**Indian Institute of Technology Madras**
**NPTEL**

**National Programme on Technology Enhanced Learning**
**Video Lectures on**
**Convective Heat Transfer**

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# Lecture 5
# Continuity Equation

Good morning so in today's class we will look at the some of the fundamental laws which govern convective heat transport and we will try to derive the first which is the continuity of mass or that is your conservation of mass or continuity equation .

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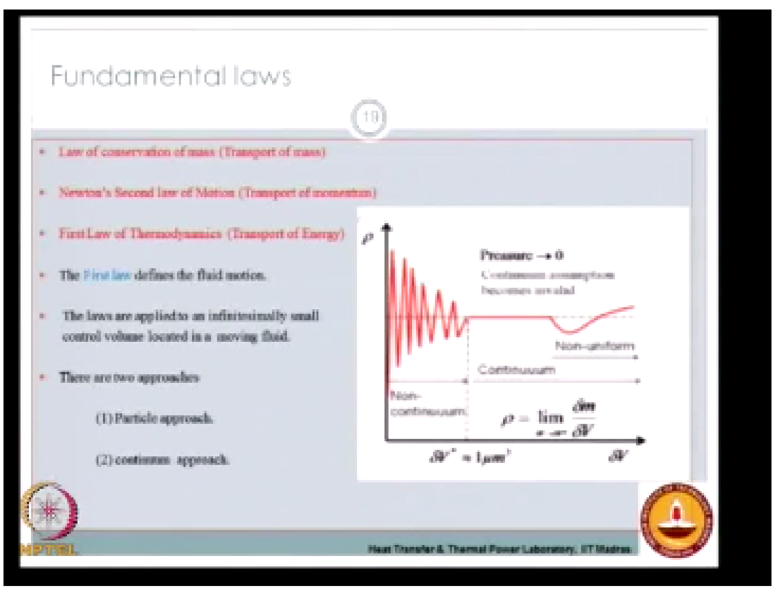


So the three fundamental laws which are governing the convective heat transport are the law of conservation of mass which is your transport of mass which is also called as a continuity equation and the transport of momentum which is nothing but your classical Newton's second law of motion this is also called the momentum equation momentum conservation equation and finally for convective heat transport we will also look at the first law of thermodynamics applied to an open system which is called the conservation of energy or transport of energy.

So all these laws will be applied to an infinite similarly small control volume, which is located in a moving fluid and when you look at therefore when we define a control volume of course there are different approaches to solving these governing equations either as treating the fluid as particles individually and tracking each of these particles in writing the Newton's second law for the motion of the particles and the energy of the particle and if you look at the other approach.

Which is called the continuum approach which we'll be following in this particular lecture we define a control volume in a particular in a moving fluid and how we define continuum here is that if you plot the ratio of dense the mass by volume that is if you define density as mass by volume and then you plot this ratio for different volume of the control volume that is for different $\Delta$ V that is the different controls.

Volume sizes if you plot this below a certain volume which is maybe approximately about one micron cube you find the plot of density looks very distorted like you can see lot of fluctuations in defining the value of density below this certain critical volume and if you use volumes which are greater or which are larger than this critical volume and you calculate the ratio of the mass divided by the volume that is you weigh the particles in that particular control volume divided by the volume that it occupies you find after that this becomes more or less a constant okay.

So this signals that we have moved from a regime which is non continuum where you have few very few particles or very small control volume where it could not statistically define an average property such as density sufficiently where your particle approach could be valid could be used whereas if you look at volume sizes which are greater than this critical volume there the density the definition of density carries a certain meaning and you find the density becomes a constant so therefore here the continuum assumption is definitely valid so we will restrict all our derivation equations to a control volume which is satisfying continuum approach.

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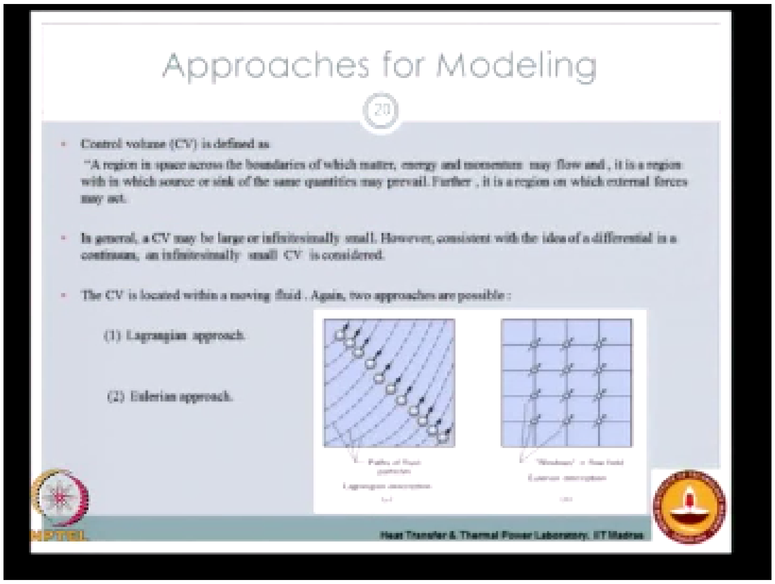


