Lecture 32
Extended Graetz problem with
Wall flux boundary condition

So today we will look at a different kind of a solution to the extended grades problem so the original grades problem was done as I said for the case of a plug flow with a constant w temperature and that was extended later by sellers to a parabolic velocity profile and we also saw the solution to that so now we will quickly visit the last of the extensions that is with a constant w flux boundary condition okay.

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So we will look at the extender so this is of course the grades problem for the thermally bore developing region for the constant w flux boundary condition either it can be with a plug flow velocity profile or a parabolic flow velocity profile if you take the case of for example a plug flow kind of a velocity profile you start with your energy equation U DT / DX is = α into 1 / R into D / DR of R DT / DR okay.

So this is your basic energy equation let us what we will do here is we will not try to non-dimensional the temperature because we have a constant w flux condition and therefore and any attempt to kind of define a non-dimensional temperature will be futile because if you define your ϑ something like T - T w / TI - T w so there it was useful and then you had a constant w temperature okay.

So you could take out of the differential and little cancel off but in this case your w temperature is a function of X so therefore you cannot define your non-dimensional temperature this way so it is better to keep it in the dimensional form okay and then try to solve so what we will do however is to non-dimensional the coordinates for X and R okay so we will write that in terms of Z and so we will define Z as X / R 0 divided / pettily number and my non-dimensional R is H which is R / R 0.

So if you substitute this into the given equation energy equation so you get u DT / D Z so from here you can write your X as Z into R 0 into pettily number now pettily number is nothing but Ro into UM into D / or you can say UM D / α okay so this I can substitute in place of X so this becomes R 0 UM D and the α goes to the numerator here so this is = α 1 / H into D / D H into H DT / D H of course they are not here and here cancel so I have a hard knot square which I have to multiply it outside so my α cancels here.

And I can also cancel off my R 0 so this I can write as to 2 times R 0 so this becomes R 0^2 which also cancels off and therefore I get.

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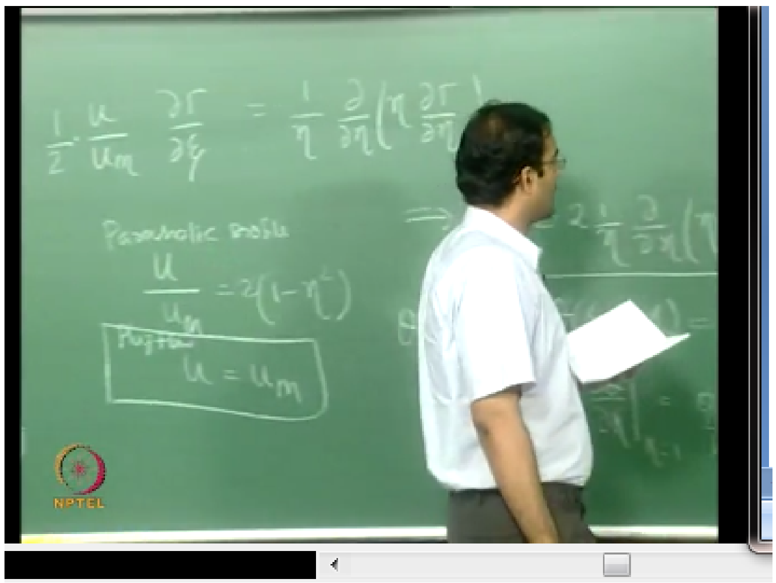

U/ UM into 1 / 2 DT / D Z which is = 1 / H D / D Z into H into DT / D H okay now come now comes the velocity profile which you want to use if you want to use a fully developed parabolic velocity profile then you substitute for the appropriate relation so you know that U/ what is the fully developed profile you buy UM is = twice 1 - H ^{2} in terms of the non dimensional coordinates so directly you can substitute for u / twice um directly as 1 - H ^{2}here if you want to go for a parabolic flow okay.

However if you want to go with the classical how great started with a plug flow then you can say that your U is = U N so this is for a parabolic profile and this is your plug flow or slug flow whatever you want so therefore let us assume for the time being that yours is a plug flow so substituting this you get DT / D Z is = twice of 1 / H into D / D H DT / DT okay so this is how your energy equation can be written and now we have to state the boundary conditions okay.

So now if you substitute the parabolic flow you have 1 - H^2 here okay 2 cancels off you have 1 H^2 so till here it is fine so now we have to state the boundary conditions for the constant w flux case okay so T corresponding to Z = 0 and any value of H TI and coming to the boundary conditions with respect to H here you have a constant w flux at R = R 0 so therefore you should rather write DT / D R or DT/ D H at H = 1 is = Q 4 / K this is your condition okay.

So this is your w flux which is K DT / D R at H R = R 0 this is basically DT / D H Z = 1 so this is a constant and what is the other boundary condition at H = 0 T should be finite R DT / D H at H = 0 should be 0 so this implies symmetry in the profile okay R this also is equivalent to saying P at H = 0 should be finite okay so now you see the problem so you want to define an high grain value problem here however the direction of the high grain value problem you have a non homogenous boundary condition.

