Spray Theory and Applications
Prof. Mahesh V. Panchagnula
Department of Applied Mechanics
Indian Institute of Technology, Madras

Lecture - 07
Size velocity correlation

Good Morning. Let us continue our discussion of Drop and Velocity Size Distributions as applicable to sprays.

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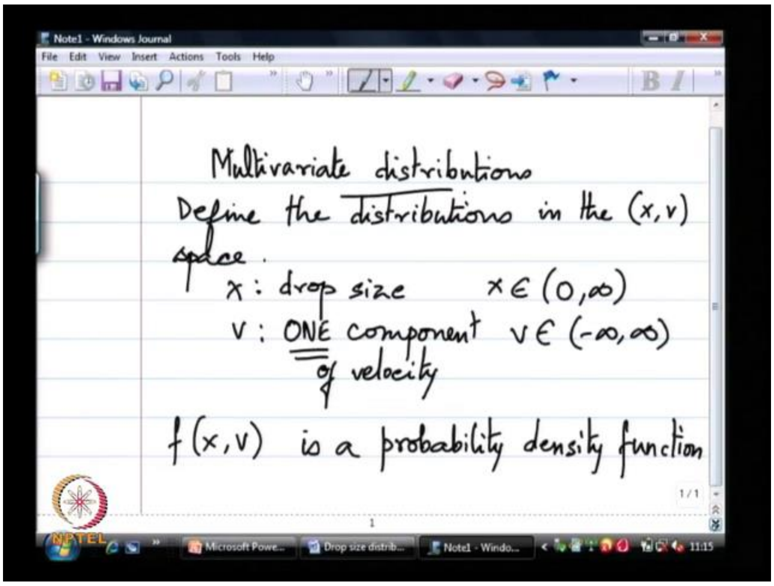

We are going to define these distributions in the x, v space, where x is the drop size and x is any number between 0 to infinity, it is a positive number. And v is will just take to one component of velocity, and because the components of velocity can be either positive or negative it goes between minus infinity to infinity. I mean without knowing anything about of given spray this is sort of the limit I can set on it.

Now I define a function f in these two variables f of x comma v is a probability density function.

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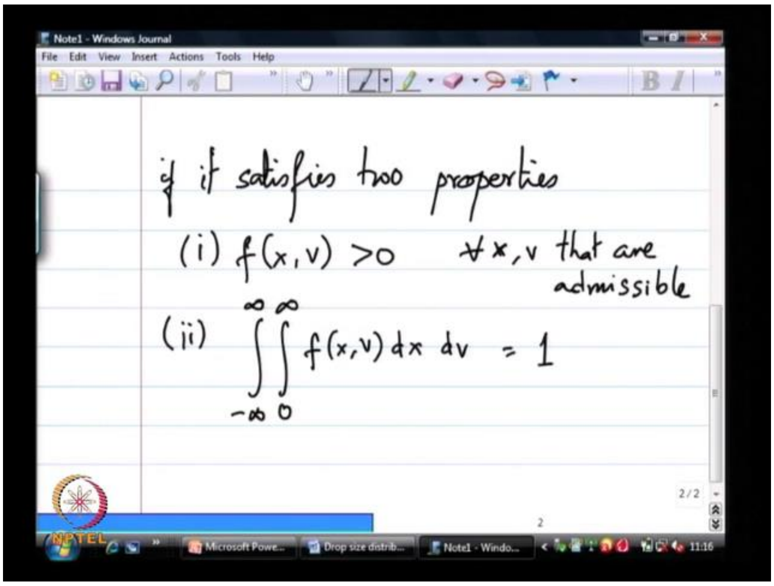

If it is satisfies two properties; one f of x comma v is greater than 0 for all x and v that are admissible. And the second condition is simply that if I was to integrate the probability density function over the set of limits that has to add up to 1. And just to recap.

