Indian institute of technology madras
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National programme on technology enhanced learning
Video lectures on convictive heat transfer

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Lecture 12

Laminar External flow past flat plate (Blasius Similarity Solution)

Good morning to you in the last few lectures about six or seven lectures so far we were focusing on deriving the governing equations describing the flow and heat transfer of course we have done that in a Cartesian coordinate framework as well as the coordinate free framework based on applying the Reynolds transport theorem and finally so we were looking at approximations to a two dimensional incompressible flows to navier-stokes equations one is casting that into form of stream function vorticity equations.

Which we have derived and if you apply that to a typical problem which is for a external boundary layer flow okay so for that we can use some scaling arguments do some order of magnitude analysis and then conclusively show that certain terms can be dropped out provided that you are dealing with sufficiently high Reynolds number flows okay so under that approximation we have derived what are called as the boundary layer equations.

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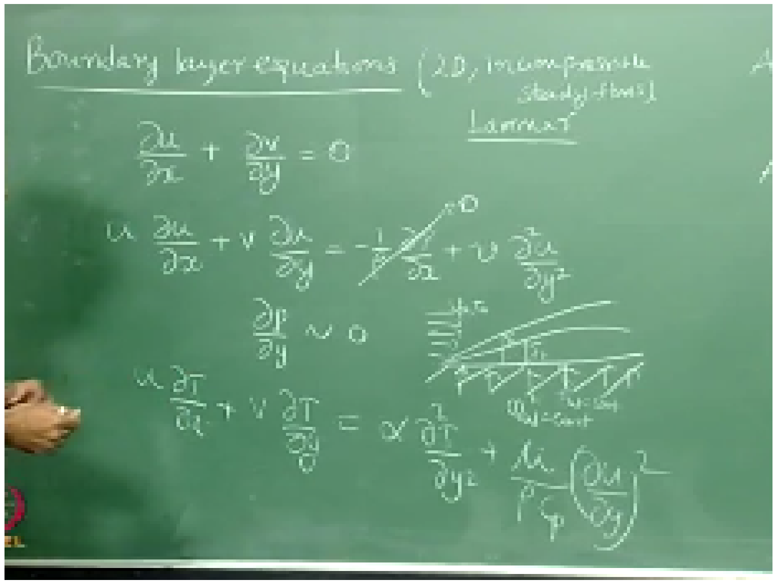

Okay so let us summarize the boundary layer equations which we derived in the last class okay so let us write it in a dimensional form not the non-dimensional form so finally the continuity equation stays as it is okay so the boundary layer equations what we are writing here is for 2d incompressible and what steady state flows okay, so of course and this is also we are assuming there laminar okay that is another important assumption so something. I have to tell you about this when we derive the navier-stokes equation.

We do not really derive one set of equation for laminar flows and another for turbulent flows navier-stokes equations are valid for all kinds of flows okay so the problem is in solving those equations numerically so most of the times if you are applying the same set of equations for turbulent flow and turbulence has a lot of features you have to resolve all kinds of scales in turbulence length and time scales and therefore numerically it may not be possible to do that kind of resolution so then what we do in order to model the turbulent equations turbulent flows.

We construct what are called as Reynolds averaged navier stokes equations okay so we averaged the equations that we have and we can do different kinds of averaging of course for steady state it does not matter we have ensemble average for unsteady problems we can do time averaging and when we do this averaging we are trying to get information only on the mean properties of the flow such as mean velocities mean pressures and mean density we are not interested in all the microscopic turbulent fluctuations and turbulent quantities okay so therefore we are going to when we apply these kind of equations to model turbulent flows.

I think when Professor Koehler comes. I think he will discuss how we averaged the corresponding equations and then we get the nulls averaged navier stoke's structurally everything will look similar only apart from the laminar stresses you will also end up with turbulent stress term okay and the problem is how do we close this particular turbulent stress okay so that is all the different turbulence models are about okay so they are there are different there are many books written on turbulence modeling many so many the hundreds and hundreds of papers published on that so this is a very important issue most of the practical flows are turbulent okay asses the if you solve the navier-stokes equations as it is there is no problem.

But most of the times you cannot resolve all the scales and therefore you have to approximately model what the what are called as average quantities and that is where the Rantz equations come so far we have not discussed those things I am just looking at plain navier-stokes equations and approximation so you can say safely that this is mostly applied to laminar solutions okay so when we solve this the resulting solutions are all for laminar flow okay for turbulent flow again the characteristics become a little bit different and therefore we have to solve the rance equations and the most of the times you do not have analytical solutions okay you have to solve them numerically but few of you extreme cases have analytical solutions.