So therefore the question is how will you convert this into a high grain value problem okay so here is where we introduce a particular technique to do that I have posted the solution for a constant w flux but in a Cartesian coordinate that is for a channel flow on the Module I have worked out the solution and I have posted it you can just go through it and a very similar treatment has to be done for the pipe flow case also.

Just I will give you the overview I think after that you can go through the document and it is very straightforward process okay so what I am going to do is I am going to now introduce ϑ here which is of course not non-dimensional but I will say this is T - TI okay so therefore I can replace this since TI is a constant okay niche in that is the inlet temperature so I can replace this T with ϑ and the condition here becomes at Z = 1 Z = 0 ϑ will be 1 0 he will become TI okay.

So that is the advantage so what I am trying to do is a wherever possible I can introduce zeros I am doing it and then this thing this will be in terms of ϑ this will also be in terms of ϑ anyway TI is a constant so if you differentiate it does not matter now I am going to assume that.

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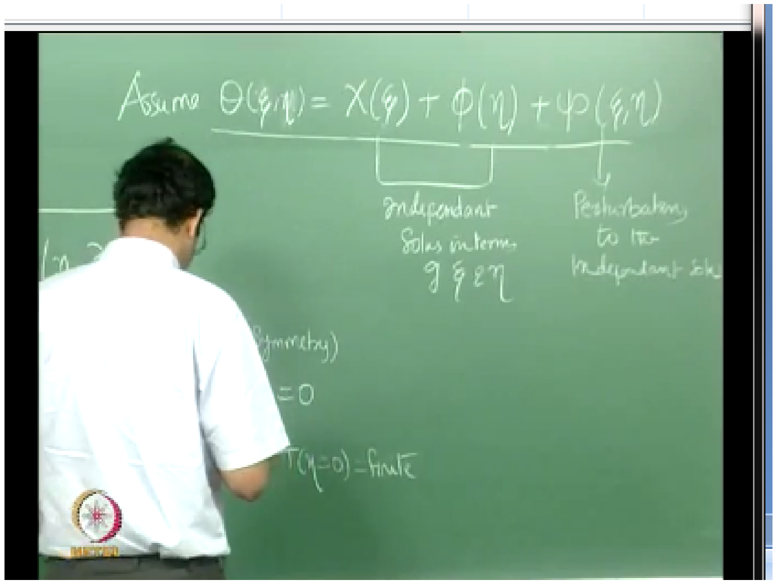

ϑ of X , Y can be written as independent solutions or sorry here it is X , or Z , H and we returned as some capital X which is a function of Z + V which is a function of only H + the perturbation which is a function of both ϑ η so see now the this assumption works well because this is a linear operator the equation is linear and therefore if you assume a linear combination of solutions that should also be a solution okay.

So you are assuming that your actual solution consists of two independent of two independent solutions one which is only a function of Z the other which is only a function of H and of course you know this is just an assumption on top of it the actual solution can be obtained if you have a perturbation to this and that perturbation is you call it as I which is of course the perturbation has to function of H and Z so that brings out the interaction between the H and Z tours okay.

Whereas this purely talks about an ordinary differential equation separately an order differential equation separately and this is a mixed solution okay so this is this is basically independent solutions in terms of H and Z and this is the perturbation to the independent solutions and since the patch partial differential equation is linear and the solution is a linear superposition of all the solutions till this will be a solution to the governing equation so now you see the advantage so once you substitute this into the governing equation okay.

So you can write this as you can separate this into two problems one where you first look at only the independent solution and the other where you look at the perturbation solution to the perturbation and then finally add everything together.

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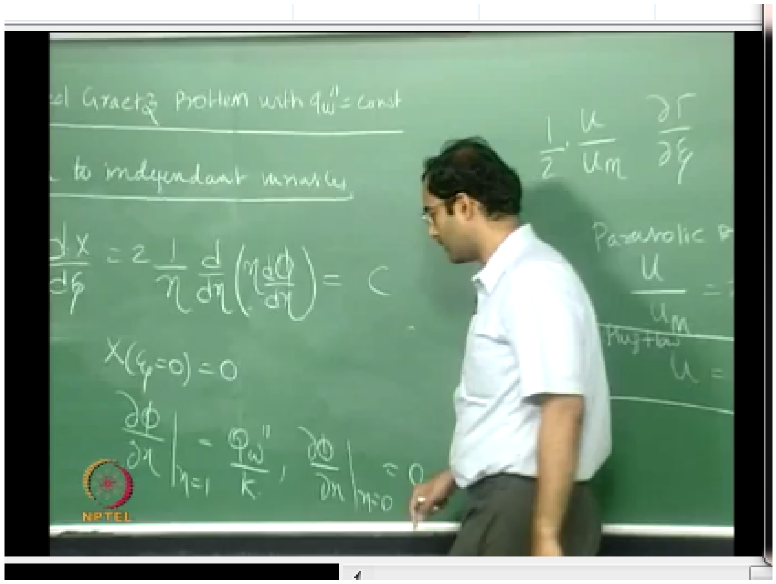

So first part will be the solution to independent variables okay so first what I say is that I have a problem now I can substitute X of Z and Phi of H separately and it will satisfy this such a way that you should have you have DX / D Z = twice one / H into D / D H into H V DV / D H okay so you first assume that your independent variables satisfy this governing equation so you write in terms of the independent variables so you get this as terms of X this in terms of V and this two can be equal if only they are equal to some constant okay.

