Spray Theory and Applications
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Lecture - 06
Discussion on pdf and moments

Good morning again. At the point where we left of last week, we were talking about the difference between probability and probability density, how probability can be a theory of probability or actually the whole concept of probability can only be applied to where the number of outcomes is finite and countable. Whereas, you need to go to a concept like probability density when they outcome is on the real axis. So, any real number between some limits is a possible choice for an outcome.

So, without going much into the theory and this is actually non-resolved question in sprays. We will for now assume that drops can be any size between some minimum size and the maximum size. The minimum size could be 0 the maximum size could be infinity, for all we care, but we will just assume that there they do have some finite limits and even otherwise nothing in what we discuss the other day is going to be different as long as there are some limits on which all the drops exists.

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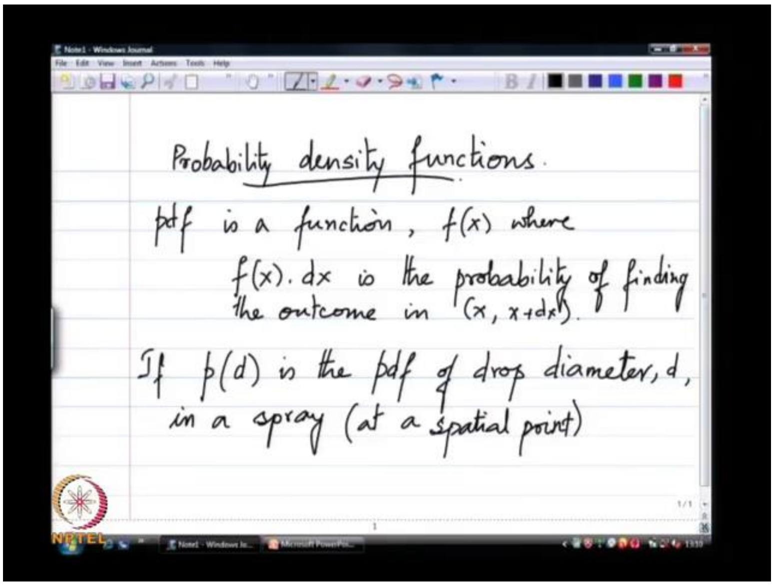

So, we will continue that discussion and start looking at sprays in some detail. So, we know that we define a pdf. So f of x is a function and f of x times dx is the actual probability of finding the value in the range x to x plus dx. So, if this f of x is defined in this fashion then f of x is called a probability density function. So, we look at the case the other day when we looked at circular hoop and we said what is the chance of one of the angular position showing upon top and really speaking the probability density could be any value. In fact, the probability density does not have to be between 0 and 1 because this is the actual probability of any set of outcomes has to be between 0 and 1, but the probability density itself can be any number; will seen in short way. In other words the function has now upper limit on the values it can take as suppose to probability, fine.

So, applying this principle to sprays what sort of intuition can we bring to the forms of the function that this f of x should have in a general spray? Let start thinking about that for a little while. So, if I will now use a slightly different notation I will say p of d, p of d is the probability density function of say a drop diameter in a spray and I will be even more specific will just say at a point for now, at a spatial point. So, this is sort of the easiest to understand. So, we will start with this.

So, if I did my temporal sampling. So, I was sitting at one point in the spray and I sampled every drop that went by me and I accrued statistics of a large number of drops and we discussed how to construct the probability density function from a large set of such drops. I did that and I got this function p of d, p which is a function of d.

