Indian Institute of Technology Madras
NPTEL

National Programme on Technology Enhanced Learning
Video Lectures on
Convective Heat Transfer

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Lecture 30

Extended Grates problem for

Parabolic velocity profile

So today we will take up a slightly different problem until yesterday we were doing looking at the fundamental grades problem the fundamental grades problem simply looks at only a slug velocity profile and then you look at the thermally developing region and calculate the solution for the temperature as well as for the nusselt number and finally for the asymptotic case where your X goes to large values you can recover your nusselt number for the thermally fully developed region okay so the great is profile originally which was which was done the great solution done / greats himself was a simplistic case and it was actually 1954.

When three people you know sell ours and try bonsai I think I have uploaded the solution from report original report from 1954 on the model you can just have a look at it so these three people sell are said I will have extended the original grades problem to other cases so the most obvious case will be to look at a parabolic profile okay rather than a plug flow which was a simplistic case and do the same solution with a constant wall temperature boundary condition okay so that is the first extension they did and also they extended it to other boundary conditions like uniform wall flux and also case where the wall temperature is linearly varying okay so these are some extensions of grades problem so we will try to do a couple of these extensions the first extension that we will do today is for the parabolic velocity profile.

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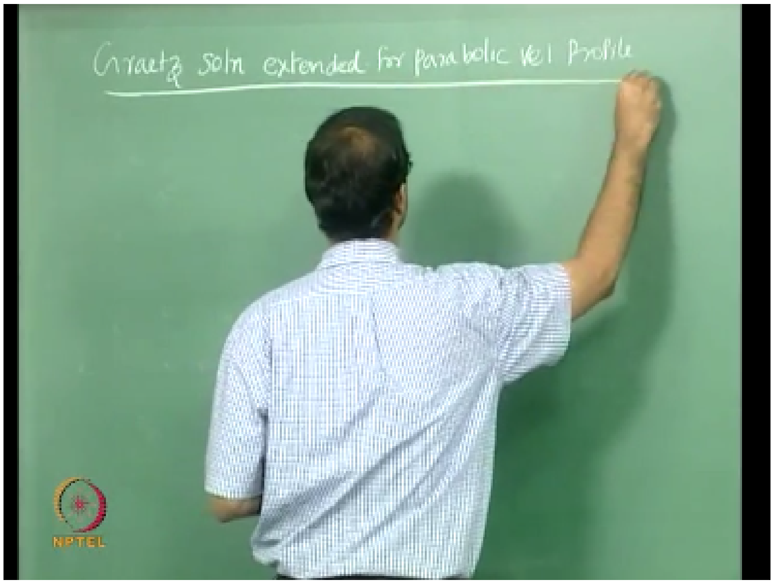

So I will call this as great solution extended for parabolic velocity profile now obviously when the great solution was extended the original great solution for a plug flow was having Bessel functions as the Eigen functions and once the extension was done it was observed that the solution was not as simple as just getting a Bessel function okay so we will look at the nature of the Eigen function how it looks and these Eigen functions have to be estimated in some way or the other that is when sellers did it they did not have access to computers in 1954.

So they tried to do a kind of no mix-and-match approach so they split the problem solution x three parts one for small values of R 1 for middle values of R 1 for large values of R they got three different asymptotic cases and they had patched up the solutions okay but nowadays you can solve the ordinary differential equation numerically directly to get the Eigen functions okay so anyway so the probably the thing is this the solution to the Eigen functions are not as straightforward as what we saw in the original grades problem so now we know the energy equation so we will write down the energy equation still remains the same that is no change 1 / r d / dr dt.

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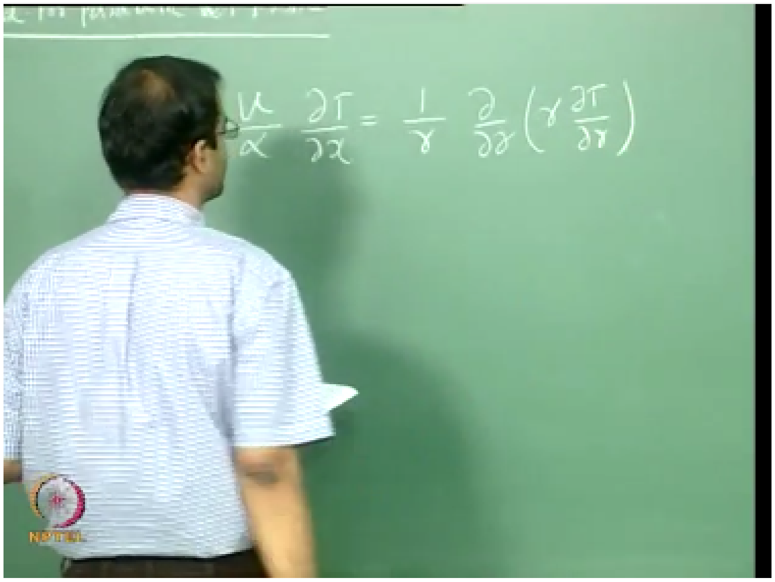

Okay so this is the form of the energy equation will be still working will neglect the axial conduction with respect to the radial conduction and still you have the convection term due to the axial velocity so we assume there are no radial velocity and no variation in the Θ direction now the question is earlier we just assumed you to be a constant and just substituted as some μ now we have to put the actual parabolic velocity profile coming from the full fully developed hydro dynamically fully developed.

Okay to do that first you can also do this in a dimensional form like what we did for the Crites problem you can substitute for you in terms of μ and leave everything in a dimensional form and then you can proceed to the solution but it will also be a little bit simpler to do it in non-dimensional form so we are going to introduce some non-dimensional variables in the beginning and then substitute those non-dimensional variables so I will introduce for temperature Θ like the way I introduced before it is T - Tw / Ti-Tw.