Which we can do okay so this is the continuity equation this is your momentum only your X momentum stays your Y momentum tells you that your DP by dy is approximately zero and finally your energy equation okay so what should be the disgust dissipation term here okay by zero but what is the ϱ CP right I am dividing by ϱ CP everywhere so this is K / P CP which is A μ by P CP times D u dy $^{(2)}$ okay so this is the only term which is staying there alright okay so these are the boundary layer equations finally all of you agree and you know why only this term stays rate okay so if not you please go back and revise so now in order to solve these equations what do we do we need definitely some boundary conditions right these are partial differential equations so can you tell me what are the boundary conditions that we can we can apply say we take the case of our flat plate the same yesterday.

We have a momentum boundary layer δ thickness thermal boundary layer δ T and the wall is maintained at a constant temperature and you have a flow u ∞ to ∞ okay.

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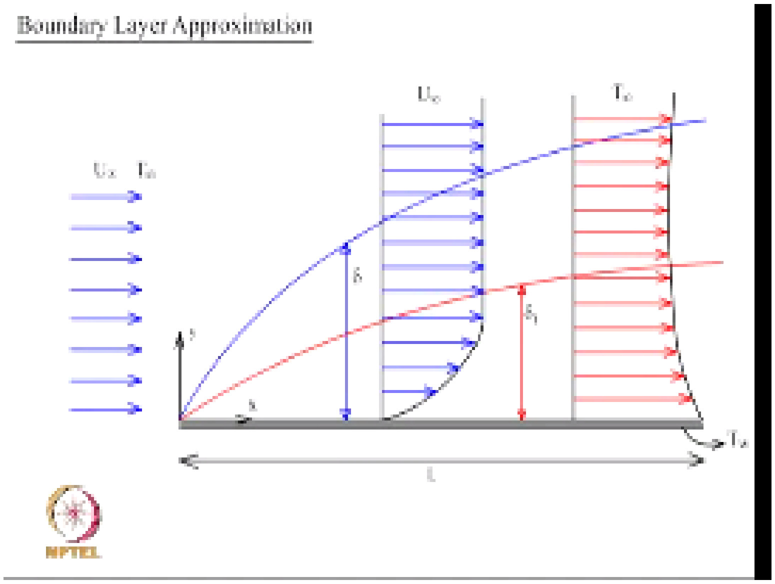

You so now you have to tell me what are the boundary conditions which we can apply for first the momentum and next the energy okay so at y =0.

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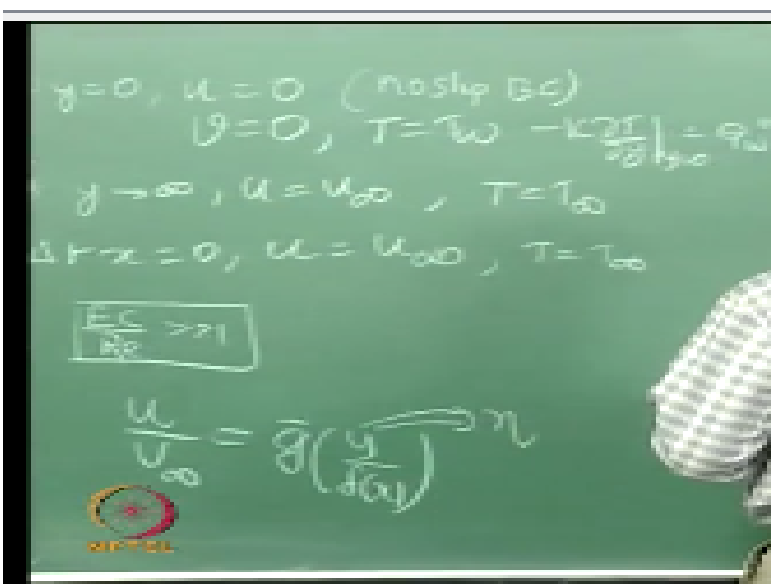

So how many boundary conditions we need in the X direction for example the momentum as far as the velocity is concerned along the x direction how many boundary conditions 3y okay so if you are too much bothered about okay. I am just asking for velocity okay so 2 2 along the X Y so why do we need to and how about in the why 2+2 what is this combination why 2 in X Direction what is the order what is the order of the PD in X Direction first order what is the order in y direction second order okay so any so how many boundary conditions does a PDE need equal to number of what is the order of the PD rate.

