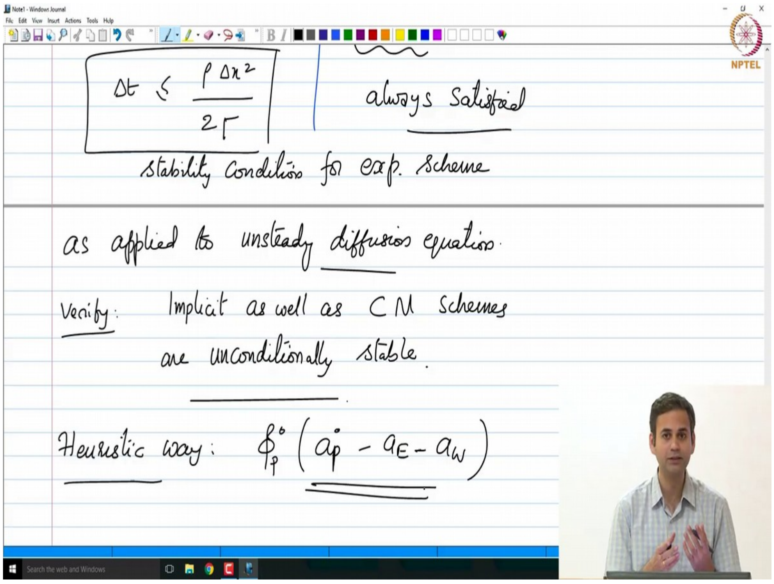

So, this is the stability condition for explicit scheme as applied to unsteady diffusion. So, it contains Γ, the density, Δx^2 and so on. Now, without proof we or I want you to kind of verify using the same analysis that both the fully implicit as well as Crank Nicolson Schemes are unconditionally stable ok.

So, this is something you will verify later fine ok. Now, we have done this Von Neumann Stability Analysis, but then we have also talked about this other analysis which is the kind of the heuristic way right heuristic way where we said the coefficient of Φ_P^o right which was $(a_P^o - a_E - a_W)$ right. This value should not become negative right.

We said that in a heuristic way saying that because we are solving a diffusion problem and increase in the Φ_P value at the previous time step should not cause a decrease in the Φ_P value at the current time step right.

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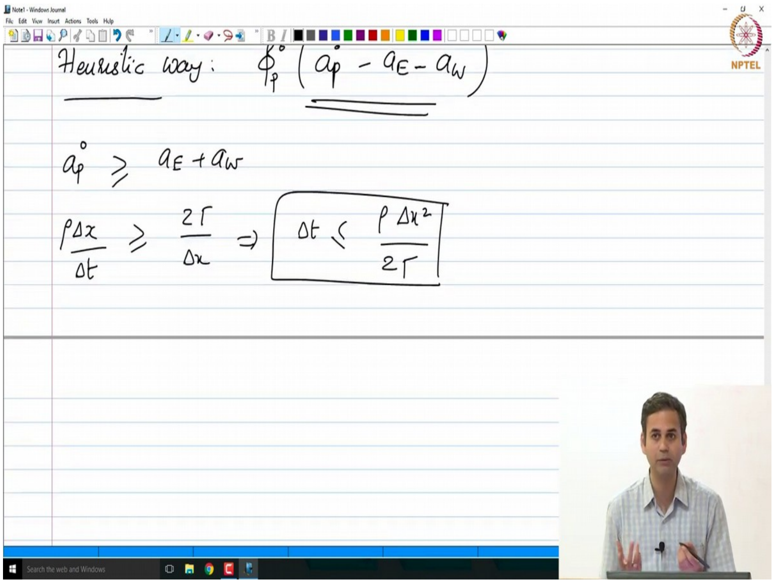

So, in a heuristic sense we said this thing if we use that analysis we can. So, we plug back what is. So, from the heuristic analysis what do we want this coefficient to be? Always positive; that means, we want a_P^o to be always.

Student: Greater than.

Greater than or equal to $a_E + a_W$ right. This is what is a_P^o ? $\frac{\rho \Delta x}{\Delta t}$ right. Was it $\frac{\rho \Delta x}{\Delta t}$? Yeah, and then greater than or equal to how much is $a_E + a_W$?

Student: (Refer time: 10:37).

$\frac{2\Gamma}{\Delta x}$.

Student: Delta x.

So, what is the condition on Δt ? Less than or equal to.

Student: Rho.

Student: Delta x (Refer time: 17:48).

$\Delta t \leq \frac{\rho \Delta x^2}{2\Gamma}$ ok. Do we get the same condition from the heuristic analysis and from the Von Neumann Stability Analysis ok, but the there were the Von Neumann Stability Analysis is much more formal. So, we can apply to everything. A heuristic is only kind of version certain aspects fine. Now, of course, what do we do if we have to extend this analysis to 2 dimensions?

Student: (Refer time: 18:15).

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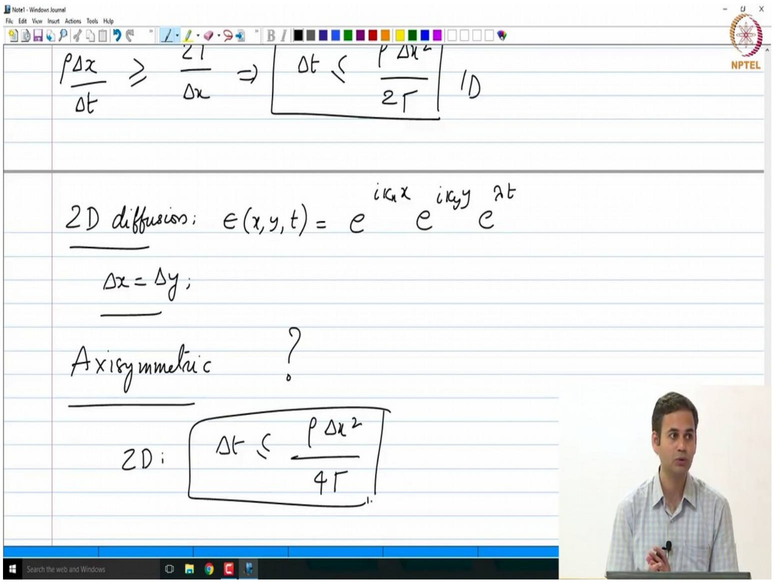

Ok, let us say if we have we are talking about 2D diffusion how do you model your error now? Your model for error $\epsilon(x, y, t) = e^{ik_x x} e^{ik_y y} e^{\lambda t}$ right. You will have wave numbers in both x and y directions that what you would use.