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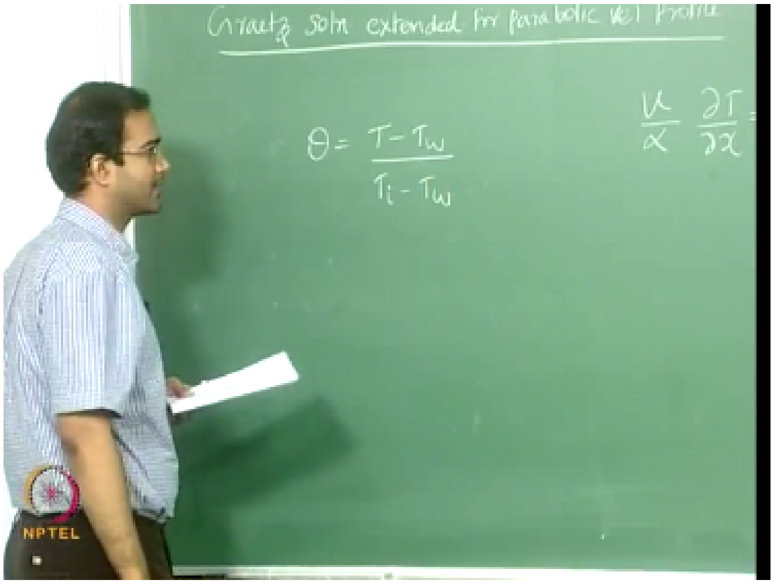

Inlet _ T 1 so that at X is = 0 you have Θ = 1 and R = R 0 your Θ will be 0 okay and at R = 0 your Θ should be finite so these are the boundary conditions for Θ are defined this way and we will also introduce a non-dimensional radial coordinate we will use the symbol H so that will be R / R 0 that is the radius of the duct okay and we will introduce another non dimensional coordinate for the axial location or axial coordinate so we will use the symbol Z for that and we will define this as x/ r 0 /Pe so this is the non-dimensional axial coordinate this is the non-dimensional radial coordinate H okay the way that we have defined this x / r 0 / Pe okay this is this is convenient because once you get the solution for variation in the X you saw that in the grades problem you had a exponential - you had something like X /V 0 / brick lane number.

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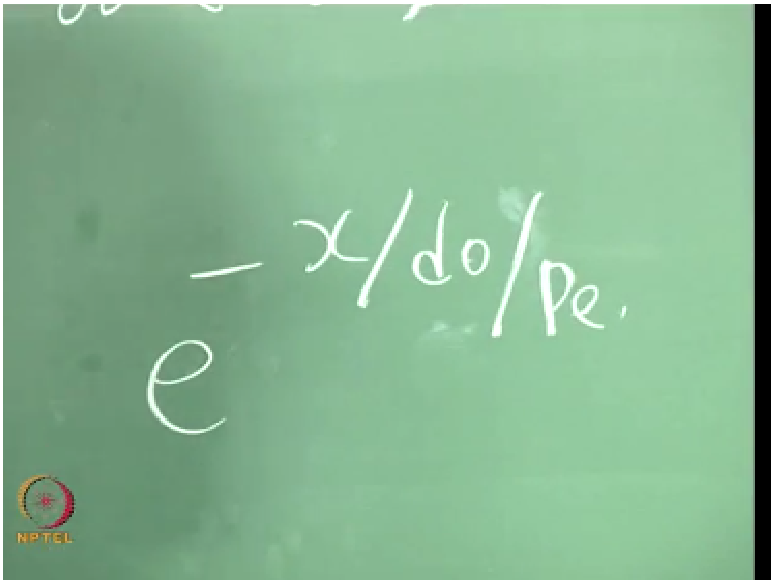

Right so in order to accommodate this variation and group this together in one non dimensional form it is convenient to define this entire group as so this group appears together okay you do not find X separately d separately and trickily number they all come as one single group for variation along X so therefore it is wise to define a non-dimensional variables grouping this together okay now there is another non-dimensional number named after grades okay it is called the grades number and actually 1 over grades number is defined as basically X /B / Pe so this is basically how the grades number is defined.

So this particular non-dimensional grouping of X / D / Pe is referred to as rates number inverse okay so it basically tells you about the non-dimensional axial variation is represented / this non dimensional number called the Wrights number inverse and therefore you can see that if I define X / R 0 / P as one non dimensional group so this is nothing but a function of grades number okay so this comes out as greats number / to exactly okay so this is a this is a one form of convenient form of grouping okay so we will stick to this particular def non dimensional formulation for H Z and Θ and therefore we will substitute this and we will see how the governing equation reduces to we will also use the parabolic velocity profile.

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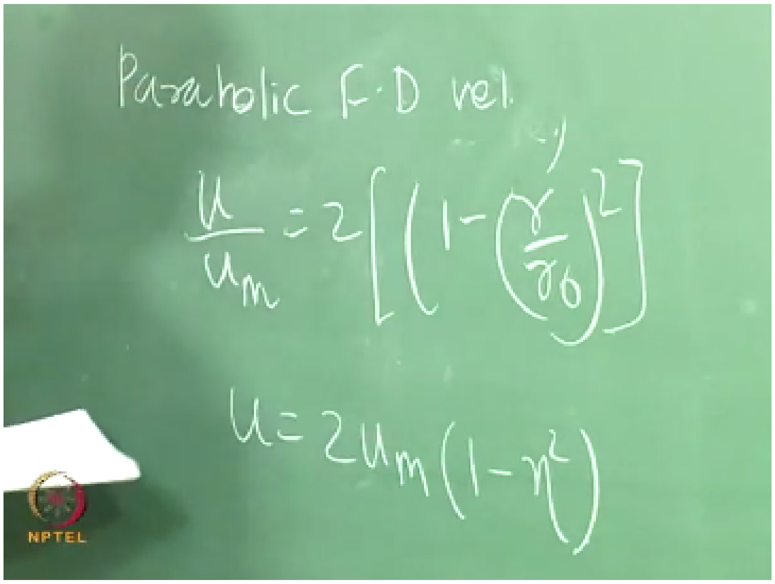

So basically that is you buy M is = twice 1 _ R / R not the whole square okay so in terms of non-dimensional radial coordinate that is 1 _ H square okay so X from here this will be already you know Pe is what this is P μ or μ D / α okay so D or D 0 you can use this is the diameter of the duct so therefore we can express X in terms of Z so this is μ D 0 / α x Z okay so you have in to R 0 which I will write it as d 0 / 2 so you have a term something like this okay or I can express everything in terms of R so I can write this as for R 0 square so this becomes 2 R 0 square okay so this is the transformation I have to do from I have to substitute for X from this expression in so it becomes in terms of Z and for our I can use this transformation so that I can substitute in terms of H.