So therefore for along the y-direction how many we need to give to okay so 1 we are saying at $y = 0$ $u = 0$ so this is what no slip boundary condition ok so this is the biggest contribution to Branton okay who has invented the boundary layer theory we discovered that the fluid velocity should be exactly identical to the solid wall velocity okay now also when you complete saying no slip and there is no vertical velocity or there is no transportation through the wall okay the vertical velocity also has to be 0 so and what is the other boundary condition along Y we can give Y okay let us say Y going to ∞ okay.

So u are $u\ \infty$ ok and so therefore you have two boundary conditions with respect to U and now what is the remaining boundary condition we need to give one in long X okay so we will say at $X = 0$ u should be $U\ \infty$ okay so similarly for the temperature the temperature also requires two boundary conditions along Y one boundary condition along X okay so what is the boundary condition at $y = 0$ either this or if I apply a uniform heat flux instead of uniform wall temperature if I same IQ all this constant so then it should be -K DT / dy / = 0 = 0 Y which what happens if it is 0 Sadia biotic boundary condition okay.

So therefore this is the condition at y equal to 0 how about at y going to ∞ $T = T\ \infty$ and $X = 0$ at t should be $T\ \infty$ okay, so this is these are the required boundary conditions to solve the boundary layer equations alright so now we will first see the solution for the flat plate flow.

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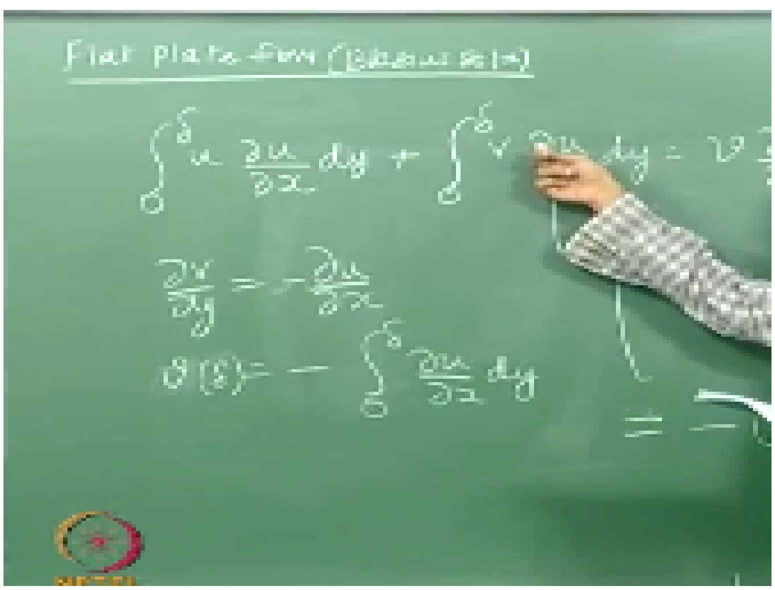

So this is also called the Blasius solution .I think most of you are familiar with Blas Blasius solution okay so this is anyway a flow problem you must have done in your fluid mechanics as well as in the heat transfer anyway for the convective heat transfer we have to revisit the flow problem before going to the heat transfer so I am going to go through the solution ok in probably in a slightly different way from how it was done before so for the flat plate case what are the approximations we can make okay still you should understand that this is a construct a complicated set of equations even if you write it for a boundary layer and of course.

We can transform this into a form which we can solve ok and we are going to do that but before we go to a complicated case where we have a pressure gradient ok so now we are going to first apply that to a very simple geometry the flat plate okay so for this case what kind of approximation that you can make and you see some approximation further to this pressure gradient is zero okay so therefore since your free stream velocity is constant okay so this term can be knocked out that we have seen yesterday and also right now we could rηin the viscous dissipation term and solve okay.

But that is a slightly complicated way of solving it and we are not going to do now if time permits towards the end of this course I will take up this particular case okay so what it means even if you have an adiabatic boundary suppose you put - K DT dy y =0 is zero and you have this viscous dissipation you will find a temperature profile okay so this is this is something which we will be taking up when we have if you have time okay otherwise for this particular case when is the viscous dissipation important as we said when you have the ratio of your record number by Reynolds number.

Which is kind of very high okay so when that can be possible when your flow speed is high because the record number is directly function of the flow velocity or a Reynolds number is very small that is extremely viscous flows or very high speed flows okay so in those two cases you have to consider the viscous dissipation okay so in the other cases the moderate velocity regimes okay you can safely neglect this term also okay so the resulting equations now look slightly much more simplified and this is what we are going to solve for the Blasius solution so before going into the solution just to give you a brief history.