Once again we can define two kinds of a control volume so one where you fix the control volume in space and then you look at the transport of fluid across the fixed control volume and this is called the Oil Arian approach which is shown by the figure B here and the other approach where you look at control volumes which are moving so in space and time so this is called the Lagrangian approach of course each of them have their own advantages and disadvantages, however the Arian approach is more conducive.

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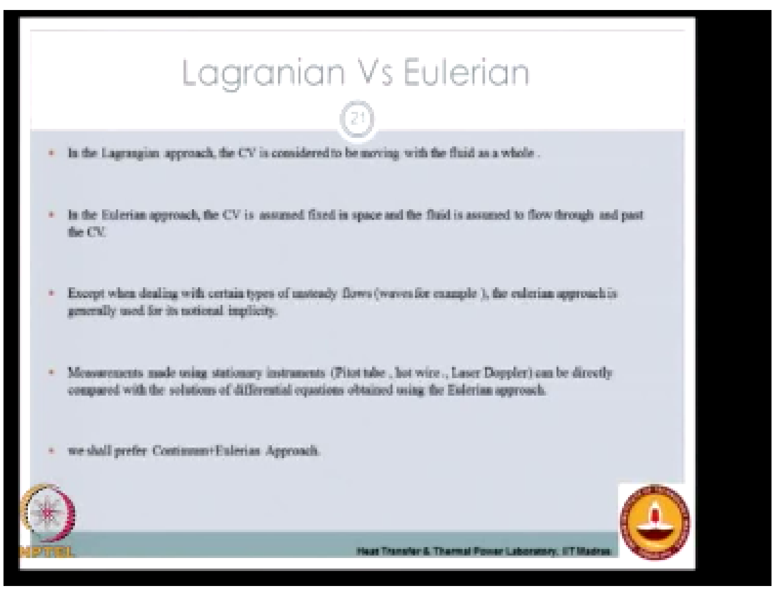


When you especially look at comparison of your measured data with whatever you have computed or whatever you have analyzed with your governing equations so in the Lagrangian approach primarily the control volume is considered to be moving while the with the fluid as a whole so you look at packets of fluid particles which are moving and you define a control volume which is basically also moving with the fluid particles and in the Arian approach you assume a fixed control volume in space and then you as you see look at the motion of the fluid.

Which is coming in and leaving this particular control wall so except for dealing with certain types of unsteady flows the Arian approach is basically generally preferred for because it is a generally simple to look at and also comparing your properties from the solutions to equations in the Arian framework to measurements made with stationary instruments most of the times .

when we use stationary instruments let such as a piton tube for measuring pressure or hot wire for measuring your velocity or using laser Doppler or particle image velocimetric so on so we try to compute the velocity field or the pressure field at any depending on a particular location so these can be directly compared with the solutions of differential equation which are generally derived using the fixed frame approach or the Arian approach therefore when we derive these governing equations we shall prefer using the combination of continuum and also the Arian approach.

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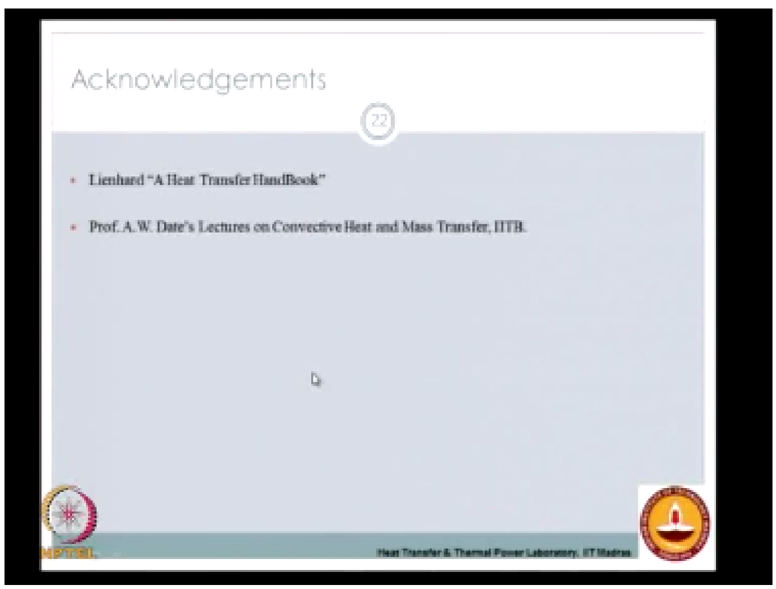


so the I would like to acknowledge for the introductory material have taken borrowed extensively from the open textbook which is available online by Leonhard John Leonhard of MIT it is a heat transfer handbook and also professor that is lectures on convective heat and mass transfer which are also available and from NPTEL so he is a professor at I T Bombay, and some of the illustrations also have used from his lectures so this is to give you the basic introduction into a convective heat transfer so today what we will do is we will go to start with the derivation of some of these conservation equations we will start with the continuity equation so for deriving the conservation equations first we will define the control volume in space.