Now, if you proceed with the same analysis you also assume that the cells are uniform. So, the Δx equals Δy right for the sake of simplicity we also assume Δx equals Δy and the diffusion coefficient is also uniform everywhere or constant and there are no source terms.

If you make all these assumptions and proceed with the analysis, what is the condition you would expect to get from the Von Neumann Stability Analysis? I am just going back here. So, that you can see. So, this equation that we have what would this equation look like? If you have more neighbors now you will get a_E, a_W and.

Student: a_N.

a_N a_S and so on right. So, what would that eventually lead to? Instead of instead of 2 you will be getting how much here?

Student: 4.

You will getting a 4 right and that 4 would essentially translate to an 8 here because you have 2 times 1 minus 2 right that will translate it to 8. So, this will become $1 - \frac{8\Gamma\Delta t}{\rho\Delta x^2}$ and that 8 would again eventually lead to a.

Student: (Refer time: 19:49).

4 in the denominator right. So, that will be essentially lead to $\frac{\rho\Delta x^2}{4\Gamma}$ if you have a 2d problem and for a 3 d problem you would get.

Student: (Refer time: 20:02).

And the denominator how much you would you get?

Student: 8.

8 gamma right. So, we kind of wrote a general equation a while back saying it will be Γ d 2 d Γ or just Γ d right where d is the dimension if it is first one dimension and so on right. So, that is what you can do. Of course, somebody asked a question what will happen in decay in the context of axisymmetric right.

Now, what will happen in the case of axisymmetric. Can you guess it from here? So, what would change or what would not change. Would you still get a 2Γ ok? That is for you to may be figure out again now do it ok. So, essentially in the context of 2D and in the context of axisymmetric we have to kind of see what will happen what will be the conditions ok. Ok alright, so yeah.

Student: Sir, in the 3D case if we apply a (Refer time: 21:05) it will be 6 times (Refer time: 21:08).

Oh sorry yeah it will be 6 times right. 2DΓ it will come out to be 6 yeah right not 8 it is only 6 times ok, but that kind of brings up a good question. So, the thing is. So, what is in a 2D what will be the time step restriction? Those this is in one dimensions $\frac{\rho \Delta x^2}{2\Gamma}$, in 2D, what will be delta t? $\frac{\rho \Delta x^2}{4\Gamma}$.

Student: 4 (Refer time: 21:33).

So, is delta t has to be. So, you have solved a 1D problem now if you want to do the same thing for 2D do you have to reduce your Δt or you can just take the same Δt.

Student: Reduce.

We have to reduce right. Now, it is only half of what we have taken in the 1D right. That is what is coming out from the stability analysis ok. So, 2D problems are much more restrictive in terms of time stepping when it comes to explicit schemes ok. So, that is what this kind of tells us fine.

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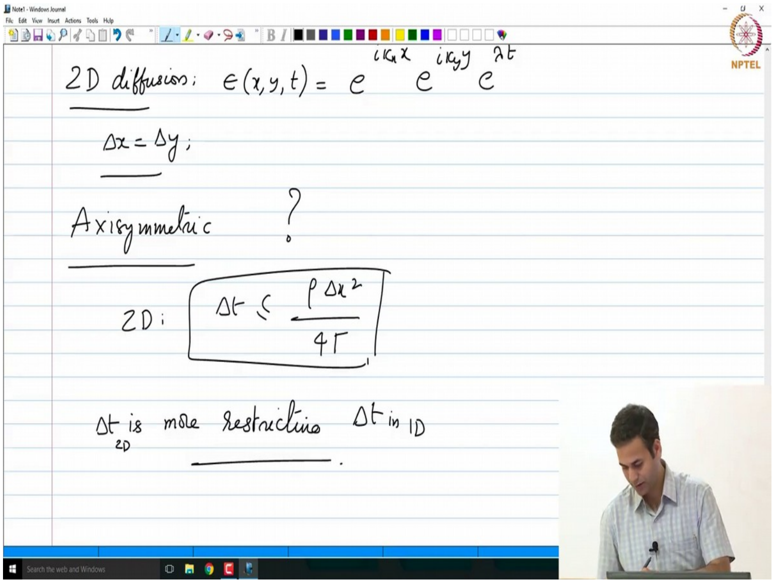

So, the Δt is in 2D is more restrictive or it is smaller than Δt in 1D fine. So, this as you can see this entire process of the stability analysis works only for if you have kind of make lot of assumptions like constant, coefficients and then those source terms and uniform cells and so on.

So, you can clearly see the complexity if these are not met right. As a result, we can only apply it to a simple problems and for practical problems of you know complex in nature we cannot really apply this rather we have kind of setup a simple problem and then see and for other complicated problems we have to kind of use a kind of our experience to kind of judge what will be the corresponding Δt and things like that ok.

But, essentially you can use this in the context of splitting down a complex problem into a diffusion convection and so on and then come up with several Δt s and then choose the Δt that is the smallest all right fine. Questions till now? No, this is clear this part is clear.

So, that kind of finishes pretty much about the different aspects in terms of truncation error, stability analysis all these things that we have discussed. Now, we can move on to the diffusion equation in unstructured meshes ok. So, up till now we have been discussing structured meshes alone right, but of course, we have introduced the time stepping schemes and everything else fine.

So, let us move on with unstructured meshes diffusion in unstructured meshes. So, we I will go back to steady diffusion as of now because unsteady diffusion would kind of follow the steady diffusion and it will be very similar to what we have done. So, we probably would not do it again.