How this was originally proposed so first it was parental who pointed out in 1904 okay Ludwick parental it was a very famous German physicist mathematician and aerodynamics who was actually with the invention of boundary layer concept of boundary layer and 1904 he says that the boundary layer equations for flat plate okay can be transformed into a ordinary differential equation okay so in those days they did not have very sophisticated computers and therefore whenever anybody was looking at set of PDS that look like no they could not solve it because numerical methods were not that popular there is no computer at that time okay so they could not find a direct closed form solution to the PD.

But parental propose that if we can convert the PDE to a OD okay so then OD is always solvable even numerically you can do it by hand also okay so therefore he proposed that the boundary layer equations can be transformed to an ordinary differential equation and this transformation is through what is called similarity variable so this is the it is a great concept that he introduces that all the time we do not have to go for a numerical solution if you are clever enough to see some kind of relationship between the flow velocity non dimensional flow velocity and the property and the for example the coordinate that it is dependent on.

If you cast them in a correct form that you can transform the PD directly into a OD okay so this is called as a similarity solution and in fact it was a student blushes okay in 1908 who did it actually he who did this similarity solution for the flat plate which we are going to do so what we are going to do next is we are going to identically trace what blush is exactly did it exactly in the way that he solved it okay, so this he got the solution for the flat plate velocity profiles okay so he was a student and later on I think it was Pons thousand Pons Heusen in the year 1921 who extended the Blasius solution for velocity profile to solve the energy equation okay so it took almost a decade to solve the energy equation.

From the original Blasius solution and then in 1930 was Faulkner and scan there were two people Faulkner and scan who actually extended the similarity method of solving the PDE s two flows with pressure gradient also okay, so far we are neglecting this particular term here but they have extended for a generalized set of flows you know flows including pressure gradient without pressure gradient so all of them they constructed a similarity solution which we are also going to see okay, so these are general family of problems okay , so therefore you can see that it had a considerably long history and in fact.

In the first in this course when we are talking about external flows we will be doing all of this okay starting from the Blasius solution till Faulkner and scan and probably an extension of this two different boundary conditions okay so all these are basically closed form solutions which you can do but of course the resulting OD cannot be solved analytically blushes himself struggled with that we will be using some elegant numerical techniques to do that okay so now we will start with the Blasius solution okay now what blushes dot now if you look at the velocity profiles okay so you can draw the velocity profile something like this here if you go a little bit downstream.

So if I were to calculate the gradient of velocity at the wall D u / dy okay so what will happen to this gradient as I go downstream will it increase or will it decrease it will decrease right ah decrease why what is that it is always zero always at the wall it is zero but why the gradient should decrease we are the boundary layer thickness keeps on increasing and you have more flow which isn't raining from outside okay so therefore the profile becomes more gradual and gradual as you go down initially you have a very sharp gradient at X =0 the gradient will be infinite okay so you do not have any profile.

Which is developed okay slowly your profile develops and it becomes more gradual and it has to satisfy continuity also write down strip so therefore what he has observed if you can find a variable okay so a non-dimensional form of the velocity which we can say u by u ∞ because we can see the order of magnitude of U is of the order of u ∞ right so if we say u by u ∞ this should be some function of some variables now you can see the variables which you are dependent on it is it is a function of two variables one is your X and the other is your Y okay now when you look at when you look at δ.

You can also see δ as a function of X clearly the boundary layer thickness is a function of X so therefore now also we saw that Y scales with δ variable Y sub YY is of the order of magnitude of δ so therefore if we combine these two variables as a non dimensional group so he says that if we can write this as some function of Y over δ which in-turn δ as a function of X okay so this is a non dimensional group and if we can find this function so this should directly explain the dependence of Y on this single variable rather than on variables Y and X right so if you say Y by δ is some variable H.

Which we call a similarity variable okay so now you are u by u ∞ is just a function of H directly and once you use the similarity variable you note how do not have to solve this in terms of x and y you can just directly solve in terms of H okay that will help you in reducing the PDE to an OD so that was this basic observation okay now how he obtained the basically the relationship between H Y and δ so now we should know how δ is a function of X okay one directly you can use simple scaling arguments like what base undoes look at the inertia and this terms balance and then you can directly get the functionality of δ okay the other long way that I am going to do is helpful because any way we do not have to derive the same set of equations again okay so and let us what I am going to do.