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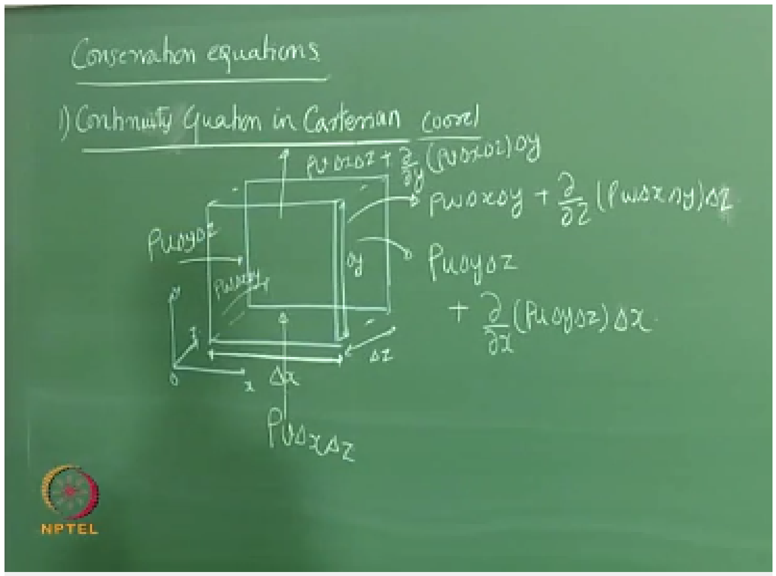


So the topic will be conservation equations and we will first start with the derivation of continuity equation in Cartesian framework in your Cartesian coordinate system so therefore we will first describe the Cartesian control volume in three dimensions.

So you can look at a look at a control volume a parallelepiped for example the origin will be from the left corner the left front corner here so this is your X Y & Z and the control volume sizes are in the X direction it has a dimension of Δ X and Y dimension it has a dimension of Δ Y & Z Direction Δ Z so these are your dimensions of the control volume and this is your coordinate system when we look at the continuity equation or the mass transport we have to look at for a fixed control volume what is the amount of mass which is coming in a certain direction and the mass which is leaving in the same direction.

So if you look at the mass flow rates for example the mass flow rate which is entering the control volume along the x direction that is from your left end of the control wall so the mass flux here will be the mass flow that is your density basically if you look at the mass flow rate that is your density times the velocity into the cross-sectional area here the cross sectional area for this particular case is Δ Y and Δ Z so that is P AV.

So that is your mass flow rate which is entering through the left control volume and in the X direction the mass flow rate which is leaving can be related to the mass flow rate which is entering the left control volume by means of Taylor series expansion so this can be written as P u Δ Y Δ Z + D / DX of P u Δ Y Δ Z x Δ X similarly the mass flow rate which is entering the control volume from the bottom plane is P V and multiplied by the cross sectional area which is Δ X times Δ z and the flow rate which is leaving the control volume from the top surface is P V Δ X Δ Z + D by DY due to the gradient in the Y Direction P V Δ X Δ Z into Δ Y .

So this is again by using the Taylor series expansion from the variable at this particular y equal to 0 you can write down what is the variable or the quantity at y equal to Δ Y using the Taylor series expansion the same thing can be done in the Z direction as well so if you look at the front phase the mass flow rate which is entering along the z direction in the front phase is P W into Δ X Δ Y and which is coming out of the rear phase can be written in terms of your Taylor series expansion + D by DZ of P W Δ X Δ Y Δ Z so these are all your mass flow rates which are entering and leaving the control volume.