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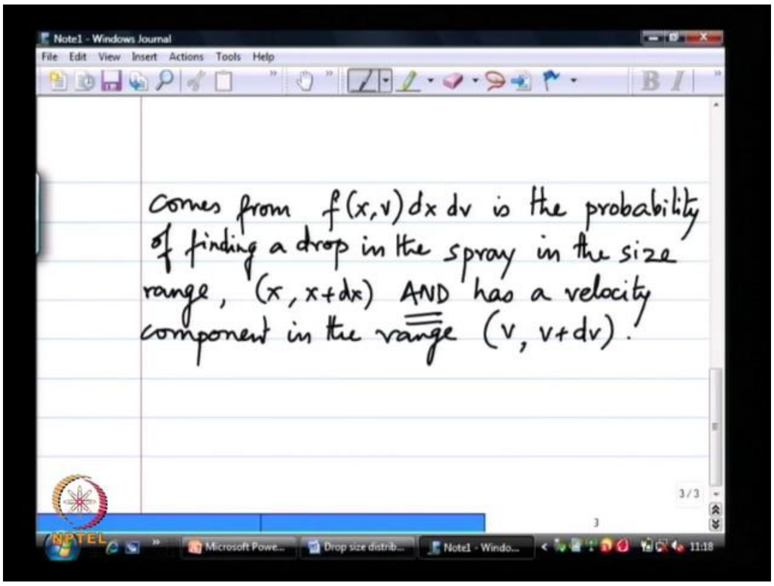

This second condition comes from our basic definition of a probability density function that f of x comma v times dx dv is the actual probability, that is a size of the drop falls in this range x to x plus dx. So, if f of x comma v is a probability density function this probability density function multiplied by dx and dv. So, if you imagine x and v as being two orthogonal axis describing this x, v space dx comma dv is like a tiny area elemental area in this space probability density function at that point x, v multiplied by the elemental area gives me the actual probability that are drop falls at that point.

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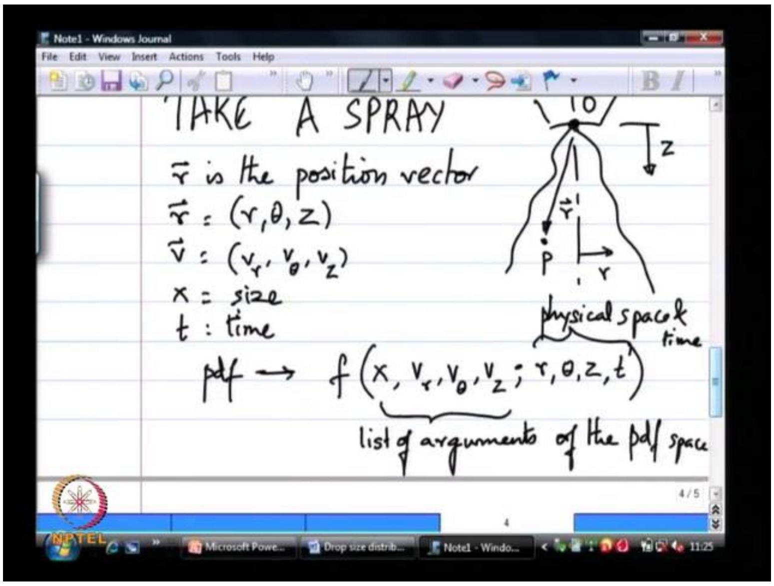

Now, if I add up these actual probabilities over the entire range of admissible values of x and v that suppose to add up to 1. That is our law of simple law of probability. Now just to set things in perspective; if I take a spray, I have a spray nozzle that is got this spray, I have in the most general description three spatial coordinates describing this spray. So, if I draw an axis this is a radial, if I just use cylindrical polar coordinates there is a theta and there may be a z coordinate. So, I have an r theta and z describing this, so if this is my origin point, every point in this spray has a position vector, I will write it more compactly. In this coordinate notation that is easier.

So this is a position vector in three coordinates, and in each of these three coordinates the velocity vector may have a v r, v theta, v z and if I simply take the size and if I say size is one scalar quantity that describes the physical measure of the volume in the drop, and t is time. If I take a very simple unsteady spray at every point P, the point P itself is described by r theta and z it is spatial coordinates. At that point I have a probability density function in x, v r, v theta, and v z. So, the pdf is essentially a function in x, v r, v theta, v z, these are the arguments of the probability density function.

