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Video Lecture on
Convective Heat Transfer

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Lecture 39
Approximate Method in Natural Convective Heat Transfer

So good morning today we will continue our discussion on the approximate solution which was attempted originally by Squire and in this case. We write down the integral formulation of the boundary layer equations for natural convection.

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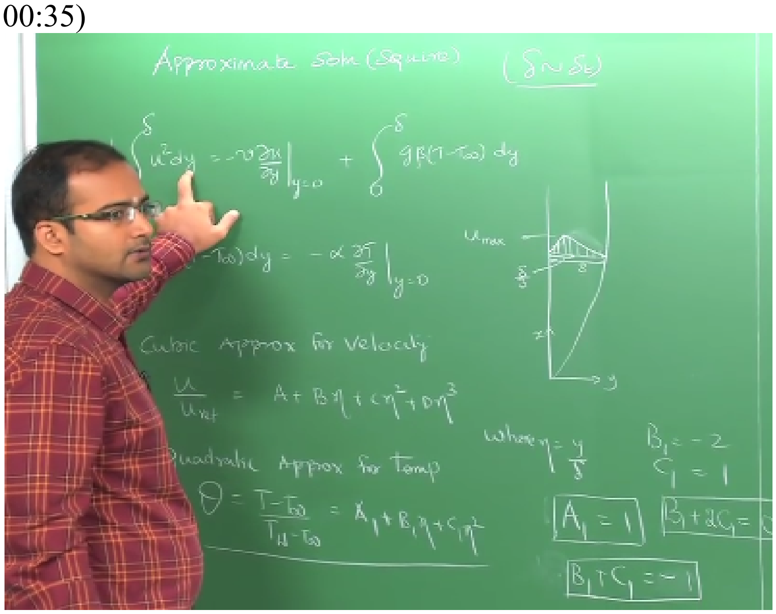

So we have the momentum integral equation and the energy integral equations which was derived and the approximation comes in the form of making a guess for the velocity and temperature profiles .So in this case since the natural convection problem has a Maxima in the velocity profile somewhere within the boundary layer right something like this so therefore we have to be careful to choose the more or less reasonable reasonably good approximation for this variation.

We cannot therefore predict this with a linear profile or even a quadratic profile for that matter so unlike the external force convection boundary layer therefore the choice of velocity profile here has to be at least in the you know the minimum order that we have to take is a cubic polynomial okay and similarly when we look at the temperature equation here we can make approximations ranging from linear all the way to in a quadratic or cubic like the way we did the external force convection so for the present case .

We will consider a quadratic variation in the temperature profile okay and the other approximation that Squire did was because we have basically several unknowns here we do not know the reference velocity we do not know the momentum boundary layer thickness and also the thermal boundary layer thickness however we have only 2 equations therefore he made the assumption that the two boundary layer thicknesses are approximately the same.

So this is valid if your Prantle number is close to 1 it can be between 0.7 to something like 1.2 1.3 which most of the gases are within this particular range of prantle numbers and therefore this is not a bad approximation to make okay now in order to find these coefficients we have 4 coefficients in the velocity profile and 3 in the temperature profile we have to write down the suitable boundary conditions for them.

So let us list out what are all the boundary conditions in terms of velocity and temperature so what are the important boundary conditions.

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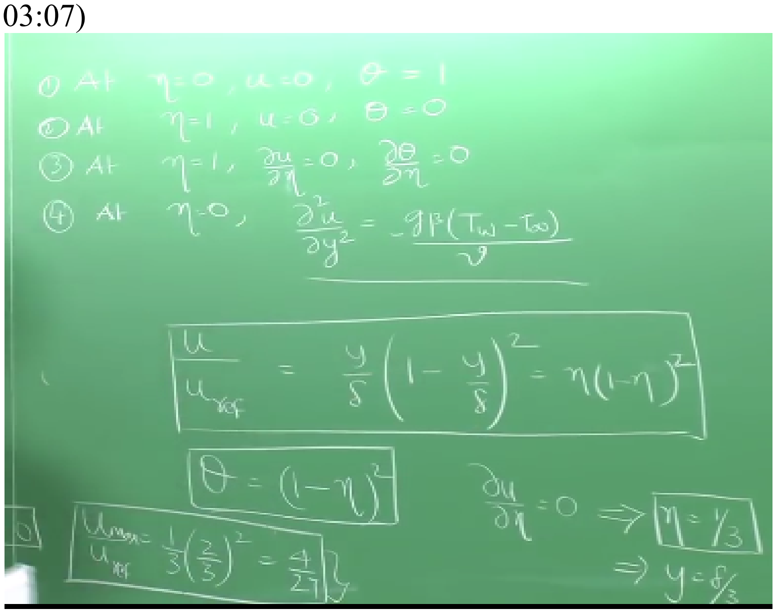