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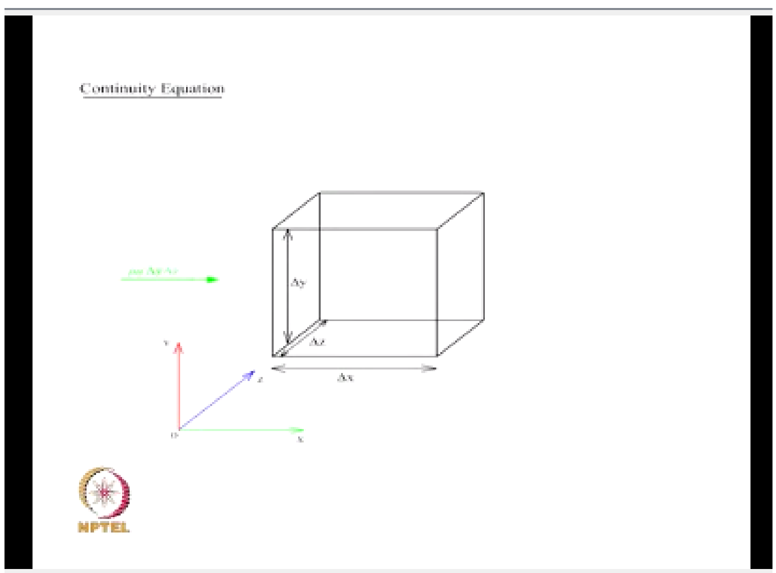


You therefore we should first define the conservation of mass so what we will say is that the change of mass inside the control volume should be equal to the mass which is coming in minus mass which is leaving the control world so that is the net change in the mass that is entering and leaving the control volume should be equal to the mass which is accumulating R which is depreciating in the control world.

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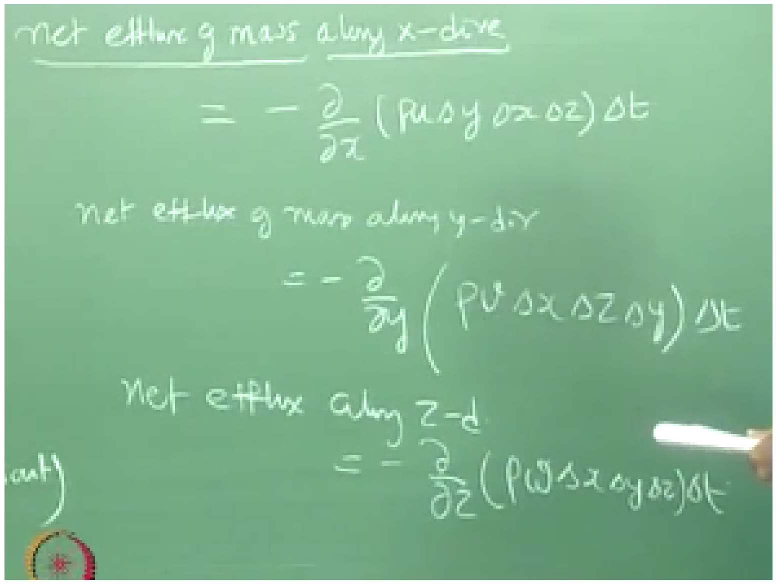


Therefore if you look at the net flux of mass through the control volume boundaries or control volume surfaces which is basically your mass in minus mass out I can subtract in each direction in X direction for example the mass which is entering minus the mass which is leaving is minus d by DX of P u ∇ Y ∇ X and ∇ Z so this is your mass flow rate net mass flow rate which is leaving in the X direction so to get the net mass which is leaving you have to multiply that for a time Δ T so this gives you the net mass which is leaving along the X direction so similarly you can write down the expressions for the flood.

The net mass which is leaving in the other directions as well so along y direction so you have minus D by DY so you have mass coming in minus the mass which is leaving which is P V ∇ X ∇ Z ∇ Y into Δ T which will give me the mass so this is the flow rate multiplied by the time or duration over which you are monitoring the mass so that is your Δ T so similarly your net flux along Z direction will be minus D by DZ of P W times Δ X Δ Y Δ Z into Δ T.

So the change of mass inside the control volume will be if you look at the rate of change of mass so that is your density times your control volume itself which is Δ X Δ Y then so this is density times the volume which is the mass so the rate of change of mass which is happening and if you want to look at the change of mass to multiply it by the time duration Δ T so this is the change of mass inside the control volume and that should be equal to the net mass in minus mass out so therefore summation of all the mass flux massive fluxes in all.

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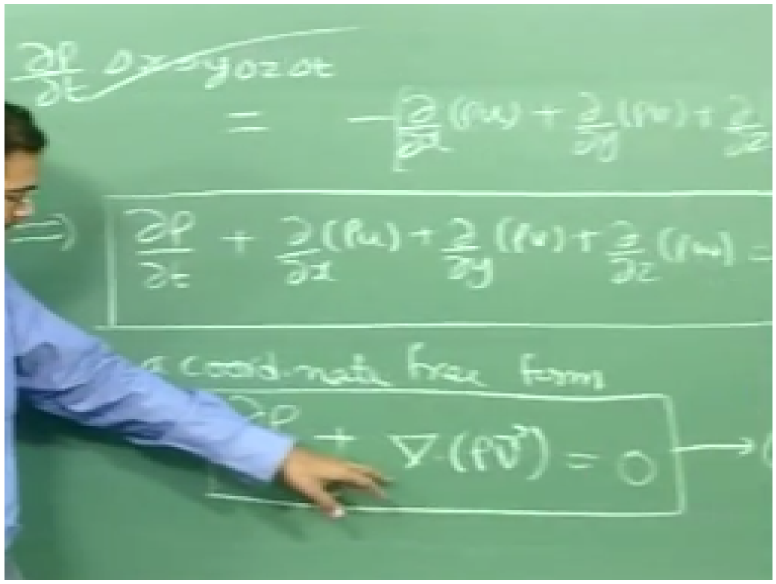