Now so what I am going to do is I am starting with the X momentum equations and I am going to integrate it along Y from the wall surface all the way till the boundary layer thickness δ okay.

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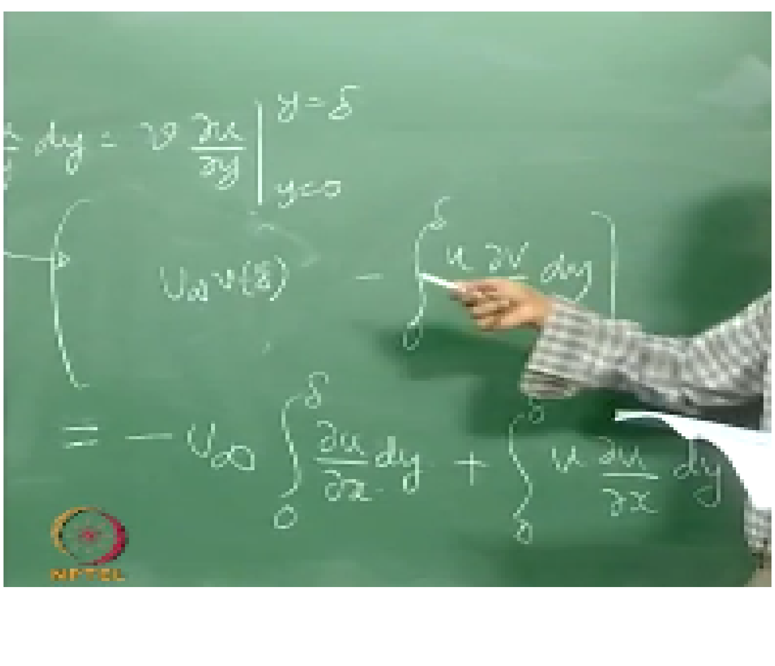

So if you integrate it out so directly you reach to this particular form and what I am going to do now is a little bit jugglery here to eliminate in terms of V and construct that in terms of U so how do I do that I want to confirm the continuity okay so your continuity says that you are d u by DX +- DV / dy okay so I can write for V I can integrate this and I can say this is - integral 0 to δ D u / DX into so integrated 0 to δ over dy and now we can so this is your kinematic viscosity so we can substitute for so I can substitute into that okay ,so what I am going to do a little bit more I am going to integrate by parts this particular term I can write it as d by dy u v dy integral 0 to δ write- 0 to δ u DV / dy /dy right what is that I think this is right ok which one okay.

So I am integrating by parts okay if I do that this I can directly write it as UV between the limits zero and δ okay so this will give me of course at zero you are vr0a δ u is equal to u ∞ so I can say that this is u ∞ times V at δ okay so - this particular term here and of course from continuity I know dv/ dy can be written in terms of d u by DX okay so therefore this can be further written - u ∞ so V or δ is this right here this is my V at δ right so I have integrated it from 0 to δ and at 0 V is anyway 0 so V at δ will be this I can substitute for V of δ from here so this will be - u ∞ times 0 to δ D u DX dy okay again DV by dy as- D u by D X so this will be + integral 0 to δ u D u by DX dy okay.

So this is how my second term so now you can see I have eliminated V completely I have written everything in terms of U ok is that clear which part integration by part okay so basically I have integrated by part here okay and then the first part if you integrate it you will end up with between the limits zero and δ at zero the velocities are 0 but δ U is nothing but u ∞ multiplied by V at δ okay now from the continuity equation when you integrate it between 0 and δ your V of δ can be expressed like this I am just substituting for this directly here and DV by dy is nothing but - D u by DX okay .

So I am just using continuity to plug in ok so this is the resulting equation so if I substitute for this term and then write the complete equation so you have to tell me how the resulting equation should look like okay so already you have UD u / DX there is another UD u by DX okay.