So then you can find this is a ordinary differential this is a ordinary differential equation this is a first-order second-order OD directly you can integrate and write the solutions for V and X okay so therefore you have two independent solutions and the boundary conditions for this particular problem in terms of X you assume that X corresponding to Z = zero is = zero okay and corresponding to fee you assume that your DP / D H at H = 1 is = Q / K and the other boundary condition is your DP / D H at H = zero is = zero okay.

So you apply these two boundary conditions to this now you get the point why I have written it that way so once the non homogenous boundary condition goes to this the remaining part which is the perturbation will have a homogenous boundary condition okay because once I have found the solutions to the independent variables the solution with respect to fee we will take the non-homogeneous boundary condition now therefore the second part will be the solution to the perturbation .

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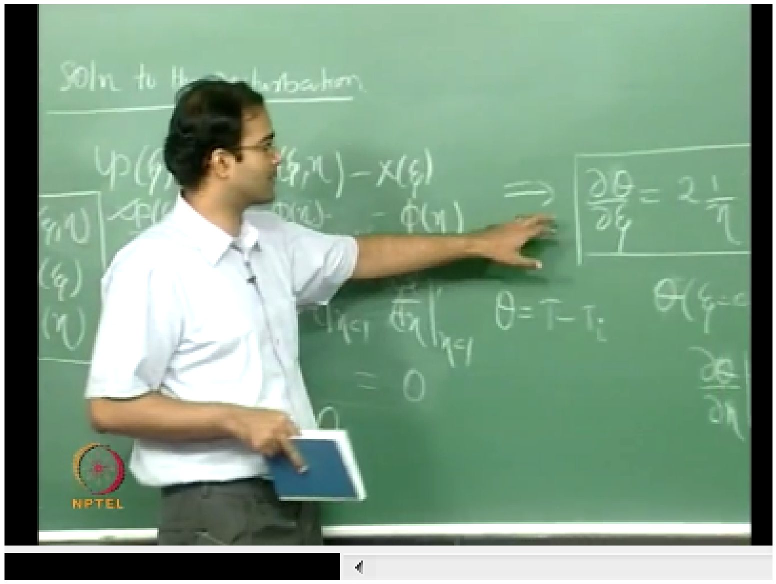

The second part will be solution to the perturbation here so if I write the equation for the perturbation I can just give the boundary conditions so the boundary conditions for say can be written as ϑ - X - V of ϑ okay so now corresponding to this fact of course when I, when I look at DC / D H at H = 1 okay so this will be D ϑ / D H at H = 1 - D π / D H at H = 1 so both will cancel out so therefore this will be 0 because your D ϑ / D H is what QR / K and also D π / DT at H equal to 1 you have forced it to be QR / K so therefore the boundary condition one of them becomes 0 directly.

And the other boundary condition DC / D H at H = 0 anyway both are 0 so that will also be 0 okay so now you see that you have reduced your non-homogeneous boundary conditions to the two you are now solving actually for the perturbation which has now homogeneous boundary conditions and now this perturbation sigh can again be solved / separation of variables okay so now you can say that I can assume that my side η , H is actually some X of Z and some Y of H and then I can proceed with my I can I can now proceed with my regular solution okay.

So what is the remaining boundary condition that is at X at Z = 0 so therefore at Z = 0 your side at Z equal to 0 ,H will be ϑ Z = 0 which is 0 - X of 0 which is again 0 - you have P of H so therefore the boundary condition at H = 0 becomes - P of H okay so these are the boundary conditions so you have the boundary condition at Z = 0 then with respect to H you have two homogenous boundary conditions so therefore if you substitute into this you separate the variables so in terms of Y you will get the high gain value problem okay.

So and that will have two homogenous boundary conditions any guess what will what the high gain value problem will be here will that be a vessel equation it will be a Bessel equation and you already know the solution to Bessel equations we have to just combine the Bessel solutions and then you now the only difference between this and the constant w temperature case where the temperature was 0 at R = R 0 here you have both gradient 0 boundary conditions okay.

So this is this is the difference and after that the solution is straightforward so this is an independent solution that you will get from separation of variables and you already have the solution for X and V which is a straightforward PD to integrate and then finally you superpose all the three solutions and you get the final solution for ϑ okay so this procedure I have very clearly illustrated in that example for Cartesian coordinate system you please go through that and in the assignment I will ask you to do the parallel thing for the duct flows okay.