The three  directions so that should give me so D  by D P by DT into $\Delta$ X $\Delta$ Y  $\Delta$ Z $\Delta$ T left-hand side is equal  to minus D by DX of P u +D by DY  of P V + D by DZ of P W  multiplied by $\Delta$ X $\Delta$ Y $\Delta$ Z  $\Delta$ T so this is common for all the  reflux in each direction so I am taking  them out so you can cancel this straightaway and this will give me my  general form of the continuity equation  in the Cartesian coordinate system.

So in a coordinate free representation  you can express you can instead of  deriving this for different coordinate  systems you can write this in a  coordinate freeform as follows you can  take the derivative with respect to time  and write it as it is now the spatial  derivative that is D P u DX D P VD  by D P VD Z can be expressed as  divergence operator operated upon P V  vector where V vector is basically UI  + VJ + W K in the Cartesian  coordinate system so therefore in a  coordinate free representation you can express the continuity equation as this  particular form so let us call this as  equation 1 and this is equation 2 .

So  therefore you can use the necessary  divergence operator in any coordinate  system whether it is Cartesian or  cylindrical or spherical and expanded to  that particular coordinate system that  is one way of simply writing down the  equations in that particular coordinate  system now what I am going to do is write down some simpler forms of the  continuity equation making several  approximations so what will be can do is  we can also rearrange equation 1 in the  following manner.

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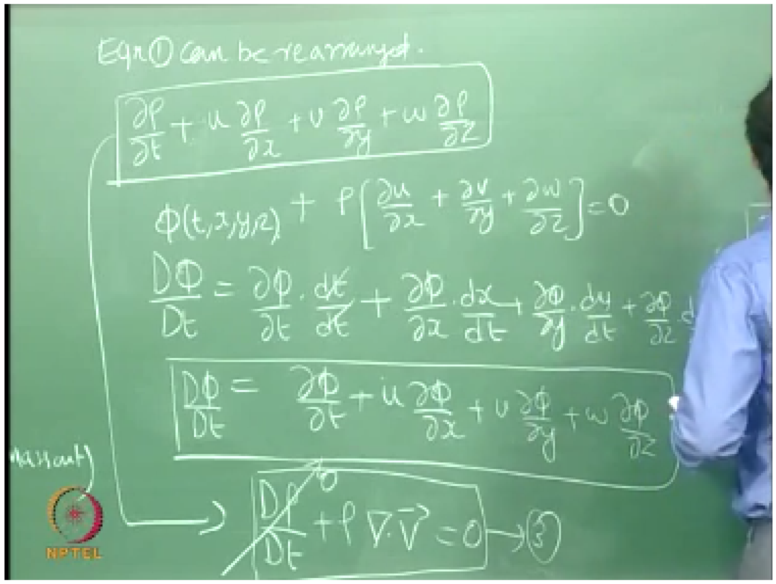


So we can write D P by DT + we can expand the derivative of P u we can split it up as u into D P by DX + V into D P by DY + w0 by DZ + you can take essentially P common and you can write this as d u by DX + DV by DY + DW by DZ is equal to zero that means I am separating this into parts so I am writing P into D u by DX + du u P into DV by DY + P into DW by DZ that is this term and the first and the second third and fourth terms are basically coming from taking you out and writing this is u into D P by DX + V into D P by DY + W into D P by D Z .

So now we can define what is called as a total derivative for any variable for example fee you can write your total derivative DV by DT now if your fee is a function of time and also your position X Y Z you can expand this total derivative as in terms of partial derivative D phi by DT into DT by DT + you have D phi by DX into DX by DT + b v by DY into DY by dt and so on in the Z coordinate system now we all know that DX by DT is nothing but the velocity in the X Direction DY by DT is the velocity in the Y direction and DZ by DT is the velocity in the Z direction and this cancels of two therefore one way to represent a partial derivative of V with respect to T and in this particular form with respect to XY and Z.

Is to use the notation of a total derivative so that you can make the notation more compact and if you can see this particular form here so instead of fee we have row and other terms are similar to the terms here and therefore we can write this equation as D P by DT that is the total derivative of density + P times D u by DX + DV by DY + DW by DZ is equal to zero so this again I can use the coordinate free format I can write this as divergence of week which is equal to zero.