We can write so this is your Y and this is your X so y =0 is nothing but We write down the integral formulation of the boundary layer equations for natural convection η =0 what can we write at η =0 μ =0 and what will be the second boundary condition at η = 1 ,μ = 0 again we do not have any velocity outside the boundary layer in the natural convection case and number 3 we need totally four boundary conditions so what will be the third boundary condition .

We can write hmm where do you want to write the third boundary condition at the wall or at the boundary layer this at what η =0 or 1 what is the condition D u by dy or D u by D η =0 so before we give a boundary condition to the higher-order derivatives we have to give it to the first order derivative okay so if you apply the fourth boundary condition at the wall what it could be so $d^2 u / dy^2$ = hmm write down the momentum equation from the momentum equation tell me $g \beta (TW - T \infty) / \mu$ and then – sign okay .

So you see that the boundary condition at η =0 is for the second derivative so before giving this we have to give a first derivative boundary condition at η =1 we cannot directly jump to this okay so therefore now we have four boundary conditions we can substitute into this polynomial here okay so try to find out the four constants or coefficients I will list down the final profile okay after you do all the manipulation.

So you will end up with u by so this is the final profile that you will be getting after all the manipulation okay yeah 1- y by Δ the should be yeah it should be $(1 - \eta)^2$ here right that is correct okay now I think then second part finding the temperature profile you can do it quickly here so let us write down the conditions we need three boundary conditions so at η =0 what is your value of θ= 1 and θ= θ and then third boundary condition D θ by zero0 so apply these three conditions quickly and find out the temperature profile in terms of θ.

So what is the coefficient A_1=1 okay so what do you get finally if you put them what are the coefficients B_1 and C_1 so B_1 will be - 2 and C_1 =1 so therefore the temperature profile comes out to be simply $(1 - \eta)^2$ okay you substitute it so you will be getting nothing but $(1 - \eta)^2$ this is $1 - 2\eta + \eta^2$ which is $(1 - \eta)^2$ okay.

So not now that you have the approximate profiles can you check the location where the Maxima is obtained so now that you have the approximate profile a cubic velocity profile you still do not know where the Maxima can occur which location which location at which y so can you now use this profile and find out find out two things one is the location the other is the magnitude of U max ah 1x3 okay so if you apply the condition D u by D η =0 so this is the condition for finding the saddle point okay.

So from this it turns out to be that η = 1/3 will satisfy this condition ok and now what is the magnitude u/ u reference at η = 1/3 anybody has a calculator can check this 1 /3 $(2/3)^2$ what it is 4/ 27 okay so η = 1/3 so what is the corresponding value of y so Y therefore δ/ 3 correct so if this is your boundary layer thickness Δ 1/3rd of this so we are actually not drawing this correctly here so if you take 1/3rd it will be somewhere like this and then so this will be where it is speaking .

So this location here is your Δ/ 3 and this is where the maxima exist this is the Val u max yeah you may well I mean we do not know what is this reference okay in the beginning it could be anything okay now from the profile you are getting actually where is the exact value of the Maxima if your u = u maxima I mean now this is not correct because the location is coming out to be 1 / 3 and also u /u reference is not coming out to be 1 it is coming much smaller than that so your reference is actually somewhere either this side or this side.

Okay you understand your reference is some value which is actually 4/27 th of u reference right so this is the maximum so your u buy the this is your this is your actually the value of your u can say this is your maximum your Maxima is actually 4/27 th time so your u reference is some value okay you are not worried about what it is but you are relating it with the u Maxima through this polynomial it should come out to be negative does it come out to be positive it should come out to be it should come out to be negative huh it is negative right yeah it should come out to be negative well.