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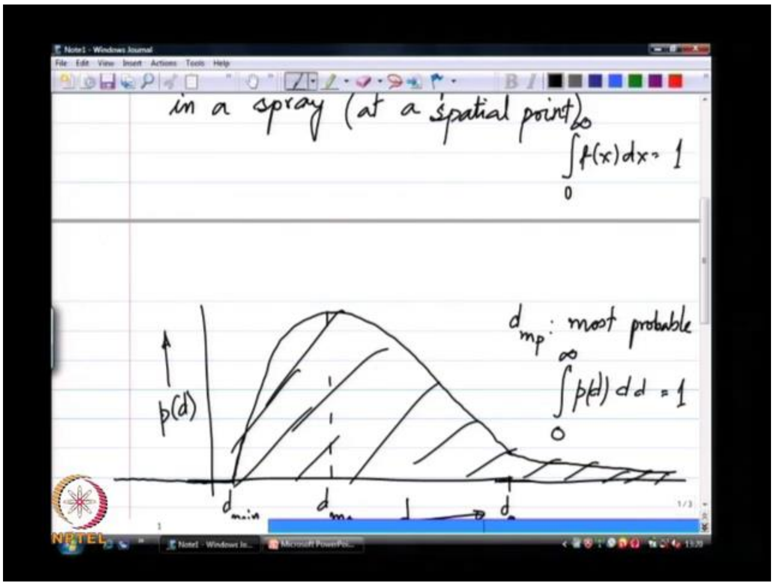

What sort of functional forms what should the graph of p of d look like? So, if d is, generally speaking what would you expect the graph to look like? Let us draw some simple features; the first sort of physical limitation is that the drop size cannot be less than 0 correct. So, on the right side, on the left hand side of the graph the probability density clearly has to be 0.

Now can I find a drop of size 0? There is no such thing as a drop of size 0, the drop is of size say some minimum a possible size in the spray. Now we will see that in we will see it may be in a couple of classes from now that, that number is actually a number that is much greater than 0, much in the sense of atomic scales. So, if one molecule of water is a few (Refer Time: 06:40) the drop of, a drop of water the smallest drop of water that you can create using a spray process will be much bigger than this size. There are reasons for that we will talk about that, but let us for now say there is a d min below which I do not expect to find any drops.

What about a d max? Let us talk of the limits if I take a typical spray we looked at the perfume spray in some detail, there is a whole through which a whole on that perfume can, through which drops are delivered into the spray right. Now is it reasonable to say that I there can be no drop bigger than that whole diameter. So, in other words if I have an inject diameter I cannot produce drops greater than that inject diameter. Strictly speaking I am wrong, I can produce drops greater than the diameter if I have other process is going on later on.

So, if two of these drops want to come together and (Refer Time: 07:53) I have no reason to prevent that from happening. So, technically speaking there is I cannot say that I cannot find drops greater than that size; all I can says the probability has to continuously decrease as the size decreases, as the size increases. So, as the size say for example, if this is my orifice, all I can say is I go further away from this d o this probability has to somehow decrease. So, that is going to have to be the form of the graph going away it has to asymptotically reach 0.

Now this is as for as our understanding without really having a spray in front of us no data nothing say now can I actually allow this graph to come intersect. So, if is this a reasonable possibility that the graph will intersect this axis beyond that the answer is that there are no drops greater than that particular size, all I can say is I cannot make this generalization for every spray. So, at the moment whatever we are saying we want to leave it sufficiently flexible to be able to apply to every spray that is sort of our approach for now.

At the moment all I can say is that this has to asymptotically approach the diameter axis. So, the probability density asymptotically becomes 0. What about the shape in between this and this? Now first of all I know the shape of the graph near d min it has to increase, it is the only way it can go, probability density also by the way cannot be a negative number, right just like probability. So, probability density is a positive number with no upper limit, lower limit is of course 0.

So, with these as the distinct now I can; there is only one possible general shape for this graph, if I say this is the starting and that is the ending there is only one way the graph can go which is if the probability density function is a continuous function and differentiable every where this is one possibility - which means that there is one point, which is the at which point, the probability of finding a drop around that point is the maximum this is not the average drop size. So, we will look at that in just a moment. This is what we will call the most probable drop size. So, you can sort of just all though this is a not a widely used way of referring to it, we will very soon make the distinction between average and this most probable drop size.