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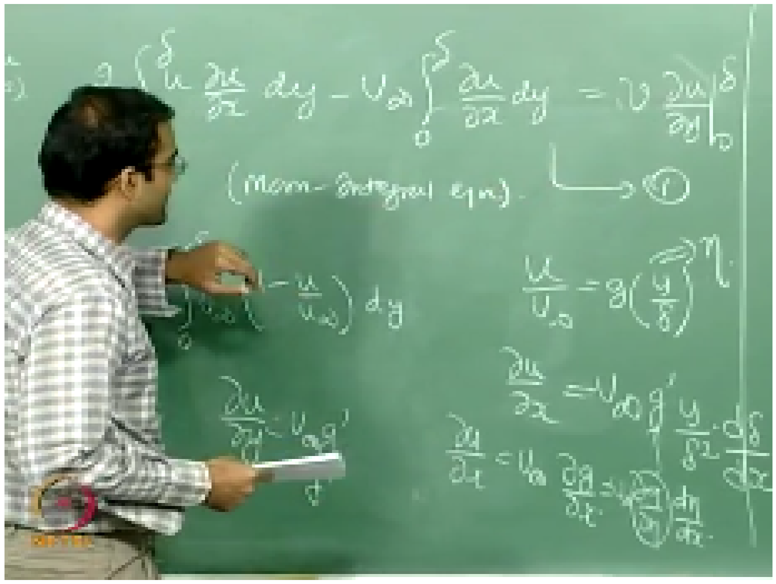

So that will be 2 times integral 0 to δ u D u by DX dy and then you have - u ∞ 0 to δ D u by DX dy that should be equal to u ∂ u / ∂ Y at now this has to be integrated from 0 to so this is the limits I am applying okay this is nothing but ok you can call this put this as equation number one and please store it in your mind somewhere that we have derived this equation because I am NOT going to do it again when we do the integral method ok so this is nothing but the momentum integral equation so that is slightly cast in another form you can say this is D u square by 2 D X and 2 2 cancels so you can write D u^2 da dy - u ∞ so they slightly cast in a different form but anyway that is basically coming from this particular equation this is your momentum integral your momentum integral is written in only in terms of you.

Your entire equation is in terms of view there is no becoming them okay and of course you are you know your momentum integral rate what is your momentum integral data how is it denoted u by u ∞ integral 1 - u by u ∞ dy 0 2 this is this is the definition of momentum integral ok or people call it as momentum thickness sometimes okay so we can you can you can do this as a nice exercise you can combine these two terms bring it to this form and then write in terms of D then by DX okay so that is another way of writing casting this equation okay so anyway so that we are not interested in solving for the momentum integral right.

Now what we are going to do we have already assumed that my u by u ∞ is a function of my G which is function of Y by δ this is nothing but my similarity variable H okay so I am going to plug in this particular solution form of solution which blushes is assumed into this particular momentum integral so I am going to do all this is finally how to solve for δ okay so first if you calculate your D u by DX from here how will you calculate u by DX so that is basically you ∞ times okay so you should go back and revise all your differentiation rules okay G $'$ and then - y by δ square into D δ by DX okay.

So how you got it you can you can say that this is some function of H right and so your D u by DX is nothing but u ∞ times D G by DX okay so this is nothing but DG u ∞ x DG / D H x D H by DX and now DG by D H is nothing but your G' okay and D H by DX so this H is nothing but Y / δ okay so then you get - u / Y /δ^2x D δ / DX okay so you please go back and revise your differentiation rules okay so once you get this now the next term that we need here what is the other term d u / dy okay so what will be d u by dy here d u / dy is much easier u ∞ into G' G' by / δ okay so but δ is a function of X you do not have to differentiate it okay so now you substitute for this D u DX in D u dy okay and you can group all the terms together which I am not going to do step by step I will only show the equation after it is grouped okay.

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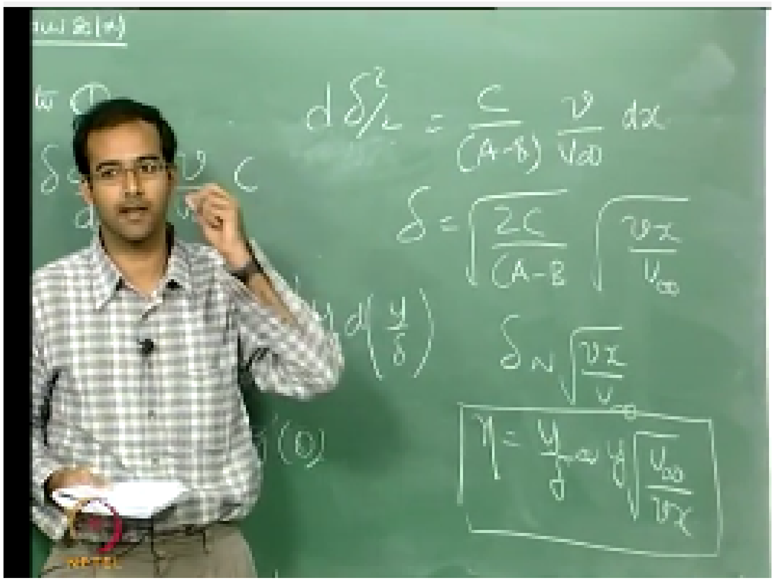