So I will call this as equation so these are different ways in which I can write the continuity equation now for steady flows if you are looking at flows where essentially you do not look at the change in any property like velocity or the density and so on so you can neglect the change with respect to time and therefore for steady flows you are left with the particular expression that the divergence of velocity is zero or the velocity is divergence free and whereas for incompressible flows you can go one step further and you can also say that.

When you are looking at incompressible fluids where the density change is not much or incompressible flows where your density variation is not much because of very low Mach numbers.

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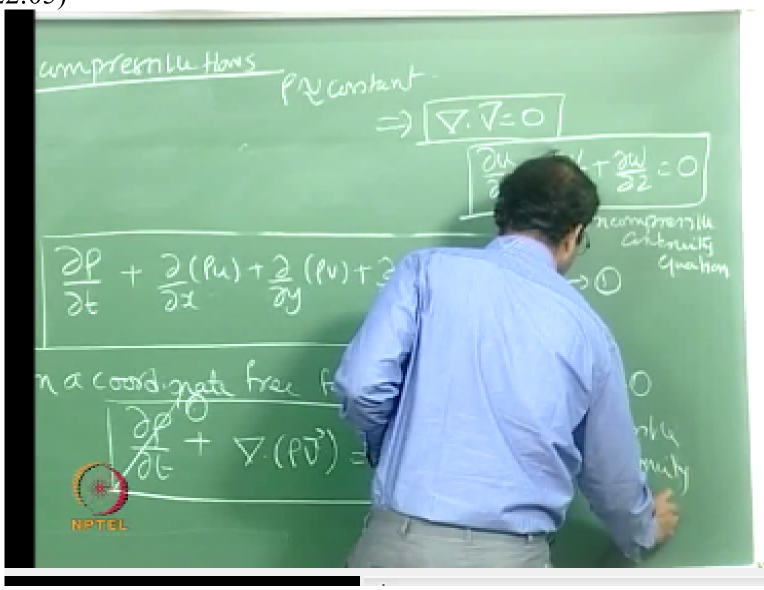


So in such cases for incompressible fluids and incompressible flows since your density is it invariant can assume density is constant and in that case also even if the flow is unsteady the derivative with respect to the total derivative will be zero because the density is invariant of space and time and in that case also you get the expression that your velocity is divergence free okay our in rectangular coordinates this means D u DX + DV DY + DW by DZ is equal to zero so this is your incompressible form of the continuity equation this is your incompressible continuity equation.

If you are looking at compressible continuity equation but in steady flow then you neglect this derivative with respect to time and then this becomes your ∇ dot P V equal to zero becomes your compressible continuity equation okay so this is so far whatever we have done is all in one coordinate system which is the rectangular coordinate system one way of writing this in other coordinate systems is to simply replace this divergence operator by the respective divergence operator in that coordinate system and get the equation in that coordinate system.

We can also do from starting from fundamentals we can derive the continuity equation in the other coordinate system I will just do this derivation in the cylindrical coordinate system for continuity to give you an example so let us define the control volume for the cylindrical coordinates.

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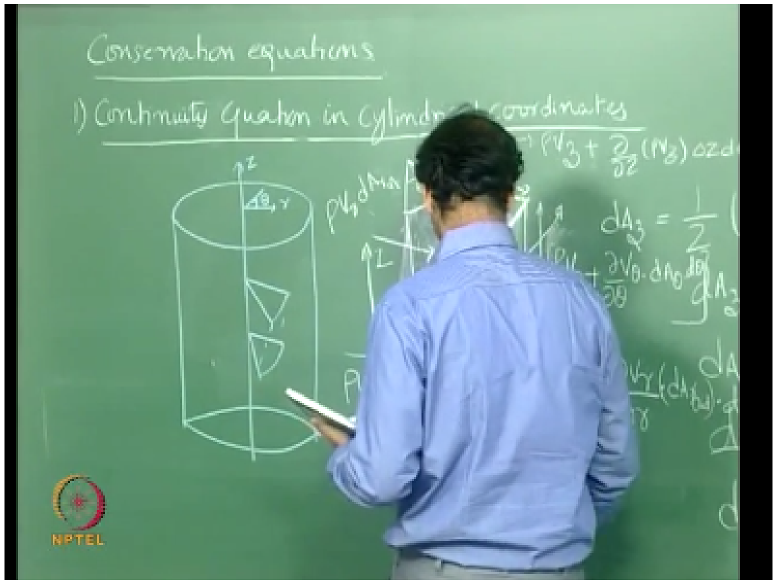


So in this case we will take a cylinder a fluid volume which is in the form of a cylinder and this is your axial direction Z you can have radial direction R and of course variation in $\theta$ direction so this is your cylindrical coordinate system let us chop of a particular control volume a particular control volume like this and we look at that control volume in detail.