Okay so this is only for when we assume a cubic polynomial if you assume a quartic polynomial this might slightly change okay but this is an indication telling you that you can have any reference velocity but if you know the profile approximately you can actually get the value of U max so in this case the U max is actually a small fraction of u reference so that means what can it say about u reference is much larger than u Max and can you pinpoint any value here which is much larger than U max no.

So this is some value that you have put as your reference it does not matter okay however what is important is finally you are able to relate your u reference with respect to your u max okay so if because you do not know strictly what is the value of u max if you want to use that as a reference you do not know what is U max exactly but you reference you know because we estimated from the scaling analysis that is nothing but √ of G β T wall - T ∞ x H this is the approximate order of U reference okay .

So from this therefore we should be able to get some idea about at least where the maximum velocity profile occurs which we have seen as 1/ 3^{rd} of the boundary layer thickness and this again varies you see the boundary layer thickness itself varies with X so the location correspondingly changes and we have also established a relation between U max and you reference okay yeah so u reference is some value theoretical value it is not exactly lying on the profile .

So this is what you have to basically conclude okay because you cannot practically identify a point which is larger than u max correct so u max is a small fraction of your u reference means u reference has to be some hypothetical or theoretical no value which is not identified in this particular profile okay but that does not matter because finally once you relate your u reference to UMAX so you can always plot it on a physical diagram okay.

So you can always now re change the reference you can use u by u max and you can use the scaling factor to scale it so it does not matter okay so what we will do next is once the profiles are obtained so you can substitute back into the integral equations the momentum and the energy integral and you can of course integrate it out because unlike the external force convection you do not have two boundary layer thicknesses so you do not have to define a Prandtl number and then say prantle number ˃1 you can therefore neglect the ratio of δ T by Δ and all that okay.

So you can directly substitute it you have only Δ this is the only boundary layer thickness but what you do not know is also your u reference okay so the exact magnitude of u reference is also not known so therefore if you just simply substitute and integrate it with respect to Y.

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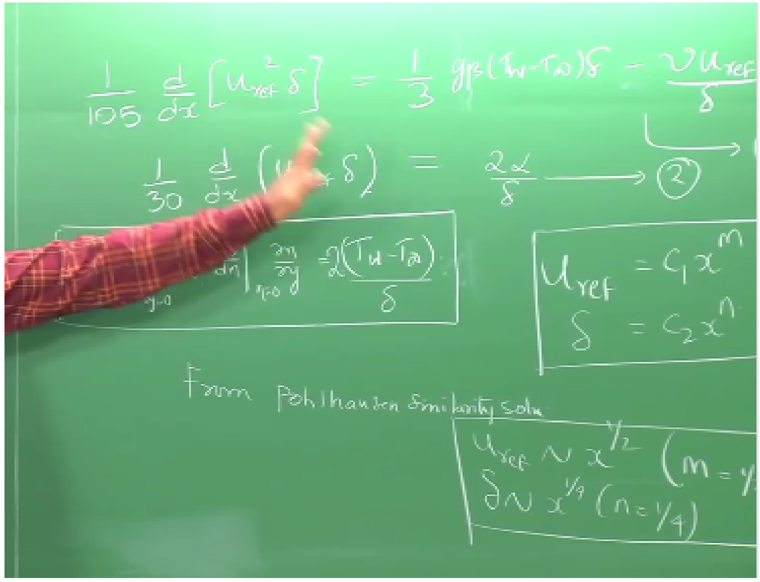

So this is the following equation that you will get so I am assuming that you will be able to do the integration so that is if you substitute for u ^{2} as u reference ^{2} x η (1 - η)^2 okay into this so integral 0 to 1 Δ x D 1 - η okay .

So this integral will now lead to 1 /105 okay similarly on the right hand side you have if you substitute for your temperature profile into this so this will be integral 0 to Δ G β into so in terms of θ you can write this as θ (T wall -T ∞) dy and you can substitute the profile for θ as (1 -η)^2 and integrate it the others are all constant G β T wall - T ∞ so if you integrate it you will get 1 / 3 as the constant and then you will have G β (T wall -T ∞) Δ so dy you can write it as Δ x D η okay and you also have -μx D u by dy at y= 0 now what will be this if you substitute this profile what will be the value of d u / dy at y =0 indicates what η= 0.