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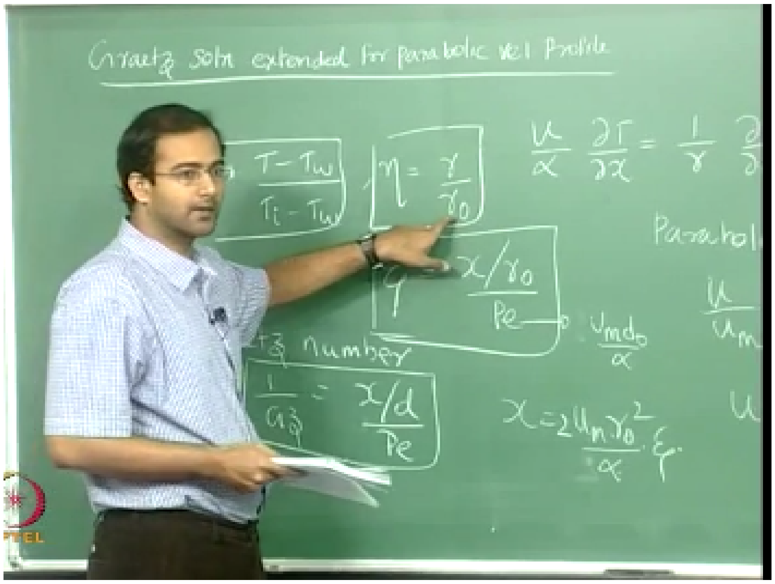

Okay so if you do that so you will be ending up with for you have already to M x 1 _ H square if we substitute the non-dimensional variables let me call this as number equation number 1the energy equation so divided / α x DT / D X so T in terms of Θ that is basically TI _ T wall D Θ / now once again X we can write in terms of Z from here okay so this will be D Z and you have the other terms you have to M are not Square and there is an α coming in the numerator okay so that should be = so when you substitute for R in terms of H so they R0 R 0 cancels between these two and then you have an R 0 square coming.

here okay so you have 1 / R 0 square okay D / D H x H D so I substitute once again in terms of Θ so this becomes D Θ / D H and then I have Ti _ v1 okay the reason why I use this Θ here because my inlet temperature is a constant wall temperature is a constant okay so it is not a function of either X or Θ so for the case of constant wall temperature boundary condition if you non dimensionalize this way so my Ti _ T h will come out of all these derivatives and get cancelled on both sides okay so therefore this cancels off you have R 0 square which cancels off here you have α which cancels off and then to M cancels off right away so finally you are left with a expression 1 _ H square x D Θ / D Z which is = whatever is there on the right hand side okay one my so okay I left one / so that should be won / H here won / H Lee / Θ D Θ / let me call this as equation number 2.

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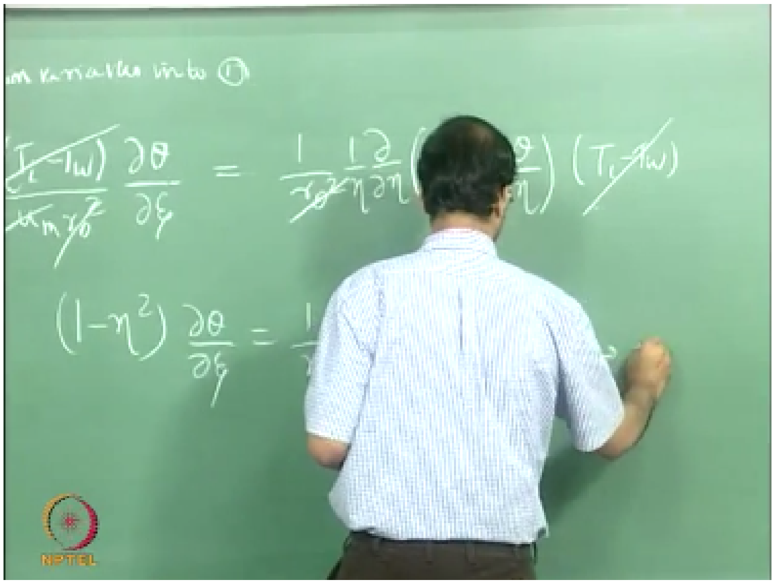

So therefore this is the non-dimensional form of the energy equation after I substitute for the fully developed velocity profile so this has become a little more compact then when I work with the dimensional form right I have so many terms in terms of U I have μ I have α all those things appearing now all those things are eliminated okay so this is a much so I always recommend that you can if you like to work with a more compact form you can define non-dimensional variables in the beginning and then you can use it for the constant wall flux boundary condition.

However you cannot define a non-dimensional Θ like this because there your wall temperature will be vary and therefore you cannot define so you have to just say t _ TI or something like that okay so you have to be careful for the case of constant wall temperature condition you can define a non-dimensional for a case of constant wall flux you cannot do that okay so but still you can for that case also you can define your non-dimensional H and Z okay that will save some put some save some effort in removing.

All this μ and variation in α and all those terms okay so now the boundary conditions for this in terms of Θ to solve this how many boundary conditions we need 3 so inter in directions in ζ direction how many Θ and how many one in Z and two in H direction right so therefore Θ now at any value of H corresponding to Z = 0 that is basically saying that T at X is = 0 so the non-dimensional Θ will be 0 what will happen at if you look at the one so if you look at the problem configuration so this is your region which you are looking now so this is your Δ T this is your Δ.

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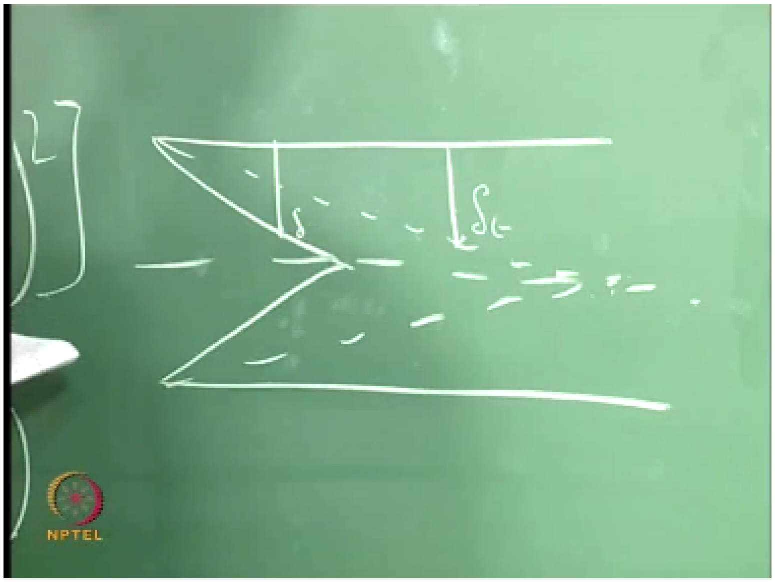

Okay and in fact what you are doing right here it is not something exactly like this but you are starting from a point where it is fully developed that is your coordinate so you can consider that your thermally fully developed region meets somewhere further down okay and this is your coordinate right here this is your X this is your R so once the velocity profile is fully developed then you start looking at the thermal entry length problem okay so you start basically heating from here or maintaining a isothermal condition so in this particular case.