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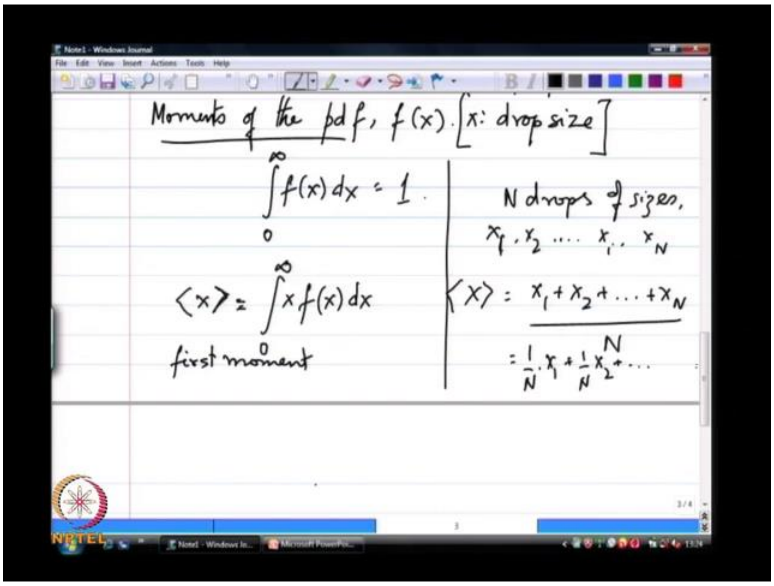

So, essentially I have this function p of d which could be; which now has these general features. If I want to write a mathematical equation that fits this there are many, many different mathematical forms that are possible that initially increase and then decrease, with or without with d min being 0, with whatever it is there many mathematical forms that are possible. As many mathematical forms, so many proponents in the literature for those mathematical forms may be this is there is at least like 5 or 6 very widely used forms of distribution that people claim are fundamental to sprays and each of them has their own valid reasons for may for those claims, but this is the shape that is generic to all of those mathematical forms. We will come to that in next class, but before that I want to define a few parameters that are not dependent on the specific mathematical form.

The whole point is that ultimately if I say what is it that I have achieve by defining this, I have defined the spectrum of drop sizes that I can encounter at one point in the spray over some period of time. How useful is this? Very useful, because I now know what this spray looks like from the inside like we said before, right. But in many instances this is too much information I want to condense it, if I have to now move to another point I have to find this all over again for that other point.

So, I want something that is more manageable and therefore, going from this full, this is the full probability density function we want to define what are called moments of the distribution. The simplest moment to define before we go to defining moments I want to make the one obvious point one more time that if I take any of this the area under this, the area going 0 to infinity of that probability density function in d d it is a little tricky, but I cannot help it, I can probable make the point over here as well just to sort of because integral 0 to infinity f of x dx because f of x dx is a probability of finding the point in the set of outcomes x to x plus dx. If I do this addition over all the say all the x that amounts to integration and that is what I get. This is the only requirement of any probability density function, alright.

So, I now want to define a set of moments of this. The simplest is what we call the average diameter you know I will move away from this d d business and I will say moments of a pdf f of x and I will define x is drop size in this case. First constraint is that f of x dx equal to 1 and then if I define a mean drop size let me write down what this is. So, the probability of finding a drop in the limits x to x plus dx is that, so that probability multiplied by the drop size itself added over all of the various size as possible is what gives me the mean drop size.

So, just to sort of illustrate the point if I have let us say different sizes x i, x 1, x 2, dot, dot, dot, some x i, x N, I have N drops of sizes then the mean drop size for this sample we all now would be x 1, x 2, that all your saying is at each drop has a probability 1 by N. So, you are essentially multiplying 1 by N times x 1 plus 1 by n times x 2. Now if some other, if x 1 and x 2 were the same for example, in this set then that would become 2 by N x 1 2 by N x 1 because we have two drops of size x 1. So, essentially what we do by average is a same as what we are doing in the sense of probability density. Now we are going to get use to this notation of using the angular brackets to denote mean.