So here we look at a sector again and maybe I should draw this in a much clearer manner I will try to use another chalk so this is this particular control volume and I am going to take a sector of this join this sector right here so now you have the three directions this is your Z direction and this is your $\theta$ direction right here and this is your radial direction okay so therefore we will start from the bottom the coordinate system can be assumed to start from the bottom right here this is the origin .

Now the first thing that we are going to do is write the differential areas in each of the directions because unlike the Cartesian coordinate system it is not explicitly seen what is the differential area so therefore will first write it down the differential area in the Z direction that is your DSE if you look at either the bottom surface are the top surface we want to look at the shaded region here so the shaded region here is your top and the bottom areas is your Da3.

So if you say that this is your radii R and this is your radiate R + V R so this is essentially the shaded area here is basically the area of this sector of radii R and the area of the sector R + D or if you subtract these two areas you get the resulting differential Arians DAC so that is the area of the sector which is enclosed by the radii R + d R will be R + D are the whole square into D $\theta$ okay similarly the area of the sector enclosed by the radii R will be $R^2$ D you subtract these two that gives the differential area between R and R + D R so if you neglect the higher order terms you can cancel off Square D $\theta$ half R squared D $\theta$ minus half $R^2$ $\theta$ and you have two Rd R divided by two so that is basically RD $\theta$ and you have D R square T $\theta$ so that is a very small term considering that these are all differential elements.

So therefore we can neglect the higher order terms and we can approximate your DAC in terms of the dimensions is RD θ and so we this is coming out of neglecting your higher order terms and similarly the variation with respect to θ direction that is if you if you look at the region which is shaded in red so that is your θ the one which is left this is basically this the other one is this so therefore in this particular case you can write this as DZ into DR because this is essentially DZ and this is basically DR so this can be written as DZ times DR and finally coming to the differential area in the radial direction if you look at the differential area at the bottom so if you if you look at the differential area of this the one which I am highlighting in white.

So this is corresponding to DAR which is entering and the other which is leaving is essentially this bigger one right here so the one which is leaving so therefore the one which is entering the differential area here will be basically our D θ into DZ or D θ into D Z because the area of this particular set the length of the sector is basically our D θ times DZ will be the area of this and the differential area in the radial direction which is leaving out here will be R + D R into D θ into DZ R + D R into D θ into D Z.

So now we can write the mass fluxes which are coming in each direction in the z is axial direction or the Z direction which is colored by the violet color here in the bottom plane the mass flow rate is basically P into VZ into the differential area of this the one which is exiting out here will be P VZ + D by DZ of P VZ into DZ into DA3 this is by Taylor series expansion similarly the if you look at the mass flow which is entering this particular phase along the θ direction this is P V θ into DA θ the one which is leaving in the θ direction from the Taylor series expansion we can write this as P V θ + D by D θ into V θ into DA θ into D θ and if you look at the flux mass flow which is entering in the radial direction .

So this is P into V R and this is entering through the differential element D R DA are in and the one which is exiting in the radial direction by Taylor series expansion be P V R + D V R by DR into DAR in into DR so this will be in terms of DA are out here okay because this is corresponding to the cross sectional area of the radial plane and the plane in the radial direction which is through which the mass flow rate is leaving and therefore now we will apply the conservation of mass and we will express the native flux of mass first the net flux of mass in all the directions.

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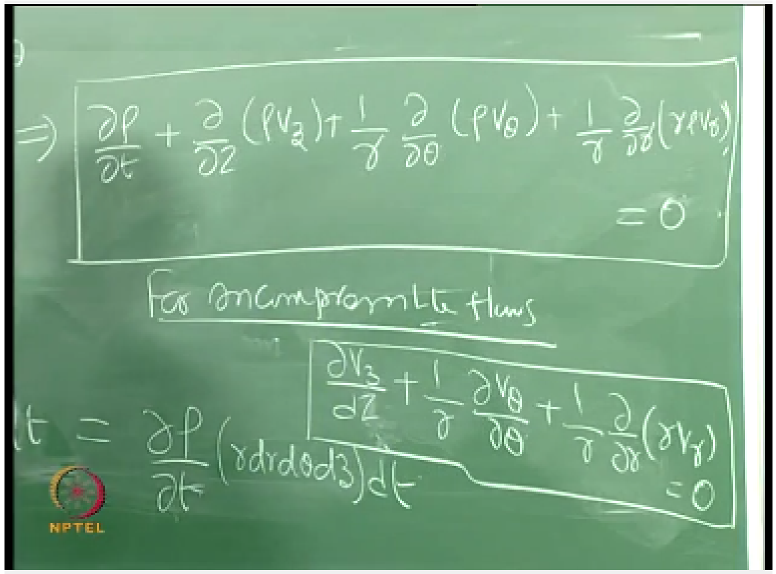

So we will add them together say along the z direction you have ρ VZ ρ VZ into DZ DA said we already know is RD θ RD R into D θ – the flux which is leaving in the Z direction from the top.