So only thing there you have cosine and sine here you will be getting in terms of the cell functions so the final solution that comes out it was done / sellers and sellers did both the case of the case where you have a plug flow as well as the case where you have a parabolic flow in the case of parabolic flow you have 1 - H square which is coming here okay when you substitute for the velocity profile you have 1 - H Square and there is no 2 here and therefore the high gain value problem in this case will be what stambouli okay.

The stambouli will kind of a problem so sellers did this and the final solution for nusselt number.

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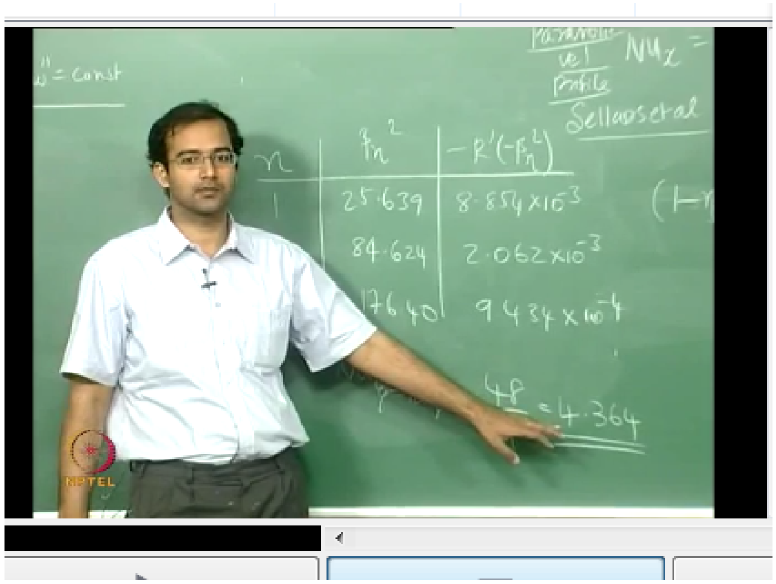

11 / 48 + okay so this our prime here is basically the gradient of the high grain function which is basically the high grain function comes from the solution of this term Louisville problem okay and these are coefficients for different values of N okay so this is β η and now he tabulated the values of β η N and R prime β η N so this is the case for parabolic velocity profile so this is the solution by cell art setup okay.

So for different values of n β η N^2 - R prime - β η N^2 okay so N 123 the values go as 25 . 639 84.624 and 176.4, 8.854 – 3, 2.062 okay so therefore this is what he has done now you can check for large values of Z okay so v large values from this table you substitute and calculate what should be the v for large values of Z so if you assume large values of Z you can say that this is an exponentially decaying function goes to zero the entire term will be very small it will be simply 48 / 11 which is exactly 4.364 just check that.

That come to 4.364 for anyone has a calculator okay so what was so do you remember this number this was the case there where we proved very first for fully developed both hydroynamically and thermally and for a constant w flux condition this was the nusselt number so now this is coming as an asymptotic solution to the sellers pro solution okay so these are the things and of course the sellers also did a case with where he did a constant linear temperature w temperature variation okay.

So he did all the 3 so he looked at the parabolic velocity profile and he looked at constant w flux condition and also linear variation in the water temperature and he has also given the solution I am not going to give that otherwise it becomes too many correlations so he has done all the three so I am just going to plot.

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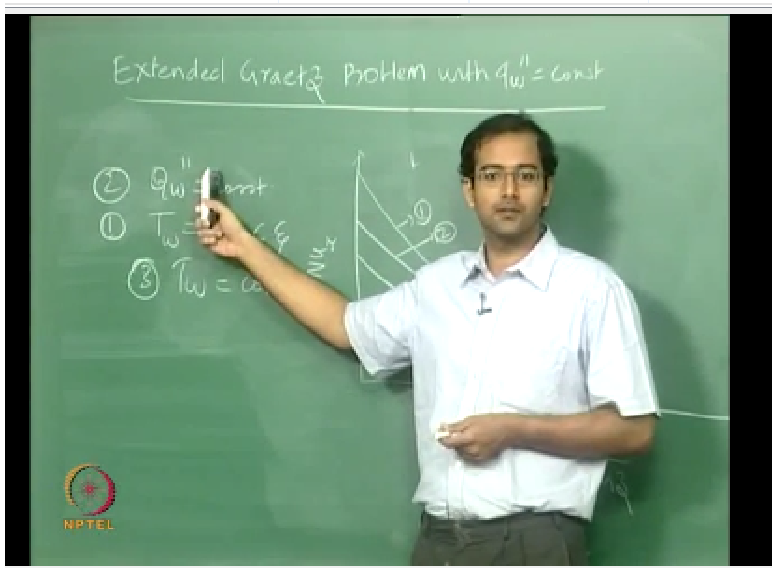

The variation of the local nusselt number with all the three different boundary conditions as a function of X / R 0 / prickly number or X / plot it as D not which is 1 / grades number okay so he has done for 3 different boundary conditions constant w flux and linear w temperature variation which was something like P w is TI + some ζ constant times H this was the linear w temperature variation and of course your constant w temperature.