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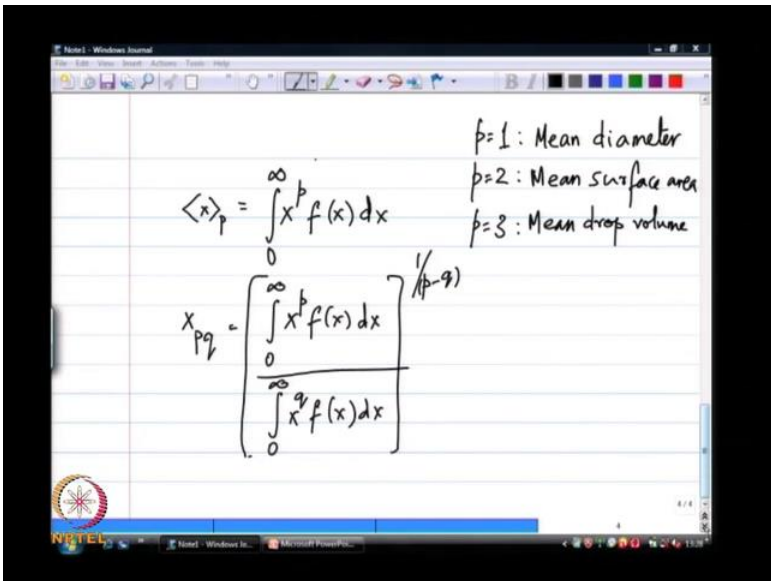

So, this is often called the first moment. I can define any moment of this distribution by doing exactly that. So, if I take x power p, p could be some number then x power p will just like if I have drops of size x 1, x 2, x 3 etcetera, then I can define if I say what is the average surface area of these drops. If I choose p equal to 2 in this case that gives me the mean surface area, barring some factors like pi or pi by 4 etcetera you know that does not I mean you are not really worry at that level of detail and we are not concerned that also usually. If I choose p equal to three what do I get? I get the mean droplet's volume.

So, I can define any of these moments you know I can define the p equal to 4, now what would that physically mean? I do not know, but I am sure there is a reason to define it and people actually use it will see in the moment one such use. So, and by the way p equal to 1 gives me the number based average. So, if I said p equal to 1 that is simply the mean diameter. I can define an average called x pq from this same f of x as the following if I say x power p f of x dx integral 0 to infinity divided by x power q f of x, 0 to infinity, if I raise this whole thing to the power 1 over p minus q.

Let us think about what this is, x power p, this is the p-th moment and the numerator is the p-th moment, the denominator is the q-th moment of this same distribution. If I take the ratio of the two that gives me sort of the value of the p-th moment in comparison to the value of the q-th moment that is essentially what ratios are, but though funny thing is if x has units of say micrometers or millimeters x power p has units of millimeter power p x power q has units of millimeter power q.

So, this ratio will actually be a dimensional quantity of units millimeter power p minus q which is like a funny unit, I do not want to deal with that. So, if I raise this whole thing to the power 1 over p minus q that gives me something in the units of diameter. So, this is all, some moment of the droplet's size distribution and it has the same units as diameter. So, I can now relate to it physically that is the only reason to do the power 1 over p minus q. Let us look at some simple possibilities. If I said q equal to 0, and p equal to 1, we get our first moment p equal 2; q equal to 0 gives me the second moment just the way we have defined.

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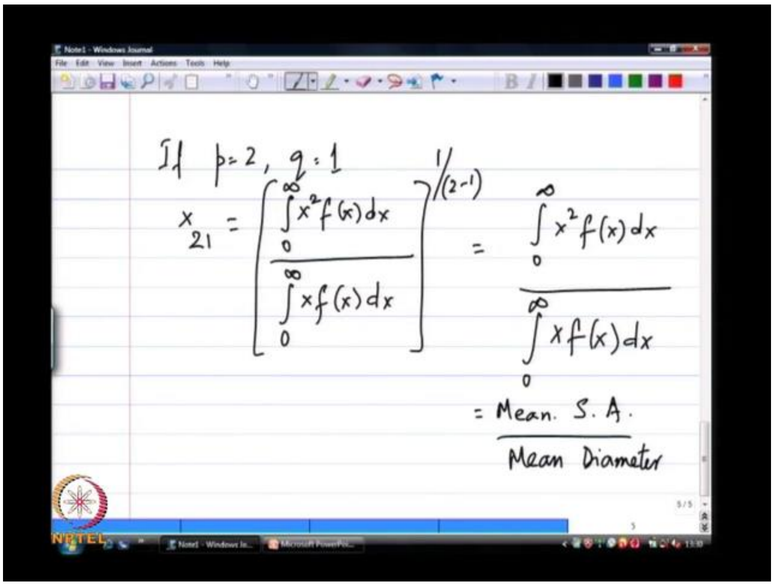