So if you substitute into one and you group all the terms. I am just going to write the final form you get something like a - B into δ D δ by DX / u ∞ times C now this a B and C are constants where a = 0 to 1 G' Y/ δx D Y by δ so what I am doing is this integral dy wherever I have I am writing this as dy / δ x δ okay so therefore the integral limits become 0 to 1 okay so you can check that you know is a nice exercise you can does not take too much of time you know you are just rewriting everything in terms of Y by δ grouping the terms together and B is to 2 times 0 to 1 G G' Y / δ xD Y / δ and you are see can you now guess and tell me what C should be this you should be able to tell zero to one.

Why it is already is already integrated and the limits have to be applied what is D u by dy u ∞ G bar by δ okay so actually when you put that all of this u ∞ can be cancelled off so you have μ times G bar now G' G' between the limits δ - δ zero divided by you have another δ and that δ also will cancel off so finally C will come out to be G' of 1 - G' of 0 okay so this because you are transforming all your Y with respect to Y by δ okay so wherever you are applying limit δ

that becomes 1 all right so this is your OD can you tell me what is the solution for this how do we solve this OD separation okay.

So what will be the solution for δ okay so I can write this as D δ ^{2}/ 2/ DX = C / a - B in two so I can separate it as you said u by u ∞ x DX alright so if I integrate it from X = 0 to some position X okay, so I can use a dummy variable here and I can say I can integrate from 0 X equal to 0 to some position where you are interested to calculate the boundary layer thickness okay so this will give my δ = 2 times √of 2 C by a - B √ x of UX /u ∞ okay so therefore you can see clearly that we have arrived at a relationship between δ and X ok so your δ therefore is a function of X through this particular relationship and therefore if you if you say your Y by δ η is equal to Y by δ so that will become Y times √ of u ∞ by u X.

Therefore this is my similarity variable okay this is my similarity variable which goes like this so this similarity variable now you can see is constructed as a function of both x and y ok so now if I do this way what it means my u by u ∞ is going to be a function of only H anymore so not on x and y respectively alright and this is the similarity variable now the same result you can get by very just one step process scaling process which you can do it yourself ok and now what I am going to do is introduce the similarity transformation now still I have not converted the PD into OD and still I do not know whether the similarity variable is correct this is just what blushes is assumed so what is the check that this variable is correct.

So when we substitute that variable into the PDE that it should get converted into an OD that is the proof that you have a similarity solution otherwise the similarity variable that you guessed is wrong correct okay so let us go and check that so I am going to see now I have to introduce a function so such that I can make the continuity equation redundant I do not want to solve this so what kind of function should I introduce stream function okay so stream function all naturally satisfies the continuity okay so therefore I will introduce a stream function such that.

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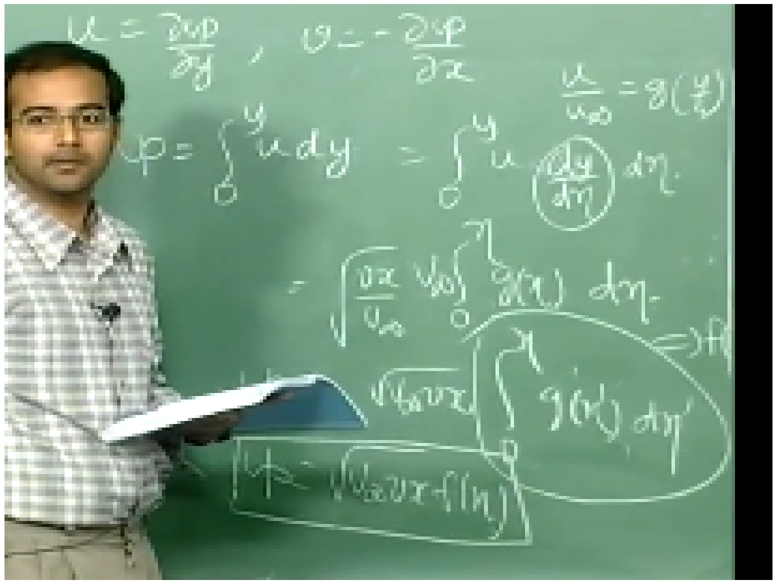

You are u =DC by dy and V = - / DX. Okay now if I take this particular and integrate it along Y so I get 0 to Y u dy and already I know my U by u ∞ is nothing but G of H or G of Y by δ so I can just substitute in terms of δ here this is 0 to Y okay u D Y by δ which is d H x δ right or I can just say dy / D H into okay so I can directly transform it this way ok so dy / D H you know so now we have constructed a relationship for H which is your Y √ of u ∞ by nu X okay so you can calculate what dy / D H is what is dy / D H √UX by okay so that I can substitute into this and I can write this in terms of G okay so this will be 0 to H so why I am transforming to H and U = u ∞.