So now I have marked here one two three so you have to tell me which one corresponds to one okay let us start with 3out of these 3 which one do you think will be corresponding to number three which boundary conditions T w is constant so that is number 3 so between 1 and 2 what could be number 1 Q w is constant how about linear variation of all temperature okay so finally whether it is linear variation of all temperature our Q w is a constant one it becomes both thermally and hydroynamically fully developed they reach the same value okay.

That is why these 2 merge and you see the corresponding value is 4.3 and for the w temperature you have 3.6 can you explain why the two cases give the same value so when you say q1 is constant in the fully developed case we have shown that the w temperature varies linearly okay so therefore it should approach the first case where your w temperature is very varying linearly throughout okay so therefore the two nusselt numbers will have to be the same in the completely fully developed green okay.

So I think this brings to conclusion all our analytical solutions as far as the internal flow is concerned and the last part today which I would like to with which I would like to conclude the analytical solutions will be the.

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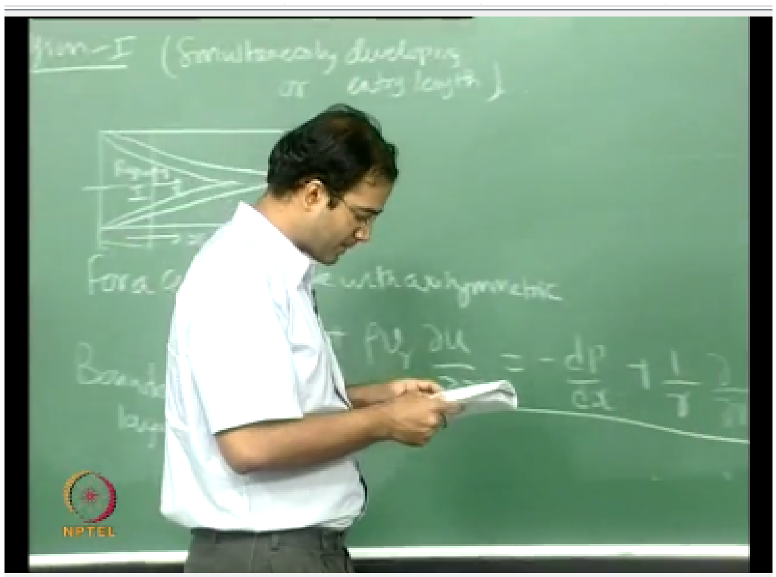

Region one that is simultaneous developing or simultaneous entry length so this is your region in the duct where your hydrodynamic boundary layer is developing your thermal boundary layer is also developing okay so this is your region one okay where both are developed so that is why it is called simultaneously developing or simultaneous entry length now this is a really tough problem okay.

So you cannot make any assumption to the velocity profiles so what the only solution to this is to solve the momentum equations and get the velocity profile simultaneously and the velocity profile will not be a constant it will also be changing with respect to X okay this is a very difficult problem and therefore if you want to solve the complete equations so you have to solve the complete lavatory equations now there are some approximations made to the solution if you say for a circular tube with axis symmetric assumption okay.

So that you neglect your variation with respect to the ϑ direction so that is your axis symmetric assumption you can write your momentum equation in the axial direction that is OU D U / DX + O V R into D U / D R will be = - DP / DX + 1 / R D / D R into R into DU / DR so this equation which we have written right here okay so this assumes that there is no variation of velocity with respect to ϑ direction and also the radial momentum is also negligible okay so therefore you write only the axial momentum equation neglecting the variation with respect to the ϑ direction.

And this equation is very similar to your boundary layer equation in fact this is your boundary layer equation right so this is your boundary layer assumptions remember in the flat plate case we had the same thing UDU / DX + VD U / DY is = if you have a pressure gradient you have 1 / Ro DP / DX + v D^2 U / D DY^2 the same way we have constructed the boundary layer equation in the radial coordinate system okay.

So this is an assumption to the actual problem because what you are doing here you are neglecting D^2 / DX^2 okay that could be important okay because if you look at the acceleration in the X direction is important and also the higher order derivative D^2 / DX^2 also becomes important which you are neglecting here and also it sometimes since its three-dimensional you could also have three-dimensional effects if you have a non circular cross-section okay.

So these are also not taken into account so in fact but you can still get some kind of an approximate solution if you solve this equation for the velocity profile and but this boundary layer assumption is valid only when you have close to the entry length if you move far away once again once the two boundary layers start to merge then the boundary layer assumption will not be valid anymore okay so this is this is what comes out of it but there was a person called Langhaar.