If I integrate it appropriately over these four independent coordinates, now the space in which the spray is characterized is this x, v r, v theta, v z space. And the physical space is r, theta, z, and time. I use a semicolon to separate the list of arguments in the pdf. So, list of arguments, I will say describing the pdf space or in which the pdf space. This would be the most complete description of a spray. If I can come up with this function f that is going to tell me the probability density of finding a given size of drop with the given set of velocity components at a given physical point in the space, and at a given point in time.

As time varies at that same physical point in space the pdf could evolve, if I have a real unsteadies spray. Simplest case let say a diesel injector puts out large drops in the beginning of the injection cycle and large drops at the end of the injection cycle; in the middle layer you may have a real fine atomization. So, this f at one point sees the variation in time. And even if I take steady spray like the one we saw in the video earlier at different points in this spray I will have a different probability density function in the size coordinate and the velocity coordinate. So, just to set we started to talk of f of x comma v as a probability density function in this two parameter space.

What we really should be looking at is a probability density function in the four parameters space of x, v r, v theta, v z that itself varies as a function of r, theta, z, and time. This is like the most general description we can think of in a statistical sense. So, we are still looking at; the moment we say probability density we are suggesting that there is in some sense a local stationarity over some period of time over or over some ensembles of cycles.

We will look at that when we look at experimental techniques, because we will take advantage of this in making some measurements of these sprays. But this is our understanding of what it is so let us now go back.

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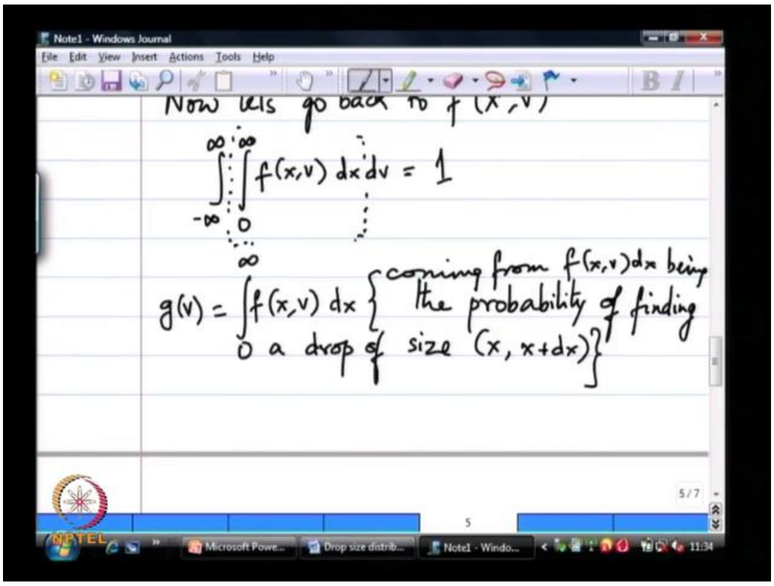

I do not want to sort of; so just to understand or the most complicated way of looking at a spray, but we will simplify it just to understand some of the physics of these distributions. We said integral; this is just coming from the idea of probability. Now if f x comma v dx is the probability of any velocity.

So, this here is a function of velocity. If I perform this integration from 0 to infinity this becomes, I will put these in curly brasses. So I want to see if I can define, like a mean velocity I want to find a mean drops size in the listing, so let us first simplify this.

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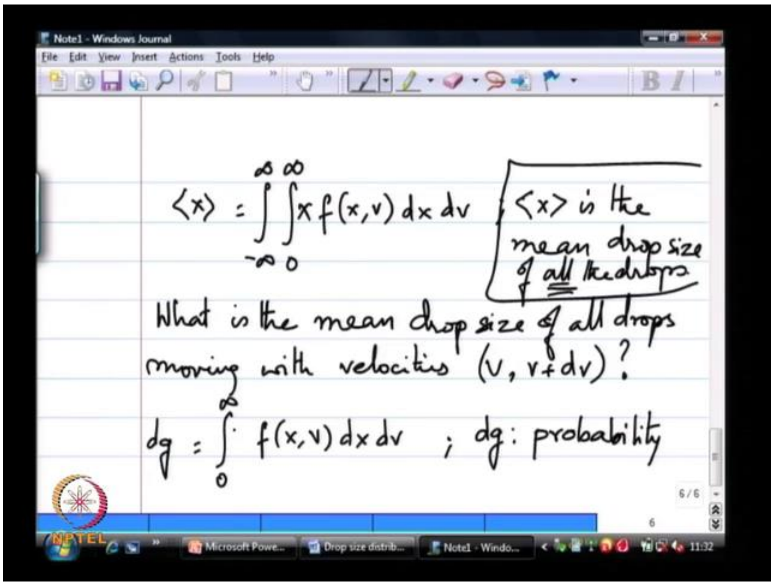