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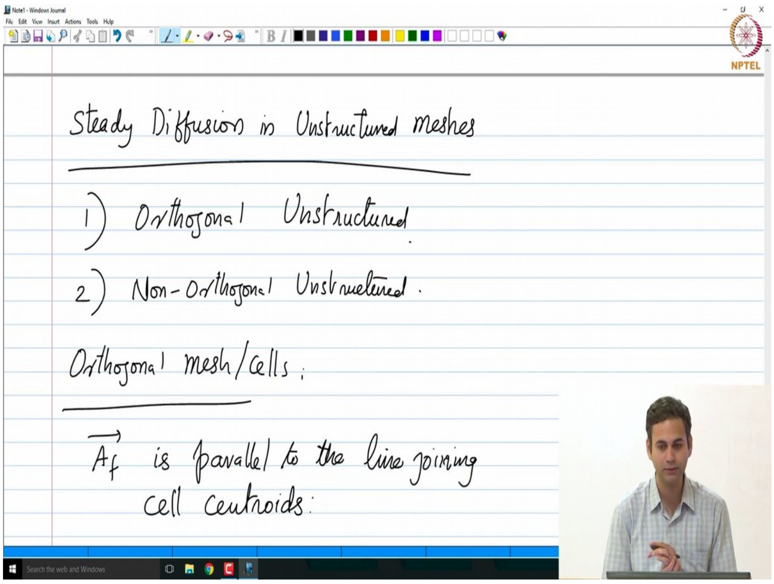

So, we were we are looking at steady diffusion in unstructured meshes. Even in unstructured meshes I am looking at 2 different types. One we would like to call it as orthogonal unstructured and the second type is non-orthogonal unstructured. So, we have 2 different types. We will first pick the orthogonal unstructured mesh and see how to solve steady diffusion on these kind of meshes. Now, what is an orthogonal mesh?

We have kind of or cells what are orthogonal cells or what is an orthogonal mesh. We have discussed this a while back in the during the initial lectures. What is an orthogonal cell? Anybody remembers or what is the consequence of having an orthogonal cell this also we have discussed till now right. What is the consequence of having an orthogonal mesh?

Student: (Refer time: 25:38) area (Refer time: 25:39) is perpendicular faces.

Area vector is perpendicular to the.

Student: Faces.

Faces area victor is always normal to the face right.

Student: (Refer time: 25:46) a dot product (Refer time: 25:48).

a dot so.

Student: a dot (Refer time: 25:50) a j dot a j (Refer time: 25:52).

a j dot. So, essentially what you are trying to say you will get diffusion fluxes only in direction normal to the face right. You will not get the other component right for x, we were only getting $\frac{\partial \phi}{\partial x}$ for y we are getting $\frac{\partial \phi}{\partial y}$. We never got $\frac{\partial \phi}{\partial y}$ in the x component and so on right. This was not happening because we have a area vector of the face in the same direction as the gradient right. So, the gradient direction is $\frac{\partial \phi}{\partial x}$ right. That is connecting what that is connecting the cell centroids.

So, the line joining the cells centroids if it is parallel to the area vector that is an orthogonal mesh or an orthogonal cell and if every cell in the mesh that you have generated satisfies this property then what you have got is an orthogonal mesh right. So, that means, the area vector of the faces is parallel to the line joining cell centroids.

So, when we talk about one particular face we are talking about only 2 cells and the 2 cell centroids right because there is a uniqueness here right. Every face has only 2 cells right. So, this if it is satisfied, then it is an orthogonal cell and if it happens to be there for every cell in the mesh, then it is an orthogonal mesh ok. Now, we are looking at in the first context as an orthogonal mesh. Let me give you an example of how an orthogonal mesh looks like ok.

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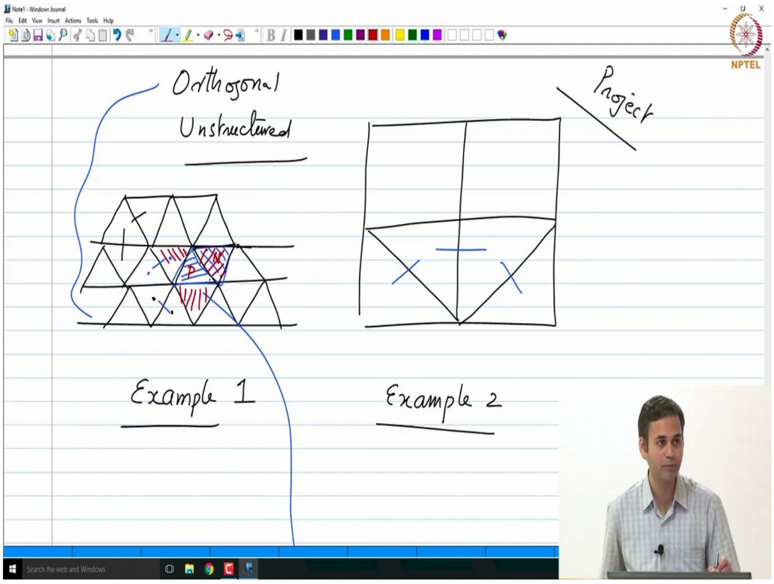

So, does it look like an orthogonal mesh?

Student: (Refer time: 27:46)

No yes. So, essentially we have let us say ich little triangles and the faces normal are parallel to the line joining cell centroids ok. So, for example, these are the cell centroids, this is the faces right and so on. So, this is an orthogonal mesh a kind of an example right. So, we have some structure like this.

So, this is basically an orthogonal mesh. Now, you may say is it structured right. It kind of looks like there is some structure, is not it? It is kind of not very easy to generate a an orthogonal mesh which is unstructured. But of course, if you have a domain that kind of ends in any regular way and that would kind of an unstructured mesh right you do not have a structure anyway ok.

What is the other way you can generate another orthogonal mesh? You can of course, have a structured mesh right which is square mesh or something and we can partition them into triangles right cut all these squares into triangles then you can probably get a orthogonal mesh right that is also possible. So, that means, let us say we have a mesh that is we start off with a square mesh. Now, I want to convert this into an orthogonal triangular mesh. Yeah, essentially I would draw this way right. This would make it a the mesh here and then which way shall I draw. Either way is fine. This way right would it make an orthogonal mesh. Essentially, you got these all should be parallel to the cell centroid directions and so on right we can do this. So, these are some examples of orthogonal unstructured meshes fine.

So, let me pick one particular cell here. So, that is essentially or 2 cells here let us say this one and this one ok. I am picking these 2 and then we will look at how to discretized our equation on this this particular cell ok. So, essentially one of this is the p cell one of this is the neighboring cell let us call this is P and N ok. So, I am drawing only those 2 here ok.