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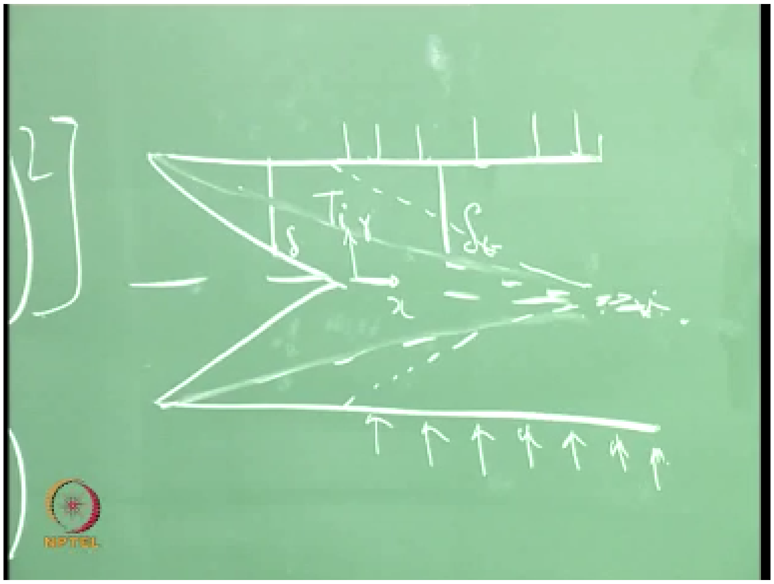

At X is = 0 this is where you are calling this as di a temperature which is entering the thermally fully developed region okay so at X is = 0 Θ should be 1 because T u should be here.

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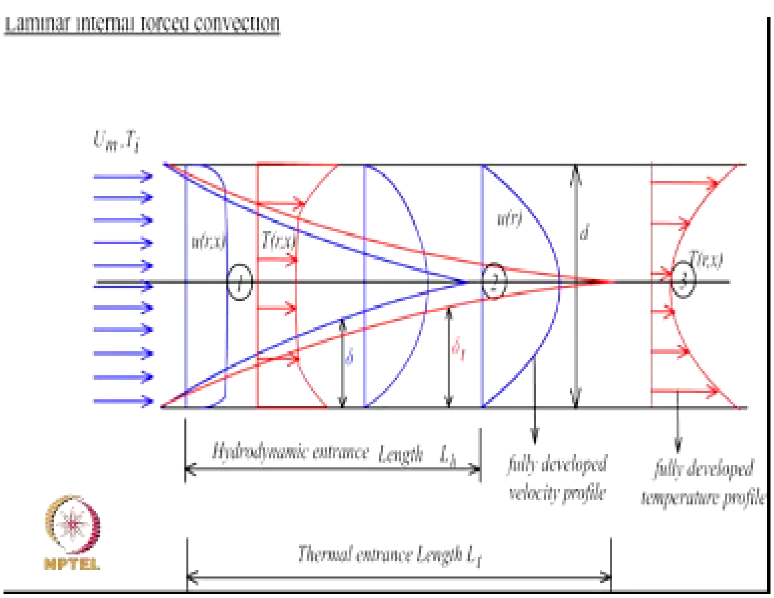

And the other boundary conditions with respect to H now H = 0 your D Θ / D H should be 0 or Θ should be finite both are equivalent boundary conditions and the remaining boundary condition at R equal to R 0 so that corresponds to H = what 1 so at any value of ζ at H = 1 what is the boundary condition 0 okay so therefore now you have a partial differential equation like before for the great is problem only thing you can see earlier you did not have this 1 _ ζ square that was a constant plug flow case now you have this extra term and the other boundary conditions are all same ok there is no difference so now we have to solve this PD so the same method that we use separation of variables will do it here okay so how do we start we say Θ Z Θ will break up x two parts 1 X which is a function of Z and capital R which is a function of H.

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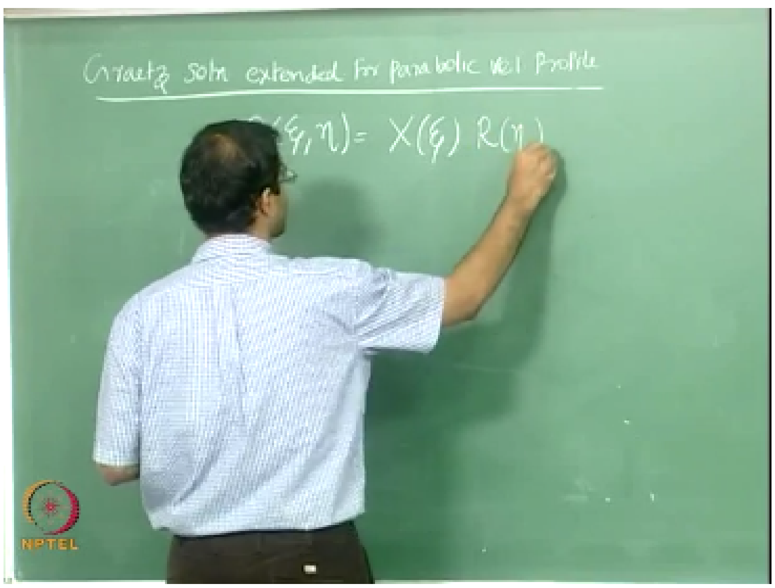

So the actual solution is product of two solutions 1 which is so these are to independent solutions ok one function of only X the other function of R or H all right so therefore if you substitute this x your equation number 2 okay assume that this is your solution and substitute so you end up getting 1 _ H square x DX / D Z and / R is constant on the other side you have 1 / H you have X constant D / D / D H x D R / D H okay.

So I divide both sides / Xx R so this becomes 1 _ H square / Xx so I will also take 1 _ H square on the right hand side so this becomes 1 / X DX / D Z this is = 1 / r x 1 _ H square x I have already a H here x D / D H x H D R / D okay so now I can see that this is a function of Z this is a function of H so this both have to be equal and therefore they have to be = a constant which is always negative okay so in order to have an exponentially decaying function along Z this has to be a negative constant okay and this is the eigenvalue so now I can write this in true form of two OD 1 where I can say DX / D Z + **Λ** square X = 0 I will call this as number three and the second whoa D I will expand this term right here if I expand it I can write this as d square R / D H square + 1 / H D R / D H + you have Λ square and this entire term I am taking it to the other side so this is 1 _ H square x R this is = 0 I will call this as my so equation 4.