If I define a mean drop size x times f of x comma v dx dv integrated from 0 to infinity and the v integral going minus infinity to infinity; this gives me the mean drop size over all the velocities. What I want is a mean drop size of all the drops that are moving with some velocity v to v plus dv let us say.

So if I ask myself the question, then x f of x comma v is first of all f of x comma v dx dv is the probability of finding a drop in the size x to x plus dx. So, if x is the size then you know the probability multiplied by the size added over all the limit gives me a differential probability of all the drops in the range x to x plus. So, this is now all the drops, but having velocities v to v plus dv.

Now this dg is a probability, because I have taken the probability f of x comma v dx dv is a probability, then I have taken all the probabilities and added them up, but I have not added up all the possible probabilities over all the range of v going minus infinity to infinity. So, dg is basically a probability of finding a drop of any size. Let us do one thing we will even erase this x, so just to get our argument straight with the probability part and then we look at the mean part. So if f of x comma v dx, and if I am only integrating in the limits over x that is giving me a differential probability of finding a drop in the range v to v plus dv.

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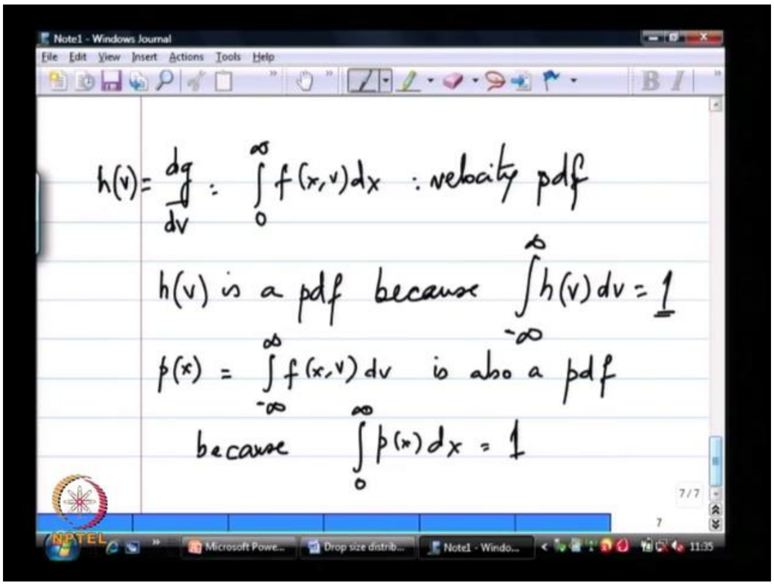

Let us first define that. So, this dg d over dv is integral 0 to infinity f of x comma v dx; this is the velocity probability density function. So, what we have on the right hand side here is only a function of velocity. I will call this some h as a function of v, h is a velocity pdf. Why is it? It is not just a function of v it is actually a pdf, because this is automatically satisfied. This is actually coming from our basic definition of this the integral of f of x comma v dx dv equal to 1 over both the ranges; if I just take this part and if I call that some new function h the integral still remains 1.

I can likewise define another function will call this p of x which is minus infinity to infinity f of x comma v dv. So, this p of x is where I have integrated out the probability in the velocity space. This is a mathematical statement; what it physically means is I do not care what the velocity is, tell me what the probability is of finding a drop in the size x to x plus dx. What the probability is of finding a drop in the size x to x plus dx and that that probability divided by dx gives me the probability density. That is what this p of x is.

Likewise h of v is physically saying, I do not care what the size is just tell me what the velocity distribution looks like. At some point to the spray or the whole spray will get to that little bit.