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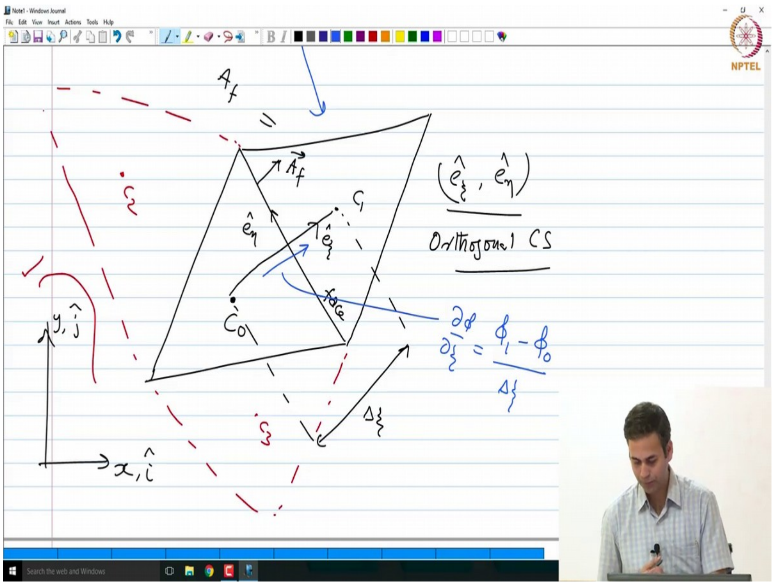

So, essentially that is cell like this that is what I have I picked this P and N from here ok. So, this is what I have drawn here which looks like this. I would like to call the primary cell which we have been look indicating till now as P as some cell with subscript 0, C_0 cell 0 is our cell P up till now.

Because right now we do not have any east west north south and then this C_0 cell which is our primary cell is now surrounded with how many cells. 3 cells right 3 neighboring cells this guy this guy 3 face neighbors right and this one ok, but as of now I am drawing only one of the neighbors. We will extend the concept to later on others as well.

So, that means, C_0 is surrounded by 1, 2, 3, 3 neighbors right C_1, C_2, C_3 I am as of now drawing C_1 cell alone, but we know that there are two more cells which are also orthogonal right which exist here right which we are which I have not drawn here, but they do exist ok.

So, I am looking at only these 2 components. Now, this is the face vector and I can of course, connect these 2 cell centroid C_0 to C_1 and this is one particular direction and similarly this is my area vector $\overline{A_f}$ right points in the direction away from the cell right.

So, so this is now parallel to the line joining C_0 and C_1 as a result this is orthogonal mesh.

Now, I can of course, also define 2 coordinate systems. One is running along the C_0 C_1 direction right which I would like to call it as $\hat{e_\xi}$ and another one running along the face which I would like to call it as $\widehat{e_\eta}$ ok.

So, essentially these are this is $\hat{e_\xi}$,$\widehat{e_\eta}$ is what kind of a coordinate system. Is this is it orthogonal or non-orthogonal? This is orthogonal by virtue of cells being orthogonal ok. So, because this line is parallel to the f. So, this is a orthogonal system orthogonal coordinate system. Of course, we have already have our original orthogonal coordinate system which was what?

Student: xy.

x$\hat{i}$, y$\hat{j}$ this is also there. So, we have $\overline{A_f}$ fine and let us define this distance between C_0 and C_1 as $\Delta\xi$ ok. Just like we had Δx between P cell and E cell now we have 0 and 1 we will call it as $\Delta\xi$ ok. Of course, similarly you will have a C_2 cell here, C_3 cell here and your definition for $\hat{e_\xi}$ and $\widehat{e_\eta}$ would be different for each of these different combinations.

So, for every face you have a local coordinate system right in which ξ is connecting the C_0 to C_1 or C_0 to C_2 or C_0 to C_3 and eta is in the direction of the face along the face fine that is understood. But, as of now we will focus only on C_0 and C_1 questions till now is this understood right. So, let us move on then what is our governing equation. What do you want to solve?

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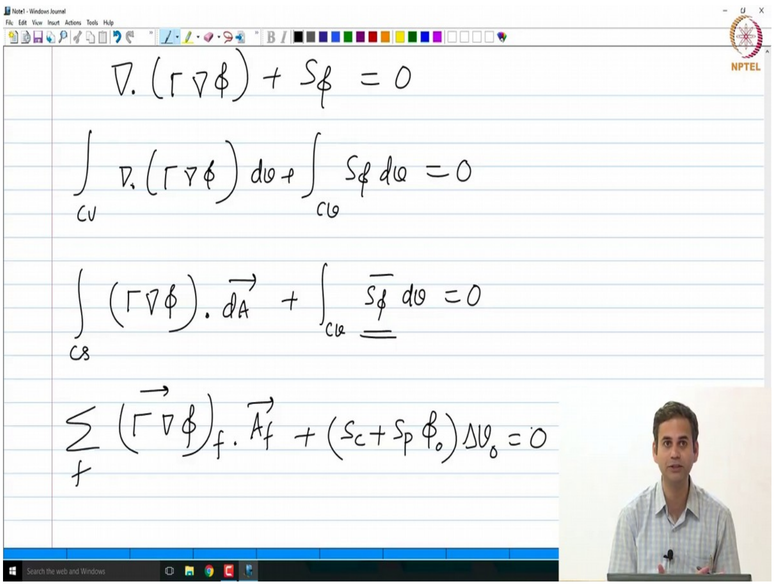

We want to solve the steady diffusion that is nothing, but $\nabla.(\Gamma\nabla\phi) + S_\phi = 0$ right. That is what we have and what is the first step. Integration on the control volume what is the control volume in this context? Which cell?

Student: (Refer time: 34:49).

C_0 right C naught. So, that is essentially $\int_{CV} \nabla.(\Gamma\nabla\phi)\, dV + \int_{CV} S_\phi\, dV = 0$. Now, we will invoke gauss divergence theorem and convert this volume integral to a surface integral.

So, this will be $\int_{CS} \overrightarrow{(\Gamma\nabla\phi)}.\overrightarrow{dA} + \int_{CV} S_\phi\, dV = 0$ alright and again saying that the $(\Gamma\nabla\phi)$ can be represented using the face centroid value right as a mean value we can rewrite this integral as a. As a what? As a summation on all the faces.