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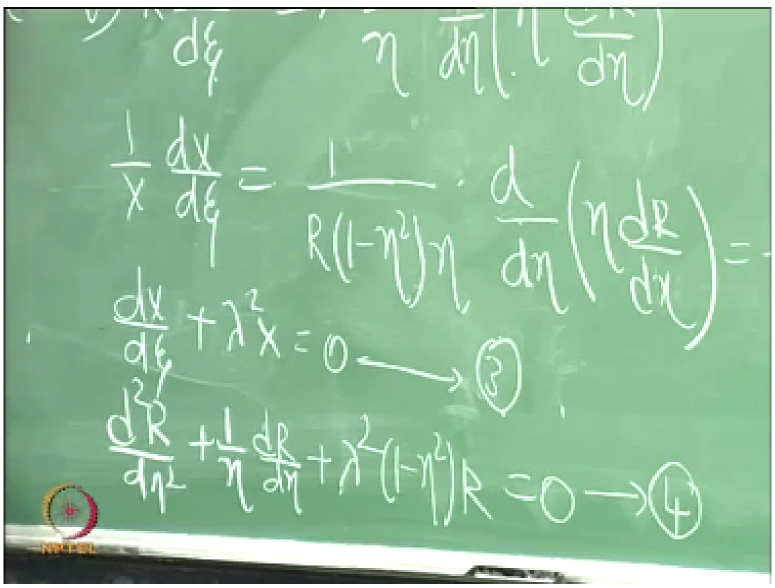

So this is the first row D with respect to X second row D with respect to capital R so I just expanded this term right here so this is e tie x d square R / DT square plus D R / D H x one okay so that is what I did and I already have H at the bottom so I divide everywhere / H and so you have this term Λ square x R x 1 _ Θ square okay so therefore I use the same boundary conditions there and with respect to R now you can see that you have two homogenous boundary conditions because with respect to Θ you can substitute.

As R x okay since X has to be finite you know it cannot be trivial therefore the boundary condition should have should apply to our right so whatever you say here so your D Θ / DX so this will be D / D H x you have X of Z x R of / so this should be = 0 at anyway so I can write this as H = 0 okay so this can be 0 only if I can say D DX / D H if you if X is = 0 also this can become trivial but the problem is if X = 0 there can be no solution ok therefore this implies that your D / D H at our D / DT or at H = 0 this can only be 0 okay right because this can be 0. only if either of this is 0.

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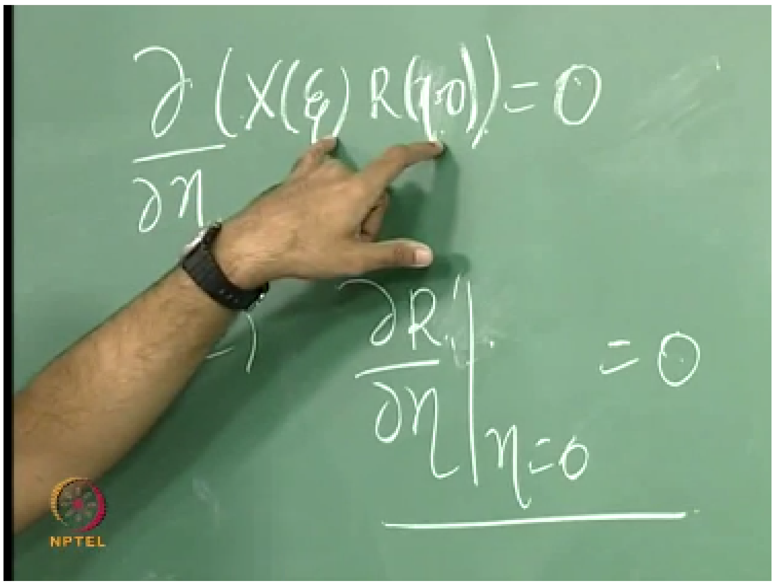

If x is 0 that means my entire solution becomes trivial okay whereas our at H = 0 can be 0 that is the only other option okay so therefore this is the boundary condition which falls on are the other boundary condition that is Θ at Z H = 1 this is = 0 the same way so once again I can say my X of Z x R of H equal to 1 is = 0 so once again X cannot be 0 so therefore this indicates my R H = 1 should be 0 so therefore these are the 2 boundary conditions with respect to so this will be anyway normal derivative not the partial one because R is a function of only H so to solve the Eigen value problem now you can see that Eigen value problem has two homogenous boundary conditions with respect to H ok so therefore equation number 4 is the Eigen value problem which I have to solve and apply the boundary condition now if you look at equation number 4 what kind of equation is there.

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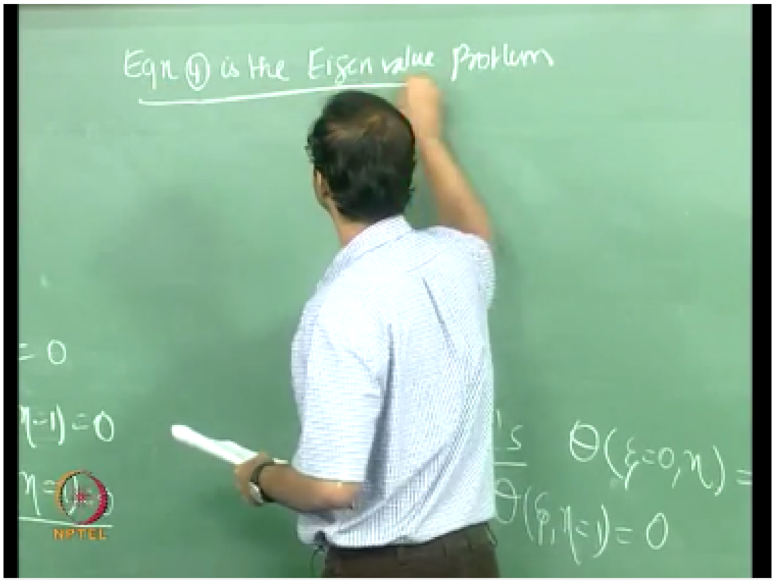

This is where you have to be a little bit careful you look at the structure of the vessel equation I have given it is very deceptive but if you look at it a little bit more carefully if you multiply everywhere with H square okay the first to the second order and the first order derivatives will appear to be similar to the Bessel equation but the term the third term here will have Λ square H square x 1 _ H square x R so that structure is not the same as the Bessel equation okay so the Bessel equation has _ some constant term _ some constant number _ nu square okay does not have - a function of X right so therefore this is not the actual Bessel equation so what we can call this as a general form of any eigenvalue problem which is called a shrm Linville equation.