But, let us say p equal 2 and q equal to 1 what do we get? Raise to the power 1 over 2 minus 1. So, this is essentially the mean surface area divided by the mean diameter.

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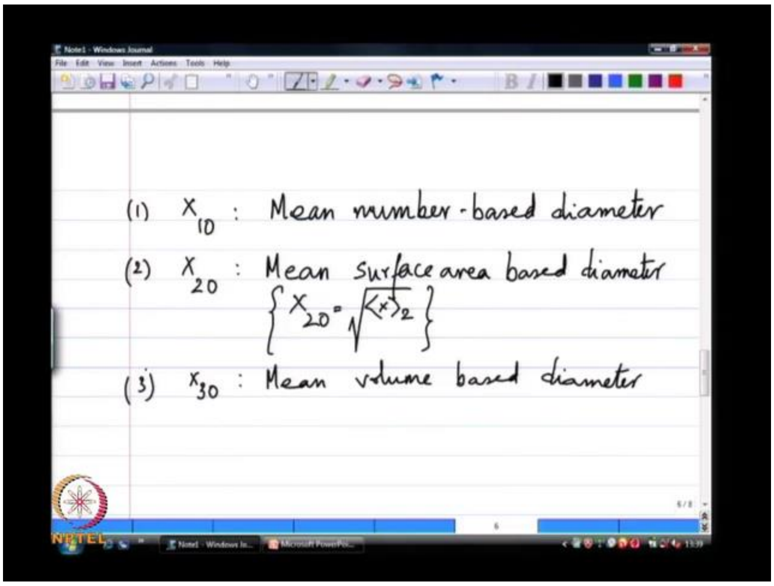

So, going back to our original distinct I want to now write everything in terms in this p q notation, x 10 is the mean diameter you can say that, x 20 actually let us be clear about this I am sorry will say this is the mean. So, just simply a number based diameter because I am just doing an arithmetic average of all the drop that is essentially what x 10 is if you go back look at this notation here of x pq. If you look at this notation of x pq the numerator is x f of x dx which is just like saying I am doing an arithmetic average over all the population of drops that I have; x 20 is the mean surface area based diameter and just for the sake of equivalence I am going to do a 1 over 2 minus 0 to make it look like a diameter.

So, I have taken the mean surface area in the spray at that point and divided and taken the square root essentially to give me back a number that has units of length of size likewise x 30. So, these are all diameters that is the point to note here, there are all they have the same units as diameter, but there all different physical quantities because now the x 30 tells me sort of there average volume. So, it tells me were the volume is concentrated.

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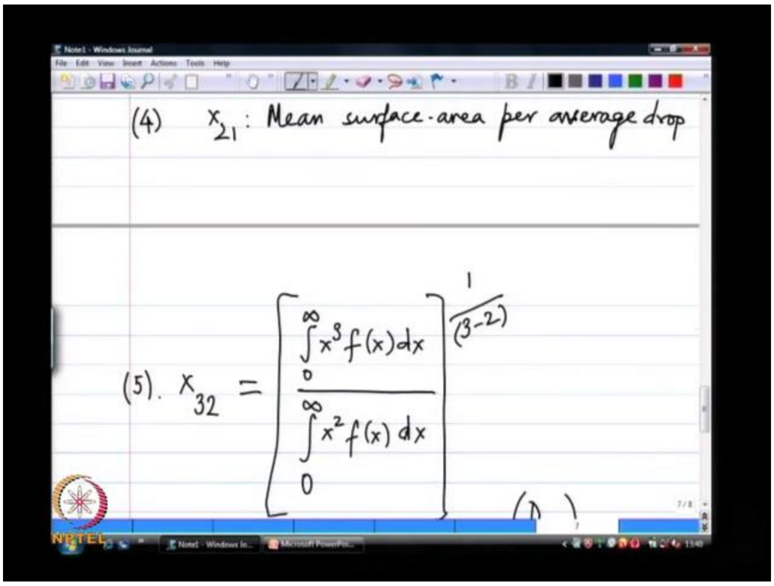

Another x 32 I want to write these out, now this has a special name this is often called the Sauter Mean Diameter.