So, this can be written as $\Sigma_f \overrightarrow{(\Gamma\nabla\phi)_f}.\overrightarrow{A_f}$ all right plus we would say that S_ϕ can be represented using it is average value $\overline{S_\phi}$ right and we introduce a linearized model for the source term. So, this would come out to be $S_c + S_p\phi$.

Student: Phi minus.

ϕ_0 right. ϕ_0 is basically for C_0 cell times ΔV_0 that is ΔV_p fine ok. We have essentially we have not done anything different compared to the cartesian coordinates still now right then let us go back and see what are these things.

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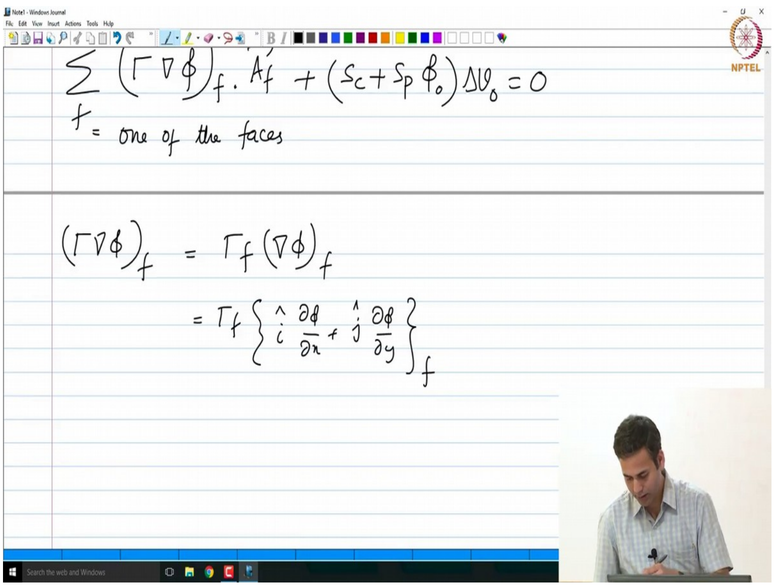

So, what is $(\Gamma\nabla\phi)_f$. Let us say I am taking one face so one of the faces. What is $(\Gamma\nabla\phi)_f$?

This is $\Gamma_f(\nabla\phi)_f$ this can be written as what $\Gamma_f\left(\hat{\imath}\frac{\partial\phi}{\partial x}+\hat{\jmath}\frac{\partial\phi}{\partial y}\right)_f$ evaluated on the faces right.

That is what we have in the x y coordinate system right. It is fine then because my so we have written in the in global coordinate system right in this one. So, because my local coordinate system which is $\widehat{e_\xi}$, $\widehat{e_\eta}$ is also orthogonal can I rewrite this gradient in that $\widehat{e_\xi}$, $\widehat{e_\eta}$ coordinates?

Student: Yes.

I can ok. So, what would that be?

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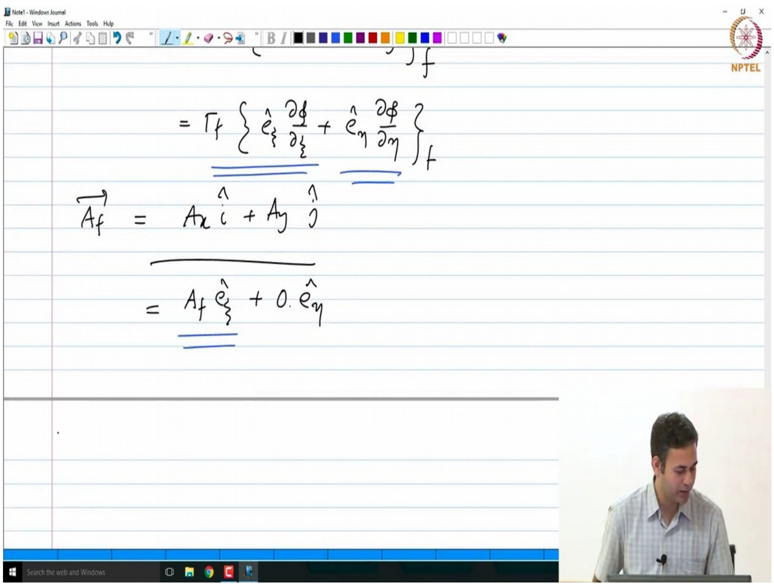

That would be $\Gamma_f\left(\widehat{e_\xi}\frac{\partial\phi}{\partial\xi}+\widehat{e_\eta}\frac{\partial\phi}{\partial\eta}\right)_f$. Is this one and the same?

Student: Yes.

It is one and the same right ok. This is fine then what about my $\overline{A_f}$. $\overline{A_f}$ of course, has 2 components right it has some $A_x\hat{\imath} + A_y\hat{\jmath}$ it has some 2 components in the i direction and j direction, but can I also write this a vector in the local coordinate system. Let us say my magnitude is A_f without t over bar, what is the direction which A_f is pointing? e_ξ or e_η.

Student: e_ξ.

e_ξ ok. So, $\overline{A_f}$ let us say if it has a magnitude of A_f, then what is $\overline{A_f}$? Can be written in the e_ξ e_η coordinates as how much? $A_f\widehat{e_\xi} + 0\widehat{e_\eta}$. Can we write this? All right, basically the area vector pointing in the ξ direction with a magnitude A_f all right very good. Now, can we calculate what is $\overrightarrow{(\Gamma\nabla\phi)_f}.\overrightarrow{A_f}$ in the local coordinate system. If you calculate that what terms survive?

Student: (Refer time: 38:56).

So, $\widehat{e_\xi}.\widehat{e_\xi}$ would survive. Do you get any other term? No, everything else this is 0 and the dot product with this is 0 right. So, we do not get. So, essentially we will go back to our similar to previous way.