So you will be getting have -μ into u ref / Δ okay so in terms of D u by dy okay so this is your momentum integral after you substitute the approximate profile similarly the energy integral also can be reduced so I will just write down the expression here 1 / 30 times D / DX u reference times Δ so that is you are substituting for you and t - T ∞ as θ (T wall - T ∞)okay so if you do the integration you get 1 / 30 and on the right hand side what do you get for - α DT/ dy at y= 0.
So what will be so this is 1 - 2 η + η^2.
So what is D θ / D η at η =0 becomes -2 therefore how do you transform this you can write therefore DT/ d y as DT/ D θ x D θ / D η at y= 0 so this becomes η =0 x D η/ d y okay so from your definition of θ your DT / D θ is nothing but T wall- T ∞ and what is D η/ D Y η =Y/ Δ so D η /dy should be 1 /Δ this is by Δ x D θ by D η is -2 so minus of this will become just 2 times okay so therefore when you substitute this into this expression so basically both sides t -t ∞ can

be written as T wall - T ∞ x θ so T wall -T ∞ will get cancelled and you will have 2/ Δ so this will be therefore 2 α / δ is that okay.

So this will be your energy integral so now therefore so we have two equations two odies in terms of what u reference and Δ two unknowns okay now how do we find these two so one thing what we can do is make a polynomial approximation for u reference and Δ which varies with X because in order to do this differentiation we have to know what is the variation of u reference with respect to X Δ with respect to X which we do not know okay .

So therefore let us make an approximation that u reference =c_1 some x^m so this is how it varies okay similarly let us make an approximation then Δ varies as c_2 some x^n from the scaling analysis when we did the similarity solution we know these constant exponents M and N do you remember what this m and n where if you from the similarity solution so from the original pole house and solution .

So what was u reference the order of magnitude of u reference was G β δ T X square root so therefore there it was $x^{½}$ okay so m should actually come out to be =1/2 but since we right now assume we do not have any knowledge of similarity solution we cannot force this condition from this okay we are doing this independent of whether the similarity solution exists or not so whatever we have done in this case we have not got any input from the similarity solution okay.
So therefore this is what we know from similarity solution and from the approximate solution we should be able to extract the value of M as 1/ 2 and what about n what is the dependence on X x / grashof number to the power 1/ 4 okay so we have 1- 3/4 that is what $x^{1/4}$ okay that is n= 1 /4 so this is what we should be finally concluding from the approximate solution so what we will do we will just let us for the time being assume some x^m and x^n so we do not know these exponents.

Let us put them into the equations 1 & 2 here okay then in order to balance the order of equations on the left hand side and right hand side the terms we have to equate the powers to be the same so you cannot have in order to dimensionally satisfy the equation the order of X on the left hand side and right hand side should be the same on the left hand side you cannot have x^3 and the right hand side you cannot have $x^{1/4}$ then dimensional it not be consistent okay so what we will do is we will substitute this into this equation and then equate the order the exponents on both sides and then from there we will deduce what will be the value of m and n okay.

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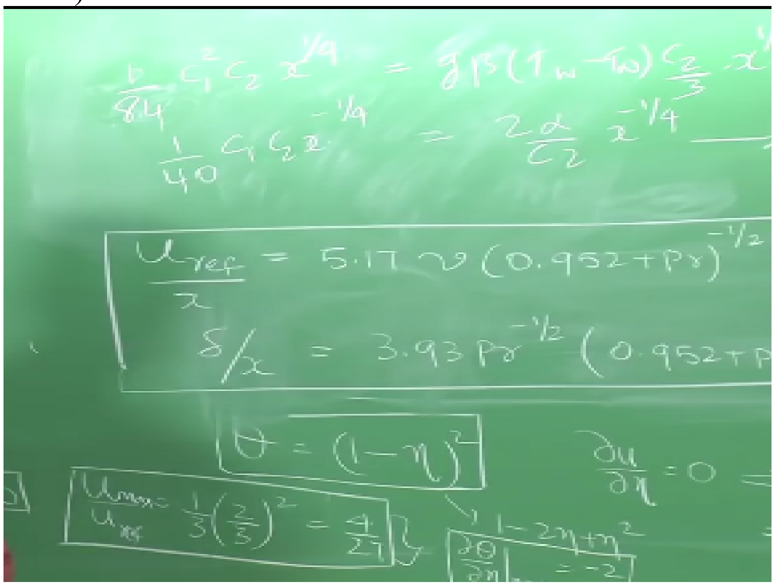

So if you substitute c_1 x^m into the u ref and C_2 x^n the momentum integral will be what is more important is the order of X okay do not have to worry about the other constants which are multiplying them because they are all just constants okay so all this but only the order of x you have to pay attention okay so that has to be equated the same on both the sides so equation 1becomes 3 and 2 becomes 4 so from this therefore in order to make sure that they are dimensionally consistent from Equation 3 what is the condition 2 m+ n- 1 should be = n which should also be = m- n okay now from the second from the equation number 4 m+n-1 should be = - n okay .