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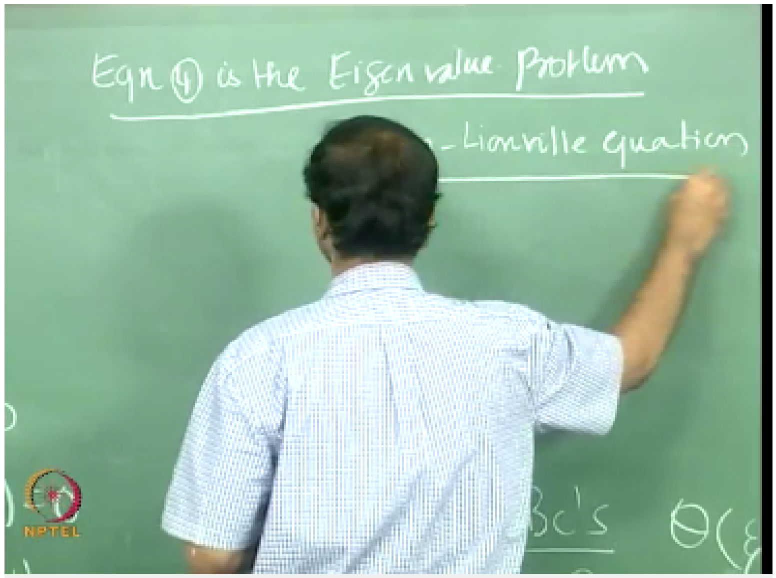

Anybody has heard about this equation shrm -Linville problem Sturm and if anybody has taken partial differential equation course anything to do with differential equation how many of done only one so you must be dot term Linville equation because that is the most fundamental part when you talk about separation of variables any general eigenvalue problem okay to for which you can find the Eigen functions it is generally called as a Shrm- Linville equation that includes even the Bessel equation.

Okay so this is a generic name given to any Eigen value problem that includes all kinds of equations all kinds of second-order OD is all kinds of Bessel equations whichever that you have encountered okay so that is the general category of Shrm-Linville problem so therefore we cannot pinpoint this to any particular familiar equation okay this is not a Bessel equation but it can be generally called as some shrm- Linville equation okay so and I will just give you the generic form of the Shrm Linville equation and yourself can see that it is generally of this particular form okay where these are the this is the weight function.

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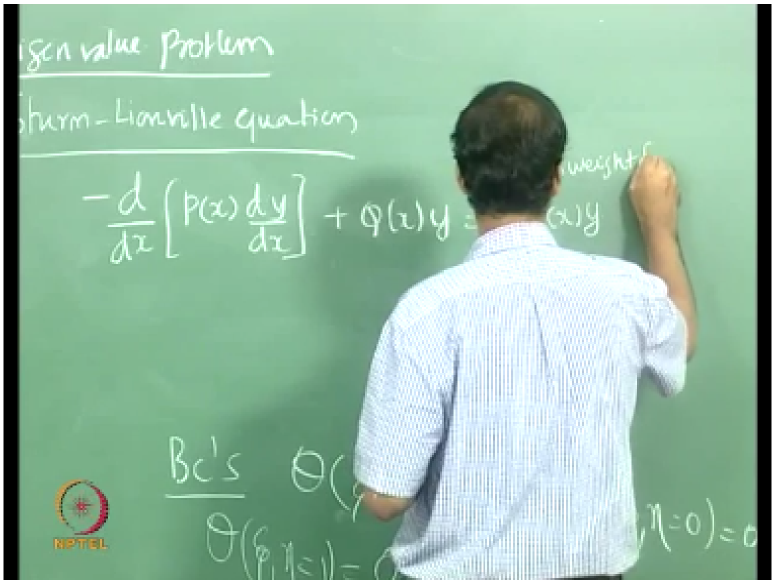

So any ordinary differential equation which can be cast x this particular form okay form system level system of equation and this includes most of the all the eigenvalue problems okay even this equation right here we can just cast it in this particular form okay so for this case you can put it as _ D / D H H dr / + 0 = Λ square x 1 _ H square x H x r so you just expand and see so this is H x d square R / D H square okay + you have D R / D H okay so I'm multiplying everywhere / H okay plus you have Λ square 1 _ H square x H right so I can write that equation in this particular form so this comes to us term kind of an equation.

So now if you compare the coefficients okay the coefficient here p of x is nothing but H and Q of X is nothing but 0 and of course Λ here is Λ square okay and if you look at this weighting function this is a function of X here so that is this has to be a function of H so this the entire thing is the main function or weight function okay so the property of any Eigen function as I said yesterday it has to be orthogonal okay that includes even Bessel functions and any general Eigen functions which are of the Shrm Linville type have the orthogonal property and for this term Lu will kind of equation set the orthogonal property.

Will be you have ∫ A to B Y of n x X Y of M x Xx W of X DX okay is = ∆ MN so this is the orthogonal T okay so what it says when M is not = n this Kroc Necker ∆ this will be 0 right when M is not = n this will be 0 therefore they have to be orthogonal means if m = n then only this will be = one so in that case this will be /square x the weight function DX ∫ should be = 1 if M is not = n this ∫ will be = 0 just like what we saw a stood if you multiply Bessel function JN x JM y okay if M is not = n so there will be 0 the ∫ product will be 0 okay so this is the principle of orthogonality for any Sturm level problem so here you have identified what is the weight function correct so in this particular case what should be the orthogonality condition so if I integrate what should be the limits of the ∫ 0 to 1 good and then instead of why I have my are RN of / x RM of H.

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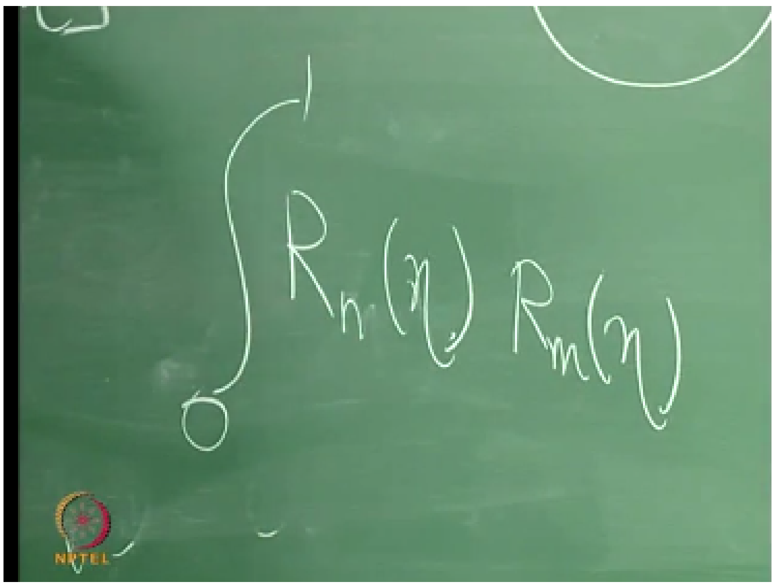

And what is the weighing function here H x 1 _ H square x D H this is nothing but your chronic adult this is the Δ function I hope all of you know what a Δ function is so it is = 1 at only one value everywhere else it is 0 so for the case where you are m = n then this is = 1 otherwise it is 0 okay so therefore this is my orthogonality condition which I will use where should I use the orthogonality condition when we solve.