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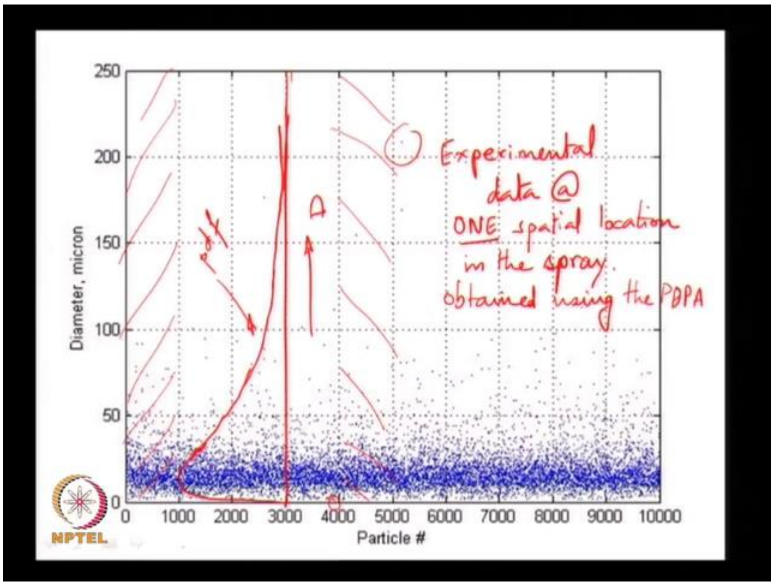

These are in some sense two simplifications of the full f of x comma v in one coordinate or the other only. So, let us look at some real experimental data to see what this actually means. This is data that was obtained using an instrument; this is experimental data at one spatial location in the spray. Now what we see here, mean there is about 10,000 dots on this graph, each dot is 1 drop. So, this was an instrument called the PDPA. Well, it is called the Phase Doppler Particle Analyzer, we will look at that again and later on in the class, but let us just say it is a particle counter.

So, I have a way of counting every particle going through that instrument at that point and figuring out what it is size is as it goes through there. So, I took a sample of 10,000 drops at one point, typically this may be like half a second of real time in a reasonably. We looked at these number right, I have simple perfume if as about a million plus drops. So, to sample 10,000 drops is hardly any time. So, what we did is we took the each drop and you see each drop and it is size, you can see how the size never really goes to 0. Let us say the smallest drop I know in this sample is probably about 1 micron. The largest drop is slightly less than 250 microns; actually it is slightly greater than 200 microns, you can see that is the largest drop, the smallest drop may be somewhere here.

So, if I now take the probability density of this. I am going to draw it on this axis. You can see how the density is 0 at 0, the density quickly rises reaches a maximum and then in fact since this is a experimental data it actually become 0, let me just make that line a little straighter.

So, if I now look at what I have essentially done is I have taken all of the drops and essentially creating you can imagine a histogram first. So, I have created a histogram and from that histogram I have constructed what looks like a continuous curve, so this is my probability density function in the drop size space. We will look at a step wise procedure of doing this in an example as well, but for now that red curve that I have drawn on top of these blue dots you can see sort of qualitatively indicates a density of these drops in this vertical coordinate called diameter, that the vertical coordinate is like a diameter and how densely are these drops fact in that space, is this red curve is an indication of that.

You can also see that it satisfies all of the general criteria we set out for an actual spray pdf which is saying that the probability density should be nearly 0, should be 0 for drop size being 0. The probability density should tend towards 0 as the drop size becomes large. The probability density should achieve a maximum at least at one point. For example, for all I can have more drops at another point so that would give me two peaks on the drop size distribution that would be perfectly acceptable, we would just be called a bimodal distribution. And we know I could go on and we know there is nothing that precludes has from having more and more maxima. But in a real spray a bimodal distribution is about as complicated as we get. Typically, a single maximum point on the pdf is what is more commonly observed.