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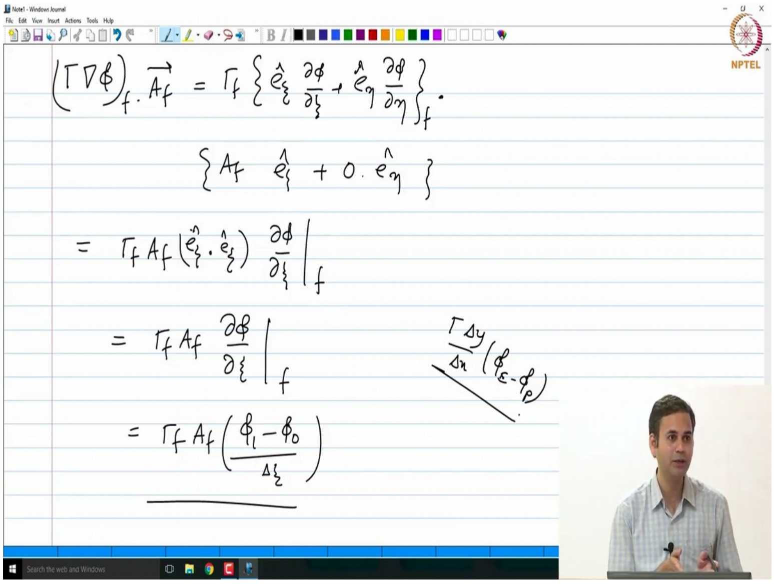

So, $\overrightarrow{(\Gamma\nabla\Phi)_f}.\overrightarrow{A_f}$ would be how much? Would be $\Gamma_f\left(\widehat{e_\xi}\frac{\partial\Phi}{\partial\xi}+\widehat{e_\eta}\frac{\partial\Phi}{\partial\eta}\right)_f.(A_f\widehat{e_\xi}\ +\ 0\widehat{e_\eta})$ right.

This is the dot product and it will give you only $\Gamma_f A_f(\widehat{e_\xi}.\widehat{e_\xi})$ that is 1 alright times $\left(\frac{\partial\Phi}{\partial\xi}\right)_f$ on the face f right that is what we would get correct ok. So, this is nothing, but $\Gamma_f A_f\left(\frac{\partial\Phi}{\partial\xi}\right)_f$ right. This is for one face that is that is connecting C_0 and C_1 right. So, we will get to 2 more terms like this each of them connecting C_0 to C_2, C_0 to C_3 right we will get 2 more terms. What is partial phi partial xi f?

Student: (Refer time: 40:36).

$\left(\frac{\partial\Phi}{\partial\xi}\right)$. So, essentially we are evaluating derivative in this direction. This is $\left(\frac{\partial\Phi}{\partial\xi}\right)$. What will be this? This will be value of Φ at C_1 that is let us say Φ_1 minus.

Student: ϕ_0.

ϕ_0 upon what will be the distance.

Student: Delta (Refer time: 40:56).

Δx I. This is basically like our cartesian thing right where we have $\left(\frac{\partial \phi}{\partial \xi}\right)$ on the east face is nothing but $\frac{\phi_E - \phi_P}{\delta x_e}$ right ok. So, I can write this as $\Gamma_f A_f \left(\frac{\phi_1 - \phi_0}{\Delta \xi}\right)$ right that is understood. Now, do you see why we have actually written in the local coordinate system why not just work with the global coordinate system?

Student: (Refer time: 41:37).

Only to write this $\phi_1 - \phi_0$ right. Because our ϕ values are available only at (Refer time: 41:42) at the cell centroid. So, we would like to have ϕ_1 and ϕ_0 to connect and calculate all the gradient right. We do not want to do interpolations and then create another mesh on top of it right.

So, that is why we kind of introduced a local coordinate system for every face and then now we are calculating gradients using the cell centroid values on this local coordinate system. Now, do you see a corresponding to the original cartesian mesh case. What was the cartesian mesh coordinates used to look like? What is the coefficients look with ?

Student: (Refer time: 46:15).

$\Gamma \frac{\Delta y}{\Delta x} (\phi_E - \phi_P)$ something like this right. So, what will be the coefficient now that we have? What will be the coefficient for A_f A neighbor?

Student: (Refer time: 42:29).

$\left(\frac{\Gamma_f A_f}{\Delta\xi}\right)$ that is the coefficient and the corresponding f determines depending on the cell neighbor corresponding values, what will be the a_p coefficient here contribution to a_p? The same value right same value will also go to a_p and also will also go to the A neighbor fine.

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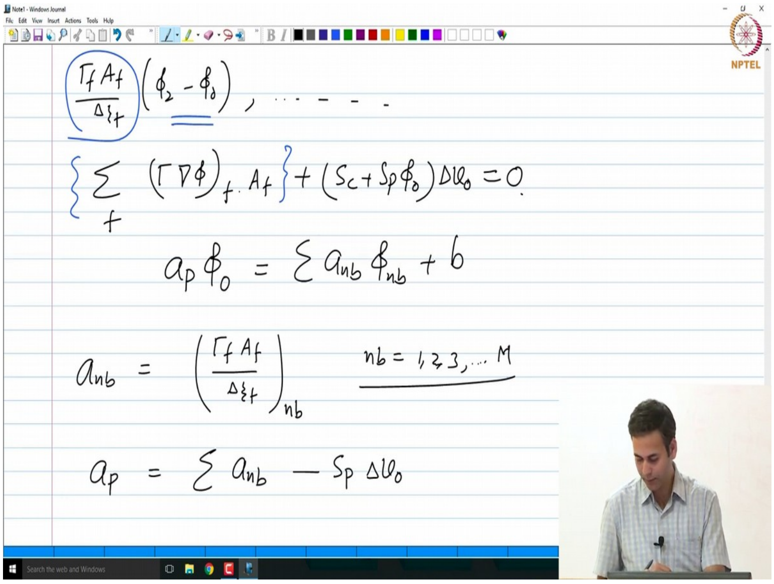

So, then now we understand that we will get 2 more terms that is $\left(\frac{\Gamma_f A_f}{\Delta\xi}\right)(\Phi_2 - \Phi_0)$ and so on depending on number of neighbors we have right. We will also get $(\Phi_2 - \Phi_0)$ and $(\Phi_3 - \Phi_0)$ and so on right. We will get all these terms. Now, if we club all of these things.