The OD when I want to calculate one of the constants right so this I keep it now so therefore you see that the solution to the Shrm Linville problems can be represented as so you can say in this case that you can write this as summation of some CN times Rn of H where n can go from 0 to ∞ so once you know the Eigen function okay so you can write this solution for Θ or whatever you call you can write this as r prime or whatever so your R of H can be written as.

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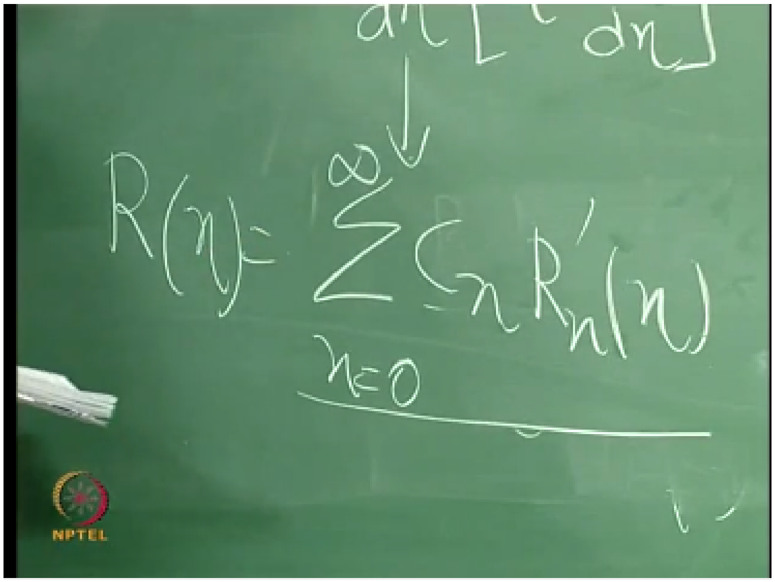

The Eigen function times the constant summed over all the values of eigenvalues okay now the question is how to basically find the eigenvalues so for that we have to it is not as simple as solving the Bessel equation because now this is a general term local problem and you have to solve the ordinary differential equation that is equation number four we have to solve this numerically okay you cannot this is not a generic this is not a particular form of any equation where you can get a ready-made solution okay so any general Shrm-Linville problem.

If it is familiar equation then you can get a solution straight away like the Bessel equation if it is unfamiliar then you have to apply these boundary conditions to boundary conditions and you have to solve for the Eigen functions which are in terms of are okay so therefore this is a little bit rigorous procedure you know I am not asking you to do it however this sellers they have done it in a slightly different way they use some approximate methods to patch it up I do not think that's a very good technique right.

Now because we have access to computers and you can solve this equation straight away numerical now the question is for solving this you need the Eigen value okay so that becomes like a constraint and the Eigen value can be related to the derivative to heart okay I am not going to detail but there is something called as a rally coefficient okay for Shrm Linville problems so which relates your Eigen value to the derivatives of r and therefore you can guess the value of

Λ right just like your shooting method you guess your value of Λ and then you apply the boundary conditions okay and then keep marching until the other boundary condition is satisfied and then again you use the derivative and then check if your Λ is correct okay so this has to be done i iteratively till you guess the right value of Λ okay so overall what I would like to say is that you have to solve equation 4 numerically to obtain the Eigen values and the Eigen function the Eigen function is basically our variation of R with respect to H.

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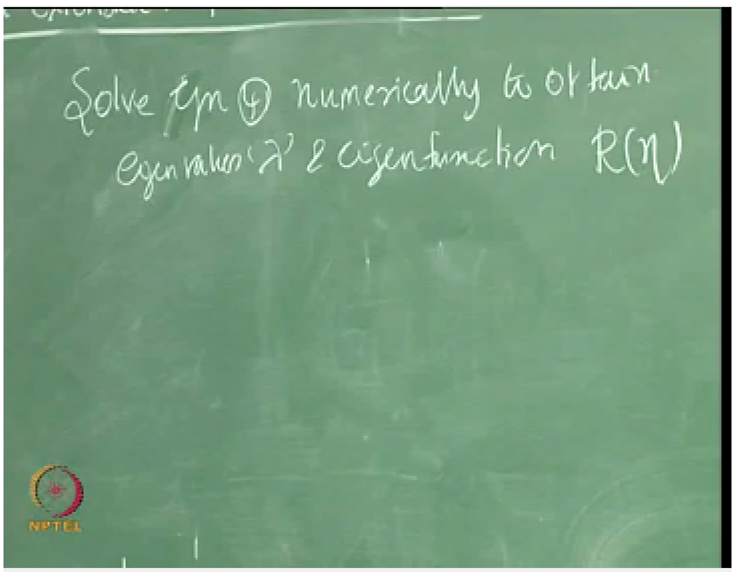

So once you do that so now your final solution so can be can be expressed now directly as Θ which is a function of Z and H now what is the solution to this equation number 3 this is a straightforward OD X = some constant C e power _ Λ square X okay so that is a straightforward solution so therefore your final solution Θ will be product of X and R ok so as I said you can have multiple values of eigenvalues and for each of this you will have a solution okay I can function so therefore you have two linearly superpose all these multiple solutions so therefore you use a summation from n = 0 to ∞ and the constants coming from this C and here you just go up to Club them together as one C x the Eigen function coming out of that that is RN of H and the solution which is coming out of this is basically Z here that is e power _ Λ square Z this is your form of your solution.

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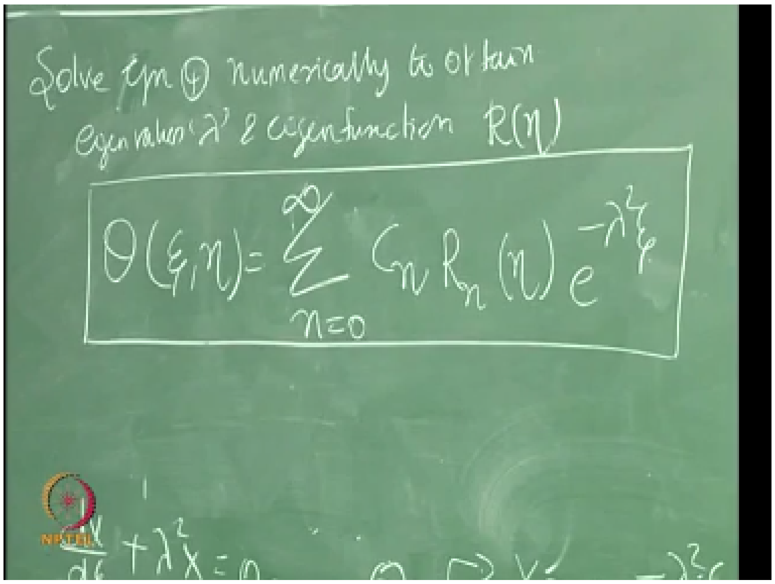

Right the product of this and this ok so now the thing is finally so now you have to do it numerically to solve for the Eigen value and Eigen function the remaining constant see can be solved / applying what? Yeah, we you make use of the orthogonality property but we have to use the remaining condition for Θ okay that is Θ at Z = 0 which is 1 okay so that is nothing but summation n = 0 to ∞ CN are n of H and for Z = 0 that becomes 1 okay so now to get CN out of this I make use of the orthogonality condition.