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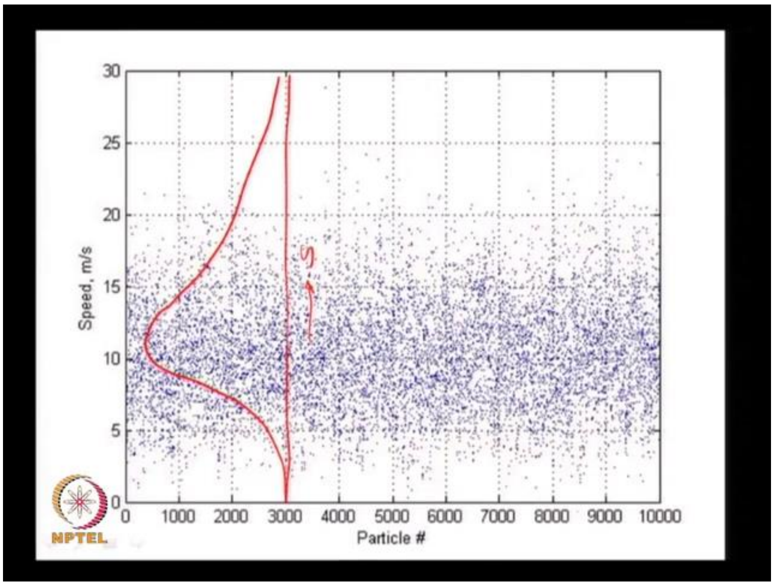

So, now if I move on to the velocity; now in this case, I have not plotted the velocity this is actually a plot of speed. The speed is slightly different from velocity and that velocity is allowed to go from minus infinity to infinity, speed is only a positive number. So, without going into much detail you can see how if I were to draw a pdf of this shows a graph that look something like that.

Now, we look at functional forms of these graphs a little later on, but essentially you can see again that it is hard to find a particle that is of 0 speeds; mean they are hardly any dots near the bottom of this vertical axis. It is hard to find a very fast moving drop either. So, 30 meters per second is about as fast as this sample of drops had. But most of the drops were moving right about 10 to 12 meters per second. This is again 1 spray at one point a distribution of 10,000 drops, a sample of 10,000 drops at that point.

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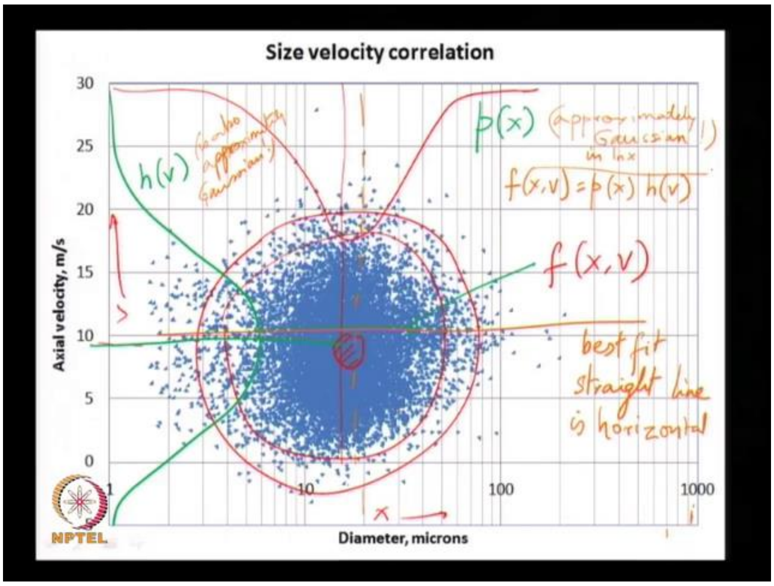

Now, if I take the size and velocity as the two independent axis now. So look at it this way, I have the diameter on the x axis and real axial velocity now on the y axis. Believe me there are 10,000 little triangles on this graph. So, every drop that we sampled, that we showed in the previous two graphs where you know the x axis in the previous two graphs was a number called the particle number it is like a imagine I slap on an Id to every particle that is coming through 1 to 10,000.

So it is just like some way to distinguish from drop to drop. In this particular graph what we have done is we ignored the particle Id; that is I do not know when the particle came. In the previous two graphs the Id was to show like a sequence of arrival as matter of fact. So, let us go back to that graph and look at it a little more carefully. If I took a sample of the first thousand drops and constructed a pdf just like this, and then took the sample of the next thousand from 1000 to 2000. Similarly if I constructed 10 separate pdf's, would the 10 be approximately similar or would they be totally different from each other. They are all going to look approximately similar. I can take all the drops in that little strip or this little strip; they are all going to look approximately similar. I have about the 1000 drops in every strip there.