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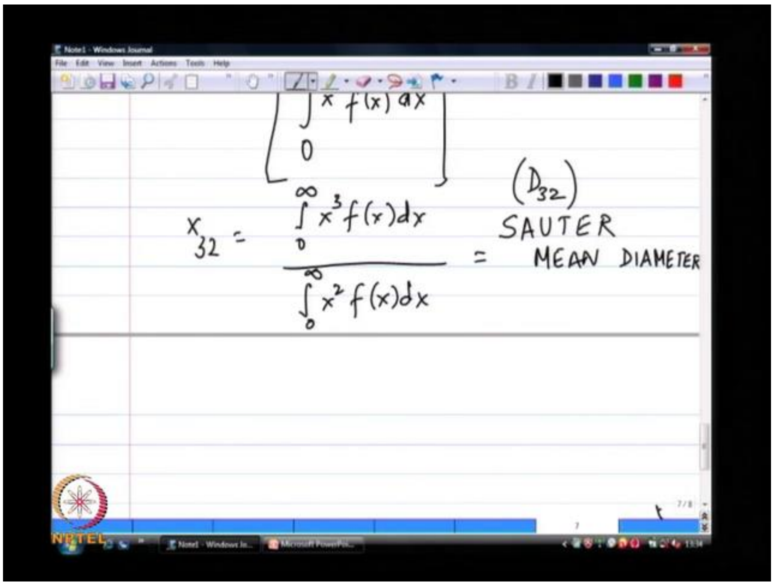

Now, when you say Sauter mean diameter in the literature they do not use x they use d for diameter, I am just using x because that is easier to do dx and these kinds of notation, but imagine D 32 you will see this written very often, D 32 is the Sauter mean diameter at that point in the spray. Now this is very widely used as a measure of a spray. Now remember it is just another moment of the pdf that is how you get the Sauter mean diameter, but what makes this number so special? It is essentially the fact that the numerator is the mean droplet's volume and the denominator is the mean surface area.

So, both those quantities being physical if I want to imagine a situation where I want to release the volume contained in a liquid drop in to a gaseous medium it has to go across this interfacial area. So, any sort of a mass transfer problem that is the simplest, imagine but it also applies to heat transfer. Any sort of interfacial transfer problem involves volume because that is your source of either mass or thermal storage in the form of rho v c p or m c p, that is the total thermal inertia or thermal storage capacity of a given drop.

So, that m, or rho m or v is an indication of the volume in the which is in the numerator and the denominator is what the mass has to go through the total interfacial area that is available to a certain given drop for that mass to become let us say evaporated or heated or whatever interfacial process has to happen. So, this is essentially also refer to as the mean, this is essentially like a mean volume per surface area for the drop or one over this is the mean surface area per unit volume you know, in our very first lecture we said the whole objective of a spray is to increase surface area, here is a direct measure; surface area of what? Surface area of a given volume of fluid, so here is a direct measure of that increase in the surface area per unit volume.