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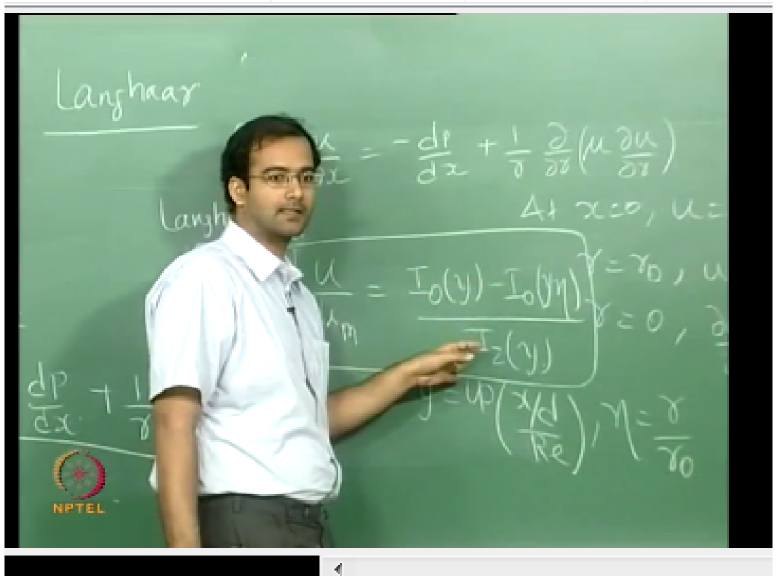

By the name Langhaar what he did was he even neglected defect of radial velocity that means we neglected this term straight away okay the big assumption and then he took only the axial velocity variation into account with respect to X and R he solved this equation numerically and so the resulting equation which is always Ro U D U / D X = - DP / DX 1 / R into D / DR into μ DU / DR so the boundary conditions that he solved at x is = 0 U is = UI.

So you had some inlet velocity which is a constant okay so when it approaches the entrance of the tube you have a inlet velocity which is constant and corresponding to R = R 0 you have u = 0 no slip boundary condition and at R = 0 the profile has to be symmetric DU / D R at R = 0 has to be 0 okay so he solved this numerical of course you cannot find closed form analytical solution because of all these terms right here and he got a velocity distribution which was in terms of Bessel functions okay.

So this has the Bessel function of the 0th order and this is the second order Bessel function the first time Bessel function of the first kind 0th order first kind second order okay so here γ is basically some spy into it is a function of X / D / Reynolds number okay and your H was R / R 0 so this was his velocity profile this is also called as longer velocity profile so you see now the velocity profile is a function of X / D through the γ as well as it is a function of H okay.

So this is the approximate profile that Langhaar got by the numerical solution to this equation and with the following boundary conditions of course he neglected lot of things here so you neglected the radial velocity and things like that so this is not a very good assumption when you go too close to X is = zero because here you have strong radial components of velocity which are in training okay.

So this is slightly this is valid then somewhere intermediate somewhere maybe from here to here okay but nevertheless it is a reasonable assumption so using this profile now you can substitute this into the energy equation so energy equation can also be solved numerically.

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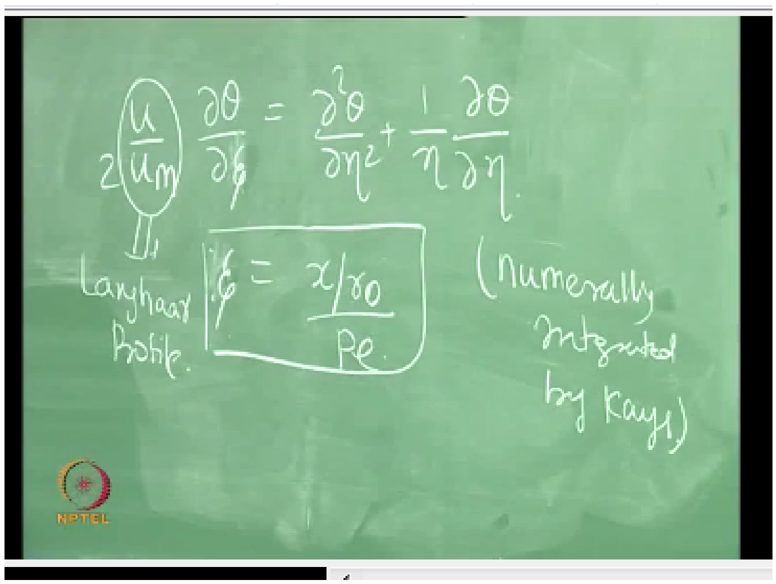

So of course you have your energy equation as U / U M into D ϑ / DX $=D^2$ ϑ / D H^2 + one / H into D ϑ / DT yeah there should be a two here if I define my X as X / R 0 / Neglect number okay so now this U / UM you can substitute from the Langhaar velocity profile okay and once again now your velocity profile is a function of both your x and y or Z or Z and H okay therefore okay so I can use the conventional variables which I used before I will call this is Z just to avoid confusion okay.