So, what that says is if the pdf constructed from any point, any sample of the population is the same I can call the spray as being stationary. That is at least in the time that I have investigated it which is this half a second nothing drastically is changing over that time. I can call this spray as being statistically time stationary. That is, if I now were to extrapolate this belief that I have or I can say is I can take a 10,000 drops sample tomorrow. On the same operating condition spray and I can expect to get a pdf that is approximately the same. It is very important belief to have without which you cannot do experiments or for that matter go further in any fashion.

So, this idea of stationarity that is if I bring on the same supply pressure, same air flow rates, etcetera I can get the same spray from a pdf sense. The exact arrival times of every drop could be different, but I am not concerned to that micro structure of this spray. The macro structure in terms of this pdf, I say it is not really a macro structure the statistical structure of this spray is repeatable; I am able to reproduce it tomorrow.

Same thing if I now come here, if I will throw out the time stamp or the Id; Id and time stamp are interchangeable. I can look at every drop in this size. So, essentially this is x in our old description f of x comma v. If I now construct the mound in this f of x comma v space that mound is going to be my f of is my 2 dimensional probability density function. In the previous graph what we could draw as one line curve now becomes a surface.

So, in this f of x comma v you can clearly see that right about this middle region there is a real peak density, and in this little ring around here there is less of a density. So, I have a peak essentially it is like a little hill that is sloping away from this peak. That is going to be the nature of this f of x comma v.

Now I want you to pay close attention to one thing, the diameter scale here is logarithmic. That means, in the log of the particle size I get what to me look likes a Gaussian curve or a normal distribution curve in the diameter space. The velocity coordinate here is linear. So the f of x comma v, this is now real experimental data right; f of x comma v tends to show a Gaussian like behavior in the log of the particle size and Gaussian like behavior in the linear, in just v.

So if I were to do this, if I was to create a probability density function of this graph essentially I can see this is my p of x from the previous graph. The probability this is my f of x comma v. So, these dots in some sense, if I was to process them appropriately I will get f of x comma v. If I ignore the velocity coordinate and only do the binning in the x coordinate what I get will look like that top red curve that is my p of x. If I ignore the diameter coordinate and only do the binning in the velocity coordinate, you can see how I will sort of get something that looks like this, this is my h of v.

So, if this is a graphical representation from a real data set of what p of x is, what h of v is, and what f of x comma v is. Now when can I say that there is no size velocity correlation in this data set? But that is a physical question to ask and there are mathematical implications associated with it. If all of these points that you see in this graph are sort of oriented along the coordinates; meaning, if I was to draw a best fit straight line and different levels of confidence interval on that best fit straight line. If the best fit straight line is horizontal then on average the mean velocity of a given drop is not dependent on the size. That is the real conclusion you can draw. That is saying on average the velocity of a given set of drops does not depend on the size. You can take any sample from any size part you will get the average velocity to be about the same. That is the actual precise meaning of size and velocity correlation.

I can do the same thing to the velocity axis. If I draw best fit straight line of the diameter as a function of the axial position that is to say what is the average size of all the drops that are moving with a given velocity, and if that also happens to be a vertical line then I can say that I can take any velocity; let us say if I ask the question what is the average size of all the drops that are travelling with let us say 0 to 1 meter per second versus 10 to 11 meters per second versus 20 to 21 meters per second. If these three average drop sizes where all the same that means, if essentially a best fit vertical line through this data is a straight line; best fit straight line that is vertical is nearly ninety degrees.

So, these two straight lines being oriented along the axis is effectively like saying that there is no size velocity correlation. The average velocity of any given sample of drops does not depend on the size or the average size of any given velocity sample does not depend on the velocity itself.

So, these are two different ways of looking at size velocity correlation. Now what does this mean mathematically? Mathematically what this says is f of x comma v, because they are oriented along orthogonal directions, the mean, the best fit are oriented along orthogonal directions. I can write the function f of x comma v simply as a product of two functions in each of the two independent variables.

So this simplifies my mathematics significantly. If I can do this, you remember in the most general case I have f of x comma v r, v theta, v z; instead of dealing with a function in a four dimensional space if I can deal with four one dimensional functions, is so much more simpler. So, except this thing called size velocity correlation I can actually deal with four functions in four one dimensional functions, as suppose to one four dimensional function in the most general case. In this specific case instead of dealing with one two dimensional function I can look at two one dimensional functions, and that would I can then reconstruct the two dimensional function from the two one dimensional functions.