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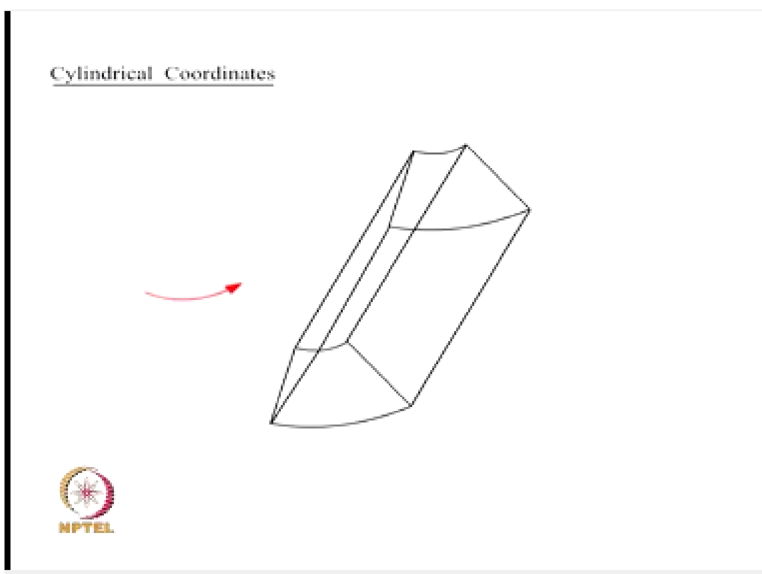


You which is P VZ + D by DZ of P VZ into DZ into R DR D- θ so this is our DR into DZ so once again i should be multiplying the entire thing by DZ here and similarly just a small correction i will be writing this is d θ and multiplying this entire thing as DA θ and this also row VR into DR into DAR out okay so this is the net flux of paths in the Z direction so that + the native flux of mass in the θ direction will be P V θ into Δ Z DZ Dr minus ρ v θ + D by D θ into P V θ into D θ into DZ DR + the net flux of mass in the radial direction which is P VR into R D θ into D Z minus P V R + d by DR of ρ VR into DR x r + V R into D θ DZ.

So I am multiplying everything by the corresponding areas and this is your native flux of mass coming from the control volume surfaces so according to the continuity of mass this should be equal to the change of mass inside the control wall so we can write the change of mass so within the control volumes all of this can be multiplied by Δ T so this will give the change of mass that is the mass coming in minus mass going out the change of mass will be the mass flow rate which is accumulating or which is deteriorating over time multiplied by the time step that will give me the change of mass inside the control volume that is d ρ by dt into the control volume here if you look at this will be d AZ into DZ.

So this will be the total volume differential volume of this control volume already we have seen d AZ is RDR into D θ so therefore the entire control volume will be r DR d-θ into DZ x dt or Δ t will give me the corresponding change in the mass okay so now you can knock off the common terms if you do that you will end up with the final expression which will be d ρ by dt + d by DZ of P VZ + 1 by r d by d θ into ρ v θ + VR by r + v by DR of ρ VR this is equal to zero.

So we can also write the last two terms together as 1 by r d by DR of R via into ρ VR so this is therefore your generalized continuity equation in cylindrical coordinate system so basically coming out of balancing the mass fluxes going in all the directions through the control volume surfaces to the change of mass within the control volume so if you simplify knock of all the common terms and neglect all the higher-order terms the terms which are very small so then you end up with this particular equation once again for incompressible flows if you neglect the density variation with respect to time and space then this will reduce to DV Z by DZ + 1 by r DV θ by DR d-θ + 1 by r d by DR of R VR is equal to 0 so this is your incompressible form of continuity equation in the cylindrical coordinate system ok.

So similarly if you can do the derivation in the spherical coordinate system as well although a little bit tedious because you have curvature in all the three coordinate axis you have a radial system there you have a polar angle and you have the azimuthally angle all the three of them coming into picture and becomes a little bit more rigorous however you can still express all of them as a divergence free operator and you can substitute the appropriate divergence operator in that coordinate system to get the equations so in the next class so we will stop here for today.

The next class will start we will continue our derivation of the momentum and the energy equations okay so for the momentum equation is straightforward once you write your control volume you have to write down all the forces acting on the control volume and the flux of momentum which is entering and leaving the control volume so we use the Newton's law and we balance all the forces to the fluxes of moment and that gives you the momentum equation and similarly we have to do the energy equation derivation which is a little bit more rigorous so we will do this in the next subsequent two lectures okay.

**Continuity Equation**
**End of Lecture 3**

Next: Momentum and Energy Equations

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