So this cannot be again analytically solved because your velocity profile again is not only a function of H but it is also a function of Z so therefore this again has to be solved numerically with boundary conditions whether it is constant w temperature or constant w flux and you get the solution for the temperature and of course the final nusselt number okay so the solution to this equation was numerically done by first numerically integrated by case okay.

For different boundary conditions and we will find the solution to all of that one by one okay he is the same case who wrote the book case and Crawford third Stanford University okay so there are Lord of solutions to basic fundamental heat transfer problems 1950s and 60s which were done by 50 60 70 s which were done by case and what happened was he found out of course solutions but this the person Heusen okay.

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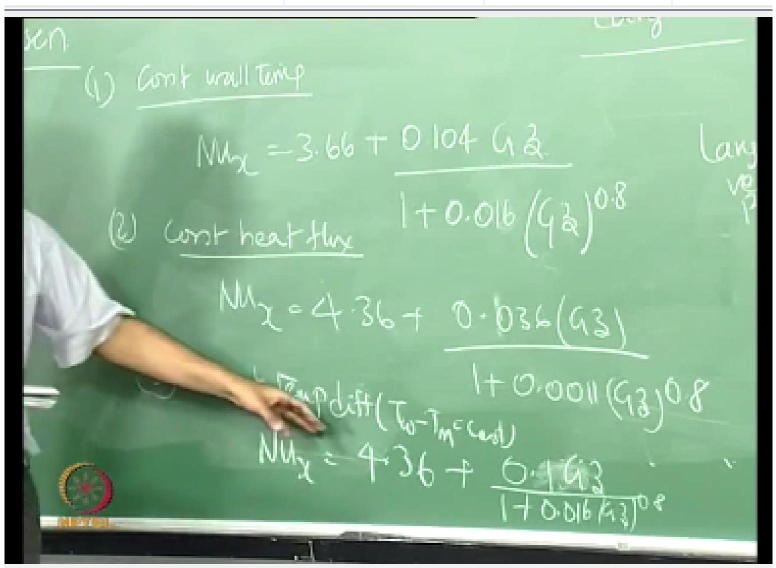

So he comes to a rescue as we saw already that wherever there is a very complicated solution in terms of Bessel functions or stambouli high grain functions this fellow house and was kind enough to do an empirical curve fitting and then give a more easier solutions in terms of only grades number okay so housing came and he took the profiles which were obtained by case and then we finally cast them into a simpler form for different boundary conditions okay.

So the first case was the constant w temperature so all these were from using the longer velocity profiles okay so still they are not the most accurate but reasonable so the nusselt number variation has expressed as 3.66 + 0.104 RE PR / X / D which is nothing but greats number okay so you can write this as directly grades number divided / 1 + 0.016 into rights number to the power point 8 and constant heat flux case so you see the limiting case where your grades number goes to 0 for large values of Z so it goes to the fully developed okay.

So 4.36 + 0.036 into bride's number divided / 1 + 0.00 one grades number or point 8 and finally for the you also did for constant temperature difference okay the constant temperature difference is basically defined as T w - TM is constant okay so for this E 4.36 + 0.036 grades number sorry 0.1 into grades number divided / 1 + 0.016 grades number 2 the PowerPoint 8 okay so constant temperature difference the simplest profile that you can think of where you can have a constant temperature difference is if you have a linear w temperature variation okay.

So that is one case which way where you can think about this and finally both the nusselt numbers go to the same asymptotic limits so these were the simplified solutions by Hausen and of course these are correlations which are much easier to work than the exact solutions are the numerical solutions which were obtained using the Langheer's velocity profile so just to summarize I do not want to now talk too much about how these illusions came because they are all numerical solutions.

And nowadays the more prudent way of doing this is to solve the complete partial differential equation using computational fluid dynamics and directly get the most accurate solution okay rather than putting so much of effort into finding the numerical solution to this approximate equations okay so as far as thermally developing profile has with fully hydroynamically develop profile is concerned we can straight get reasonably straightforward solutions analytically but simultaneously developing profiles are much difficult and therefore we need to go for numerical solutions.

So on a concluding note I will just summarize whatever solutions we developed analytically okay if you look at the analytical solution to the problems.

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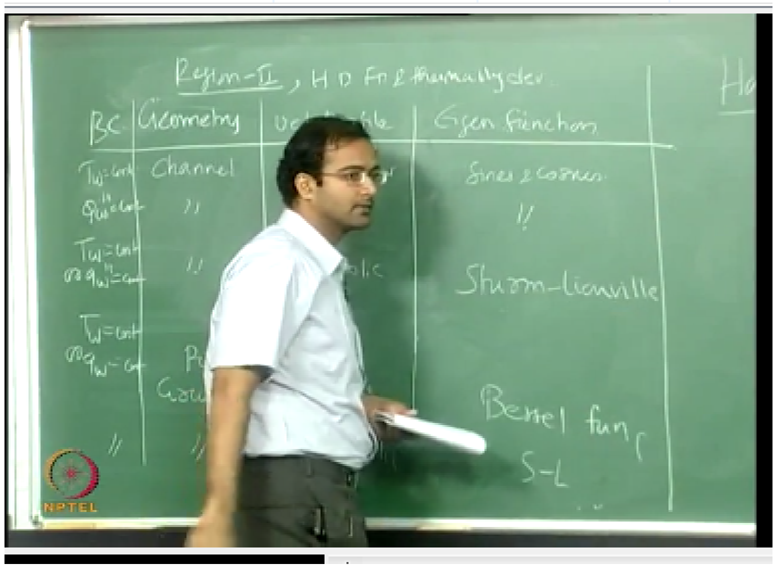