Now mind you this p of x is approximately Gaussian as you can see from this, and h of v is also; now really speaking it is not p of x that is Gaussian it is p of this is p of x, but remember it is in the log coordinate. So, it is Gaussian in the log coordinate. Now with this let us go back and look at some very simple functional forms. So, if I have to take this kind of data and actually generate p of x or h of v what I need is a mathematical function for p of x or for h of v that I can find a best fit to this data set.

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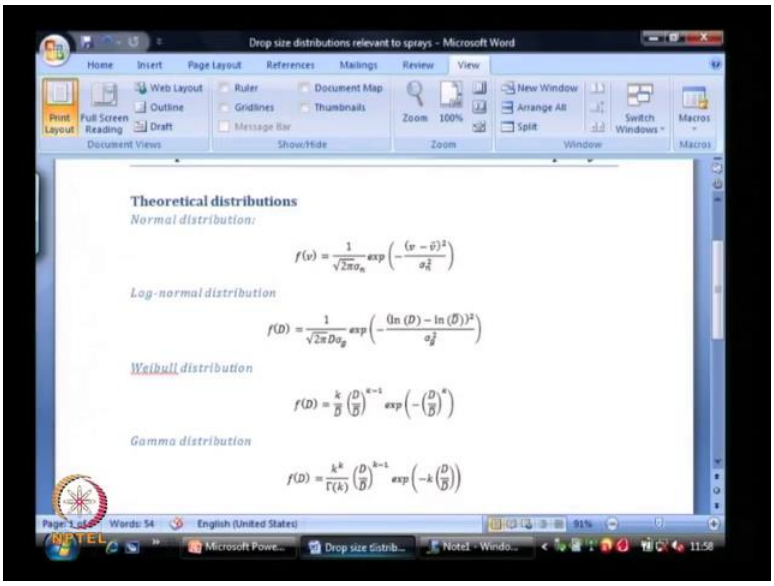

So, some of those mathematical functional forms are listed here. The simplest is called the normal distribution. So, this is like your h of v is given by this 1 over root 2 pi time sigma n in the denominator exponential of v minus v bar squared by sigma n squared. So, I have two parameters in this functional description; v bar and sigma n. So, this is called a two parameter probability density function. And for any given spray, say for example if I go back to this, this gives me the value of v bar. And like a extent of the spread gives me sigma n.

So, this is by if I have an average faster moving spray all it means is that v bar is a higher number. On average if the distribution as a wider range of velocities then sigma n would be a higher number. So, this is a functional form that I can fit to any sort of a pdf. You can imagine how a different point in this spray, because the sample of drops are moving with different velocities and different spreads in the velocity v bar and sigma n would be a function of the spatial coordinates. So, a pdf like this is a pdf in two parameters, those two parameters can become functions of r, theta, z, and time.

So the reason I put the semicolon in that 8 parameter descriptions is the pdf itself is only in the variable v, but the parameters in the pdf v bar and sigma n can be functions of r, theta, z, and t. Now, so this is one simple pdf that occurs quite commonly we will see this. The second is the log normal distribution, so it is essentially a normal distribution as you can see here except it is normal in the log of the diameter. We saw this in the previous slide, where if I was to make the particle size, if I was to plot the sample of drops in the log of x space the spread this is now my p of x looks approximately Gaussian.

So, again this is a two parameter description just like a previous case with D bar and sigma g squared as being the two parameters, sigma g being the two parameters and D bar and sigma g in this case could be functions of r, theta, z, and t. So, that would be the way to give like the most detailed level of information.

And there are errors that we will see that look like the same spatial, that look similar in form one is called the Weibull distribution. This is also a two parameter description with D bar and k as being the two parameters; likewise, the gamma distribution with D bar and k as being the two parameters.

These are just examples of functional forms that you can use for either the p of x or h of v in the previous case. And as long as you have know size velocity correlation you can choose to describe f of x comma v a simply a product of p of x and h of v.

We will stop here, and we will continue this in the next class.