So if you solve this you will be able to satisfy this with values of m=1/2 and n =1/ 4 okay so even directly from this if you equate n and n cancels you have m =1/2 and then if you again substitute m =1/2 you can get the value of n from this second equation and the same thing will satisfy also this equation okay so m=1/2 n =1/4 satisfies both correct so now therefore you see that this is correct because from these similarity solution scaling analysis that is what we have already obtained.

So now we can at least be assured that the order of magnitude of values we get from approximate solution will be correct because we are predicting the behavior variation with respect to X correctly now the next step is to find out what is the exact value okay so now since we know m and n so this we can substitute for m and n in the equations 3 & 4 and what we have is 2 equations 2 constants C 1 and C 2 correct so these constants are coming from u reference and Δ okay.

We now know m and n only the 2 constants C 1 and C 2 have to be determined so if you substitute for m and n so can you please quickly calculate if you substitute m =1/2 n = 1/2 what is the how does this what equation 3 reduced so what will be 2 m + n / 105 3/ 140 into what is the exponent of x here 1 / 4 and 1 / 84 and this should be $x^{1/4}$ right on this side we have G β T wall -T ∞ x C 2 / 3x $x^{1/4}$ - C1/ C 2 μ $x^{1/4}$ ok the powers of X should be same and similarly from this equation what is m+n/30 please check calculate once again 1/ 40.

So let us call this is 5 and 6 so you have 2 equations for two unknowns constant C 1 and C 2 okay so now if you solve them solve these two equations if you find it too cumbersome to solve you can put it in a symbolic manipulation mathematical package like Mathematic or maple you can simply copy paste these two equations and asked ask it to solve for C 1 and C 2 okay so finally you will be able to therefore get C 1 and C 2 and from that therefore the expression for u reference because u reference you have assumed as C 1 x^m so we will get the constant C 1 and M is already determined to be 1/ 2 okay.

So the final expression comes out to be u reference / X = 5.17μ(0.952 + prantle number)-$^{1/2}$ and grashof number ^{½} okay so if you group all the terms defined grashof number prandtl number so this is what you will end up with similarly Δ / X so Δ is C_2 X^n so once you get C 2 so you can also get the expression for Δ 3.93 x Prandtl number $^{-1/2}$ so therefore finally so we have the correct expression to determine the exact magnitude of u reference and Δ okay so for a given local grashof number and prandtl number you can get the exact value of u reference okay .

Now you see that u reference is actually a function of X so there is nothing like a constant reference for the entire plate so each location the u reference changes okay and similarly the corresponding variation in the boundary layer thickness is also obtained so now what is the final step so you have got your 2 unknowns so what is remaining hmm what is remaining to be determined Nusselt number finally okay.

We have the exact solution from awestruck okay now similarly we have to get a correlation for nusselt number from this so how do you get nusselt number .

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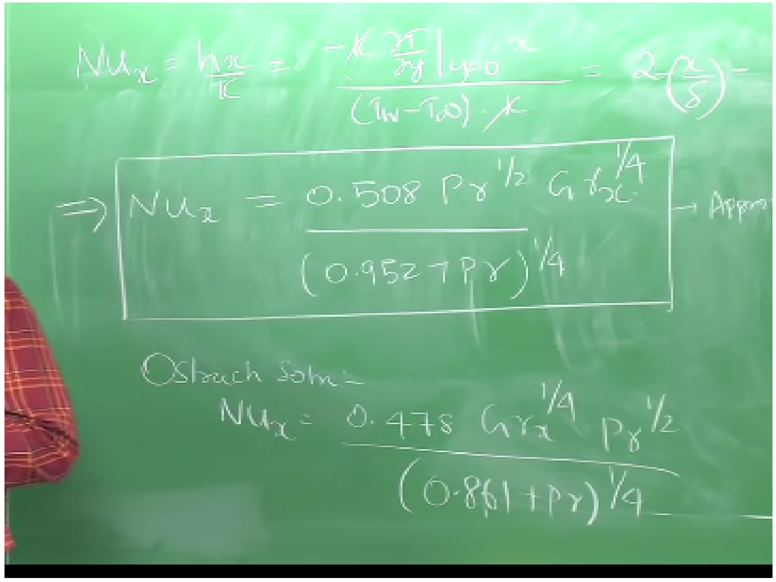