So, we have original equation $\sum_f \overrightarrow{(\Gamma\nabla\Phi)_f} \cdot \overrightarrow{A_f} + (S_c + S_p\Phi_0)\,\Delta V_0 = 0$ right. We will of course, want to write our equation in the standard form. What is our standard form? $a_p\Phi_P$ in this context Φ_0 equals $\sum a_{nb}\Phi_{nb} + b$ right. This is our standard form. So, what

I am asking you is basically substitute $(\Gamma\nabla\phi)$ from here for each and every cell into this equation right into this term and then rearrange it this way ok. Now, we have already gone through this. So, what will be a_{nb}? What will be the coefficients?

Student: (Refer time: 44:11).

Of course, this will be the coefficient right. So, this will be the coefficient; that means, this is nothing, but $\left(\frac{\Gamma_f A_f}{\Delta\xi}\right)$ and this f itself is defined for that particular neighbor right. If you are looking at neighbor of 1, 2 or 3 right depending on that this f will be defined. There is no summation here right a n b itself is this value. This is like a east a east is $\Gamma\frac{\Delta y}{\delta x_e}$. So, there is no summation here right, but of course, nb can go from neighbors that is 1, 2, 3 all the way to some capital M if you have M neighbors right. You do not have a triangle you have some kind of a honeybee structure right some kind of a polygon then you may have some 8 faces or 9 faces right. So, depending on that is this f is defined. f is the face between C_0 and C_1 or C_0 and C_2 and so on fine. Now, what about a p. What will all come in a_p? a_p is now.

Student: (Refer time: 45:15).

Is with has a negative coefficient, but it will go to the right hand side. So, what will this this will have one contribution? Now, you will get .

Student: (Refer time: 45:24).

Right. So, essentially you will get sigma.

Student: a_{nb}.

Student: Minus S p (Refer time: 45:28).

Student: Delta v (Refer time: 45:31).

$a_p = \sum a_{nb} - S_P \Delta V_0$ right. Your as usual S_P term will be there. What about b?

Student: (Refer time: 45:36).

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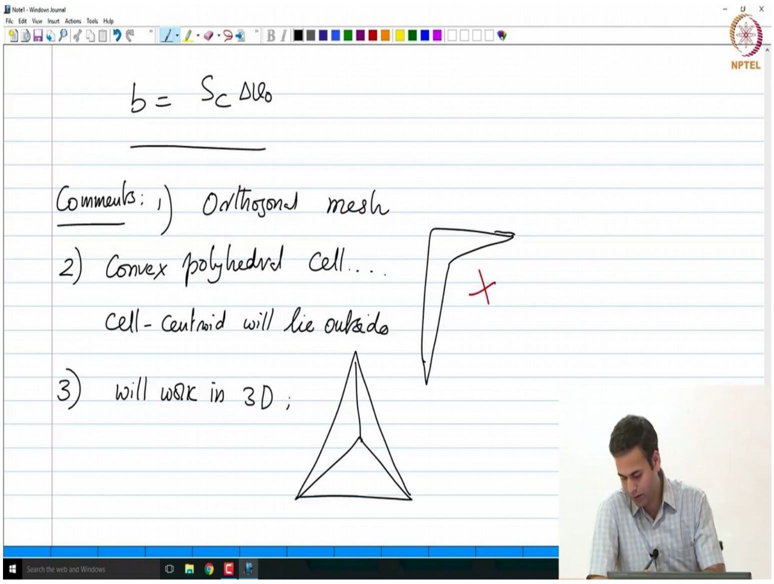

B is how much? $S_c \Delta V_0$ ok. So, the beauty is now we got a very simple equation which is the same as what we have got in the cartesian right in the cartesian mesh as? Well, only thing is we just have now coefficients which we have to evaluate based on the faces. Now, can we calculate the area vectors if we know the mesh. Is it possible to calculate the area vectors? We know the mesh right.

So, one when we know the mesh we know all the coordinates cells centroids everything right. Can we calculate the area? We can actually calculate and then we can calculate all these things all right and then work with this fine. So, then let me make few comments so that we can kind of finish this here. The comments are. So, the assumption is that this is orthogonal mesh. What we have also assumed is this is also a convex polyhedral cell. Now, what is a convex polyhedra? What is a convex polyhedra? Sorry.

Student: (Refer time: 46:53).

All diagonals. So, essentially any line connecting any 2 points inside the cell should entirely lie within the cell that is a convex polyhedra right. Now, why do say we have to have a convex polyhedra for our case? That means, we cannot have cells that look like this right. Why is that so? Because if you have something like this somehow I have generated an orthogonal mesh ok. So, we have a boomerang like this which is can give us some kind of an orthogonal mesh. Cell centroid will be outside, so what? Ok, we do not care why cell centroid if it is outside what is the problem right.

Student: Essentially (Refer time: 47:35).

Essentially, it is not a representative of the cell values. What else? What else can go wrong? Right, it will lead to negative coefficients right. We see that once it is outside all your calculations will lead to the coefficients will all lead to negative coefficients right. So, essentially that is why we cannot have we have to have only convex polyhedra right.

We cannot have cells like this because the cell centroid will lie outside if we have something like this. So, essentially it is not a representative of the cell values ok. Is this treatment limited to 2 dimensions this entire thing we have done ah? Let us say we generate a orthogonal mesh that is unstructured in 3D, can I apply the same thing?

Student: Yes.

We can do right. Only thing is that you have more neighbors. Now, right f will have instead of if it is a triangle if you 3 if we have a tetrahedron you will have 4 right. So, essentially this can be will work in 3D. So, we can probably have a tetrahedra cell and this will work in 3D also right. The same analysis can be extended ok. This will work in 3D.