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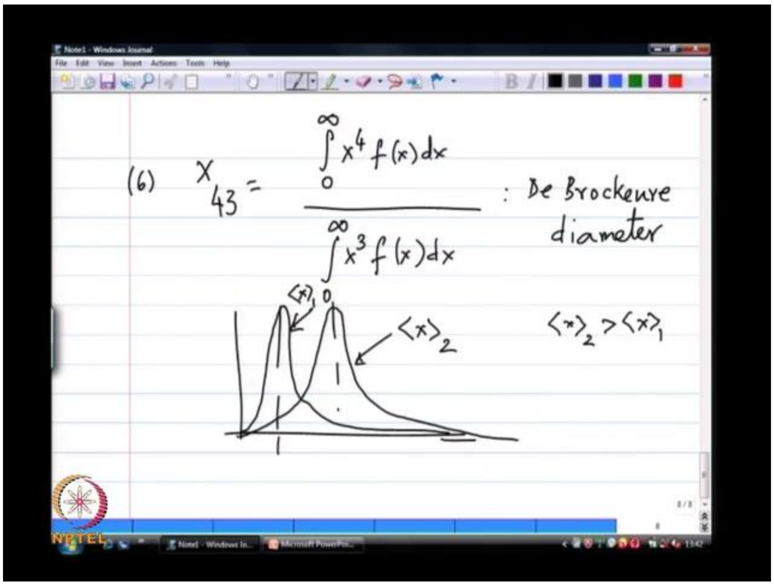

So, if you in some very crude way take 1 over this and multiplied your volume that you had in that 1 whiff that automatically gives you the total surface area that you have generated. So, that is, this is one of the more commonly used measures. Another sort of commonly used measure called the De Brockeure diameter. Notice that if p minus q is equal to 1 and automatically you have units of diameter. So, this is used in cases where in like spray drying where evaporation is important.

So, it is like x square gives you the surface area and x power 4 is a moment that is related to it, although it is really speaking Sauter mean diameter is what you will say reported in many, many different applications. In fact, some of these theories of you know handling distributions came about even before they had the idea of characterizing sprays using distributions the first origin of idea of distributions came from pulverized coal when power plants, when coal fired power plants were being built. The most you know some you must figured it out that the most efficient way of burning coal was to pulverize it to tiny particles and then burn it.

So, I now had to have some measures of what the result of my pulverization process. So, all of this distributions, all of the moments that you see here D 20, x 32 or D 32 were all developed for particle sizing in pulverized coal applications and they are just as applicable for this case as well. So, here we have, I have what I am claiming now is that using let us say I will start to number this just to give you a quick count of where we are, let us say this 1, this 2, this 3 I will also I had x 21 here which is the mean surface area per average drop. So, how much? So, that is like another measure of surface area per average diameter; this 5 and this 6.

So, we are now somehow claiming that instead of giving you a whole pdf if I give you either the 6 numbers or even a few of these 6 numbers you can have, you can get a reasonable estimate of what that distribution will look like, without really having the full functional form at your disposal. Knowing the limitations that we started to talk of, if I just tell you and x 20 and did not tell you it was from a spray really you cannot make much of it, but tell you it is a if I tell you it is a drop size distribution then you already have all these intuition that we started with, that the drop size distribution has to tend towards has tend towards smaller and smaller value as you go forward, as you go towards infinity.

It has to have an all most finite cut off although d min can be 0 there is no reason to not be 0, but we usually find it to be finite value and it has to reach a maximum value at some point and then drop off. This intuition take an alongside that number now has, now tells you some shape of this distribution like for example, I can clearly say I mean the simplest intuition to start with is to this would have a greater x bar we can already see that.

So, if I just tell you x bars of two of the spray at two different points you can imagine what these distributions would look like, they have to be something like this. Now sprays it, spray themselves spray itself is not a one dimensional entity, I am not talking special dimensions or time the set of drops at one point in the spray are characterized by more than one variable we looked at a whole list of variables in a very first class. So, I could have a diameter be a characteristic, temperature is another scalar characteristic, concentration is another scalar characteristic, velocity is a vector characteristic, I can treat the velocity has been let us say at most 3 most, 3 more scalar may characteristics scalar properties. So, I could have all these scalar properties of a drop apart from the size itself.