So I multiply both sides / the weighting function times some RM of H ok both sides and integrate so from 0 to 1 so the weighting function is H x 1 _ H square x some RM of H x D H okay I multiply this from both sides so I have ∫ 0 to 1 summation H = n = 0 to ∞ e turn x 1 _ H square I have R M of H R n of H x CN so CN will be here in 2d right so now you can see that based on the orthogonality property if M is not = n then this entire thing will be 0 okay only for m = n this will be one okay so therefore we can say this is ∫ 0 to 1 this there would not be any summation.

Because for any value of n which is not = M this summation will be any way leading to 0 so this will be CN x H x 1 _ H square x R and Square D H so if I say 0 to 1 H x 1 _ H square x RN square of H D H will be actually _ 1 / this is the actual value if you know the Eigen function you have to take the derivative with respect to H and this will be at H = 1 and this is the value of this complete product and integrate ok so this is the property.

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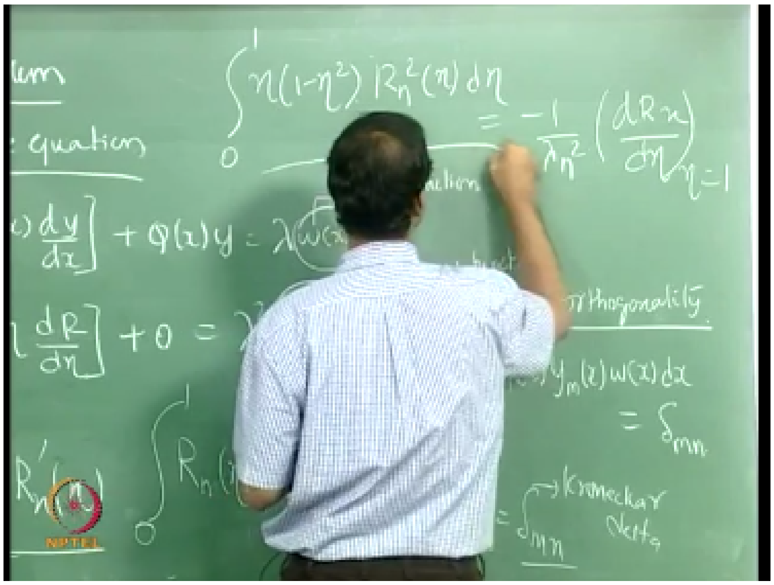

So this is also property of the Shrrm Linville system of equations so therefore you can calculate your C so your CN comes out as ∫ 0 to 1 e tie x 1 _ H square x RN of H D H divided / so the summation is gone you have ∫ 0 to 1 x H   x 1 _ H square x RN square of H DT.

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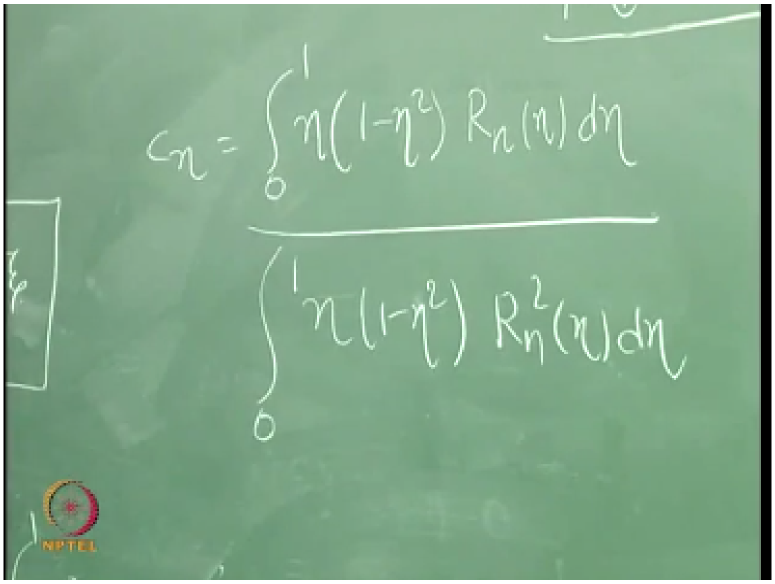

Right for m = n so this becomes our n square otherwise this is 0 so basically we will stop with the fact that you can calculate your CNE here I will give you the actual calculation of these ∫ s 2 of them I have to check whether with whether this is with respect to H or whether this respect to Λ so let me just hold on for that in the next class so I will give you the expressions for evaluating these 2 ∫ s and then from there it is mere substitution okay so once you get your constant CN there for your solution.

So for Θ is known once you know your Eigen functions and your eigenvalues okay so this people cell are set all they have with some approximate method they have evaluated the values of eigenvalues and the corresponding constants I will give those tables also in the next class when we do the solution and from there you can asymptotically reach the case of large values of X and you can recover your earlier result for nusselt number that was some 3.6 for constant water temperature with a parabolic profile so you can exactly reach that as emphatic well so in the next class on Tuesday we will complete.

This solution and so we will look at very briefly the extension of the greatest problem to uniform wall flux boundary condition there I have posted already in the Module a solution for channel flow where I assume a plug flow velocity distribution and constant wall flux boundary condition and have posted the form of solution I will like plane very briefly how to do it and you can look at that solution and you can very easily understand it for the case of constant wall flux boundary condition you have to tweak in such a way that the eigenvalue problem has homogenous boundary conditions whereas if you have a constant wall flux boundary condition that is not homogeneous okay so therefore we play around with the solution in such a way you reach a homogenous boundary condition and that is explained in the derivation which I posted on Module and you can do a similar kind of a derivation for the duct flow so what I have done is for a channel a very similar procedure for duct flow can be done so that will be a extension which will be useful to you and that will go towards the ∫ solution the approximate solutions to internal flow problems okay it is very similar to your external flow ∫ method okay that will be the last topic and with that we will wrap the internal flows laminar internal you okay.

Extended Graetz problem for

Parabolic velocity profile

End of Lecture 30

Next; Extended Graetz problem

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