So local Nusselt number Hx/ k which is -K DT / dy y =0 divided by T wall - T ∞ x K so we already have the profile for temperature and we already got DT /dy at y =0 what was it - DT /D Y at y = 0 2/Δ that is it so we have KK cancels here therefore 2 X / Δ x T wall - T ∞ mean so this is nothing but if you write in terms of D θ so I think this will be absorbed so it is nothing but simply 2 x / Δ so it is exactly inverse of this okay .

So therefore we can write the final expression as 0.508 prantle number $^{1/2}$ what would be crash off number to the power 1 /4 divided by we have 0.952 + $(PR)^{1/4}$ okay so therefore we have the final expression right so compare this with the ostrich solution what was the constant value that in the case of ostrich solution cell you point let me just inform 0.67 6 so we have 0.67 6 but we have to also divide it by 4 ^{¼} so that comes out to as 0.478 0.47 8 times rash of number to the power half into tantalum rash of number to the power 1 by 4 prantle number half divided by 0.8 6 1 + PR $X^{¼}$ okay .

So if you compare these two okay I mean of course there are some variations in the constant coefficients both in the numerator and denominator but if you calculate the absolute value for a fixed value of prantle number so then this comes out to be very close okay so this value is also slightly smaller this value is also smaller so finally you will get a very close match between the exact solution and the approximate solution okay .

So therefore doing an approximate solution in this case also we will be able to get satisfactory agreement okay and but however compared to the external force convection this is not so trivial okay you have to guess the variation of u reference Δ and then you have to calculate the exponents and then coefficient so it I s a little bit round about okay.

The external force convection it was very straight forward you know your reference velocity that for only δ and δ t have to be calculated in this case you have to make some approximations for that okay but nevertheless I mean I would suggest that if you are not interested in numerical solution so then the approximate method will give you a straightforward correlation otherwise a similar effort will lead to a new ordinary differential equation which has to be again numerically solved okay.

So the same procedure can also be extended to constant heat flux boundary condition since I think you have already done this for the external force convection you should be able to repeat similar procedure for by extending the square solution okay yes see that prantle number equal to one yes we started off because that is to make an assumption in the reducing the number of unknowns but does not mean that we will be omitting μ by α we are not saying that we should always assume μ by α =1 okay still that factor μ by α creeps up here you have α so therefore here you have μ .

So when you calculate you will get the non-dimensional prantle number okay this need not be exactly one but to make the analysis simple we used this approximation here that is all I mean that that is also telling you that both boundary layer thicknesses are nearly the same so there is no point in differentiating between the two but when it comes to the thermo physical property they are actually different.

So you are correct in the sense that although we got this we cannot use this for very low parental numbers and very large parental numbers this has to be applied the Pantanal numbers are close to one so in fact most of the natural convection problems are done with gases okay you do not force very large parental numbers are very small Prandtl numbers into you know natural convection mode right so we will stop here and then in the remaining class that is in the evening at five o'clock we will complete the rest of the natural convection parts because so far we could find exact solutions our approximate solution.

But for most of the problems in natural convection especially where you have other configurations like for example the same plate placed horizontally instead of vertical then you have more complex boundary layer growth for which we cannot find exact solution so we will look at some empirical correlations for those configurations and then for other geometries like cylinders and spheres so there are no simple solutions again okay so we will I will just give you some empirical correlations and quickly go on to the internal configuration in internal flows how natural convection happens.

once again that we cannot get a closed-form solution so all we have to rely on numerical solution or experiments so mostly we are all correlation based I will quickly go over some complex configurations and the corresponding correlations and along with that I also want to suggest you a project with the natural convection this is the final project which we will be doing so that also I will give you an idea in the evening class okay thank you.

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