Which I can take out into G of H x D H into you have √ X by u ∞ so this is nothing but your √ of u ∞ UX times 0 to H G of b η D H ok so this is your side and now I am going to introduce another function ok function which is says that this integral is nothing but this is nothing but a function of H so you can use dummy variables here and you are integrating from 0 to H so the resulting function which is a function of H M is a donut denoting it by f of H so therefore your φ = √ of U ∞ X F of H okay, so this is the relationship between the stream function which I am going to introduce and your similarity variable H okay.

I hope all of you got it is just a very simple substitution and writing in terms of bη any questions on this okay so now we have done this so just couple of more minutes and I am going to write down the velocities and derivatives everything in terms of the similarity variable H okay any questions is it clear okay so all I am doing is I am transforming from Y to H okay so I already know the relationship between H and Y so I substitute for dy by D H and I already know use a function of H this way so I substitute and finally get the relationship between an bη.

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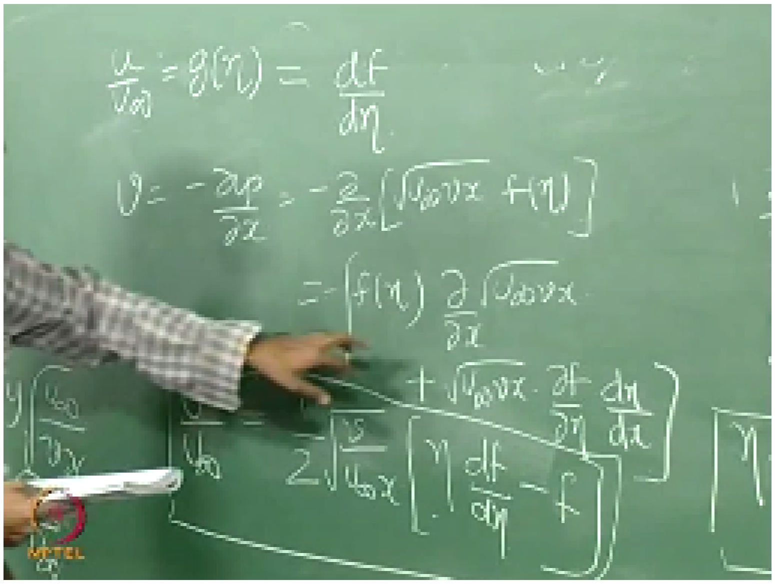

So now first U and U∞ already we know it is a function of G of H okay and your G of H is nothing but if you look at this particular definition it is nothing but DF by D H right because integral of G of η is nothing but a feet ah okay so therefore G of H has to be what DF by D H that is right now similarly my V velocity is nothing but - DC by DX so now you have to tell me so you have psy as a function of H so this we have to differentiate with respect to X okay so you have to do this and tell me anything will be that will stop today okay so if I group these terms together so finally I should be getting V by u ∞ this should be equal to 1 by 2 √ of μ by u ∞ X x so this term right here I can write this as H x DF / D H okay - F okay so you can check for yourself this particular it is a little bit of jugglery so you can cast this in terms of H and you can check whether this particular form comes out okay so this is you have to play a little bit around there okay.

So with that we will stop here tomorrow we will plug so why we are doing all this fear first getting an expression for similarity you this is your stream function as a function of the similarity variable and then in your boundary layer equations you have your velocities your derivatives all of them have to be now cast in terms of the similarity variable and how we are doing that because we know velocity is a function of psy and psy is a function of similarity variable okay so we have to use that put them in the PD and finally.

You will be ending up with a nice ordinary differential equation okay so I suggest all of you whenever we do these kind of derivation especially in the beginning where you are not used to so many differentials you please go back and pay some my no spend some half 45 minutes checking all these equations and I think slowly you will get accustomed to this and after that you will become faster okay.

Laminar External flow past flat plate
(Blasius Similarity Solution)

End of Lecture 12

Next: Numerical solution to the Blasius equation and Similarity solution to heat transfer

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