First is the geometry or the configuration and velocity profile and the corresponding eigen value problem or I will say high grain functions in fact I can also introduce boundary condition so this is for region so hydroynamically fully developed and thermally developing so now look at the constant w temperature and if you look the case of channel Club that is in Cartesian coordinate system and if you assume slug flow at a plug flow velocity profile the high grain functions will be in terms of sins and cosines okay.

When it comes to constant w flux the same thing and slug flow but what is that what will be the high grain function still it will be the same only thing you have to go by this approach whatever I describe break up into two independent solutions and perturbation the perturbation solution will still be homogenous with a stock flow will be still sins and cosines now T w is constant or Q w is constant channel flow but a parabolic or fully developed velocity profile.

What will be the high grain function what is that what function Bessel function in a Cartesian coordinate system we do not get a Bessel equation okay and when it comes to T w is constant or Q w is constant but for pipe flow with a circular cross section okay I will say circular cross section if you have slug flow what will be the high grain function that is it okay and the same boundary conditions circular duct and if you have parabolic fully developed then it will be stumped .

So please take note of this so when you solve any kind of problem whether it is a channel flow or pipe flow depending on the boundary conditions depending on the velocity profile this you may alternate between either of these kind of equations okay so one more last table and with that so therefore to summarize all the solutions for fully developed case that is for.

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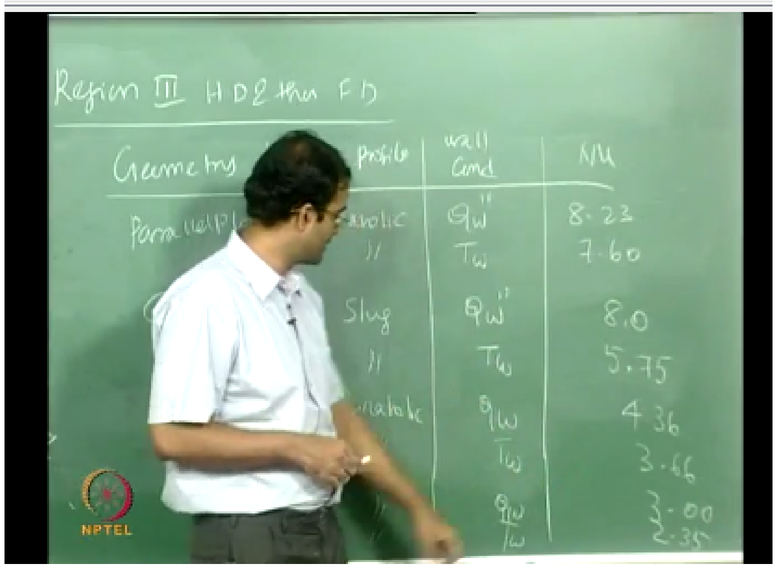

Region three and thermally fully developed so this is all for laminar flow till now okay and when Professor Kohler comes he will start looking at solutions to turbulent flow so geometry velocity profile w condition and the corresponding fully developed a self number okay we will start with a parallel plate with a parabolic profile and Q w is constant the value is 8.23 parallel plate parabolic and P w is constant 7.60.

Next circular tube slug flow and Q w that is 8.0 circular tube and then you have slug flow T w that is 5.75 circular tube but parabolic and then you have Q w you have 4.36 circular tube parabolic T w 3.66 similarly for triangular cross section triangular duct if you have a parabolic velocity profile you have Q w then you have 3.00 triangular parabolic constant w temperature you have 2.35 okay.

So this is the summary of the region 3 results we have already done circular tube complete more or less okay and parallel plate is much easier than this because you will not have any Bessel equation will have straightforward ordinary differential equation for which you can find sins and cosine functions then you can see the order in which the nusselt number decreases so it is the highest for the parallel plate case and compared to the constant w temperature constant all flux has always a higher value of nusselt number then comes your circular cross-section okay.

Your slug flow always has the highest myself number rather than parabolic flow and your triangular cross section or non circular cross sections will have lower value of nusselt numbers okay so this is to summarize all the results so we will stop here and tomorrow and on Saturday the last two classes we look at the approximate methods there is the integral method to solving the thermally developing region.

Extended Graetz problem with
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End of lecture 32
Next: Approximate method for laminar internal flows