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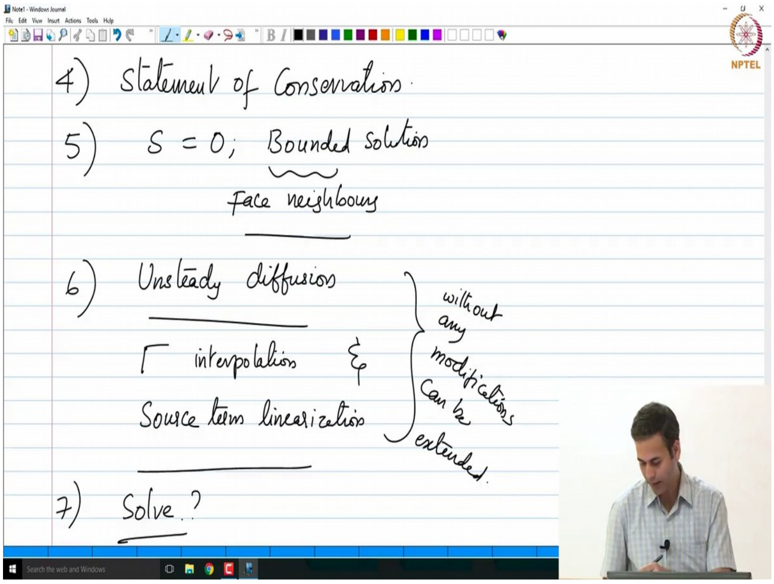

What about what about this equation that we have $(\Gamma\nabla\phi)$ this equation. This is a statement of conservation right. We have the diffusion fluxes leaving the faces or entering the faces is balanced by the source term. So, we again still have the statement of conservation ok. So, as a result the solution you would get is satisfies conservation. So, let us say if we have the source terms equal to 0 in the absence of source terms is the ϕ_0 value bounded by the neighbors?

Student: Yes.

B is 0 right. So, in that context is it bounded?

Student: Yes.

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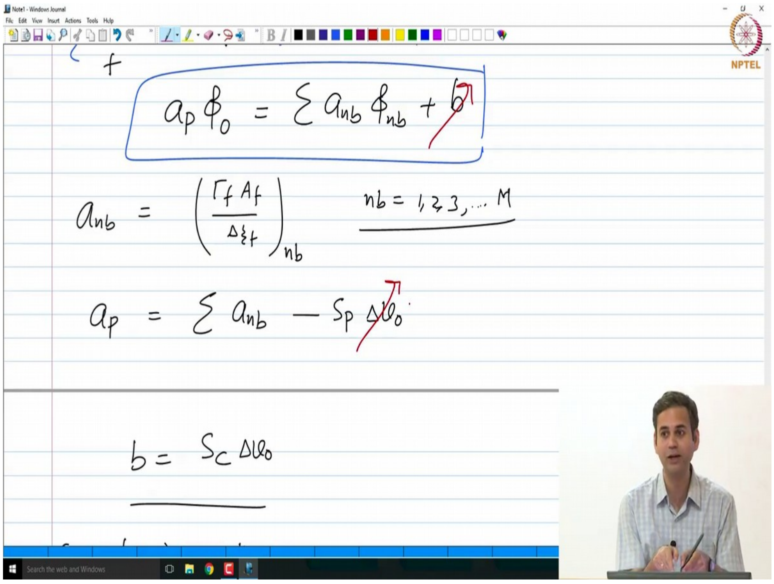

It is? Bounded because $a_p = \sum a_{nb}$, $S_P \Delta V_0$ is 0 and $a_p \Phi_P = \sum a_{nb} \Phi_{nb}$ right. So, essentially this is still gives you a bounded solution. Bounded by who? By the face neighbors only the face neighbor not the other neighbor not the ones which have a vertex in common right only the face neighbors. Fine, it is kind of bounded. Now, what about unsteady diffusion? Let us say if you want to solve this with an explicit scheme or an implicit scheme can you do this?

Student: Yes.

We can do this. Essentially, you just have extra term $\rho\phi_P^1$ and $\rho\phi_P^o$ right that extra term.

So, you can extend the entire unsteady thing we have done the explicit, implicit and Crank Nicolson to the structured unstructured orthogonal mesh ok. So, we will not do it again. What about interpolation of gamma interpolation? Can we extend this to this mesh?

Student: Yes.

We can extend it. Essentially, earlier you had this distance is defined in the x direction. Now, you have distances are defined in which direction? In the ξ direction that is all. So, essentially you do that and this thing. What about the source term linearization? Is it the same as before? It is also same as before.

So, all these can be extended without any changes right without any modifications of except for the geometry right without any modifications can be extended so far so good. How do you solve it? We got a system of linear equations, what method would you used to solve this? Let us say this is unstructured to begin with. Can we use Gauss Seidel?

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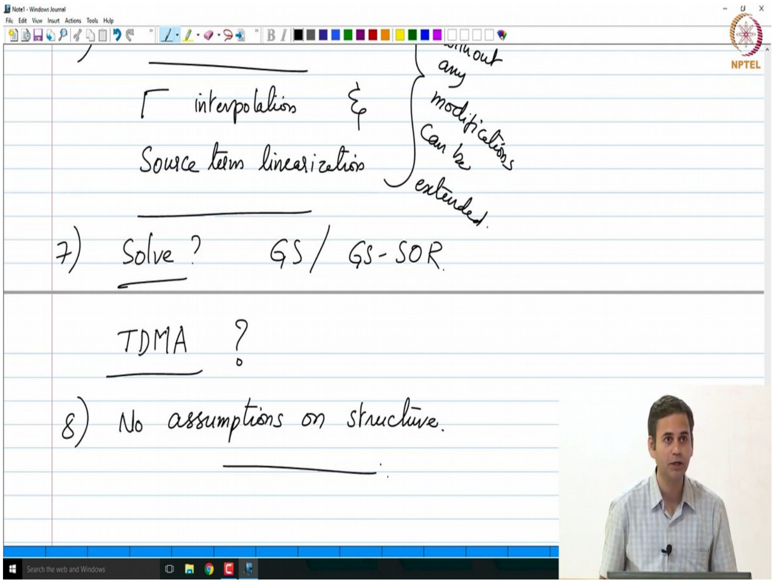

Yes we can use. So, essentially can use Gauss Seidel or Gauss Seidel Successive Over Relaxation right. This can be used. What about TDMA or line by line TDMA. Can we use this?

Student: No.

No, we do not have a structure right. As a result, we cannot use line by line TDMA, but of course, we can create some kind of a what is your line and kind of probably do it ok, but the preferred method is Gauss Seidel if you have unstructured meshes right. So, that is about it. So, there is no assumption on structured no assumptions on structure. So, essentially you can have an unstructured mesh, but it only has to be orthogonal that is all we have discussed till now. Fine, very good I am going to stop here. We will meet in the next lecture.

Thank you.