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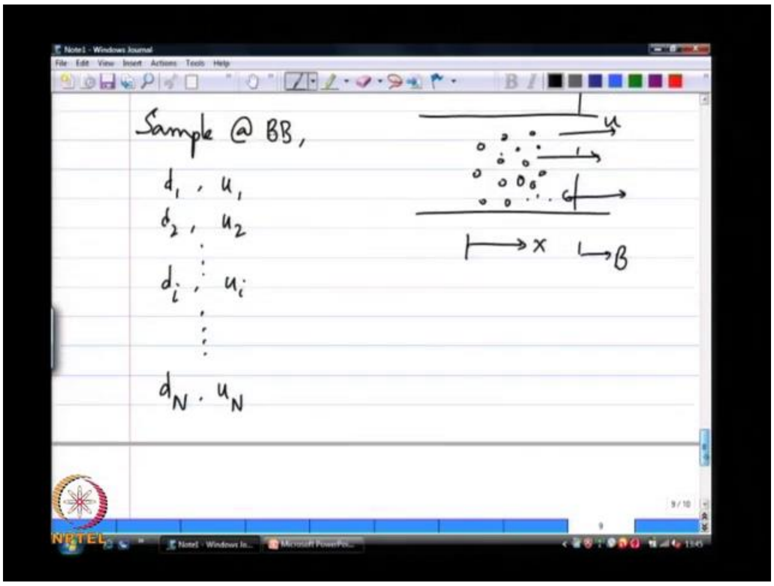

So, if I want to understand what the set of drops look like, in that space I need to go to multivariate distributions say for example, I will take a simple case of drops in a pipe let us say they are all moving only in the x direction, there they could be of different sizes, but there are all only moving in x direction. So, I have a velocity u. So, if I do what I told you before if I sit at this location y y or BB I will just call this BB, if I sit at BB and sample all the drops going by basically measure its velocity and diameter. I have N drops I can sample the diameter and velocity of every drop. I can now construct pdf in two dimensions in d and u.

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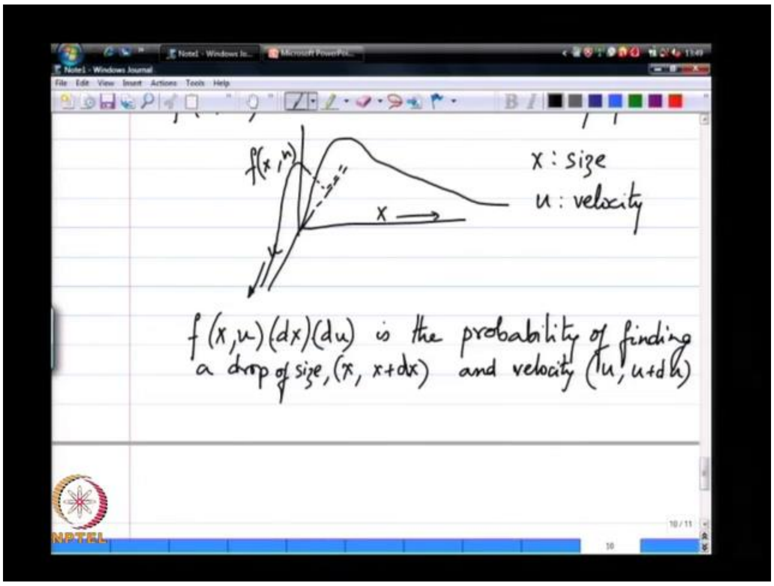

I put this in quotes using the word dimension only to show the idea that I have two orthogonal directions in which pdf could vary because the function itself is a function of two variables. So, essentially it is a two, you have two independent variables in the argument list of the function which is d and u. So, if I will try to draw quick cartoon sketch of this.

So, essentially this f of d comma u now becomes a surface. So, I could have, in this direction it could be something like this, in the other direction it could be something like that, I am just drawing sort of images, sort of shadows of this distribution and two different directions. These are all I could have this surface in three d in f of d comma u is a function of d and u and this surface gives me the total, they gives me the probability density in this two variables. Now again by doing this I only made the problem more unwieldy.

So, I need to, I am better of going back to my moments. I want to see if I can define some moments of this 2D distribution in the context of d being diameter and u being velocity. So, the simplest thing is to say that if I want the average diameter just like. So, let us before we go this far I want to rigorously define what this is, can, let us me just quickly rewrite this in terms of x I thing that makes more sense, x is size, u is velocity.

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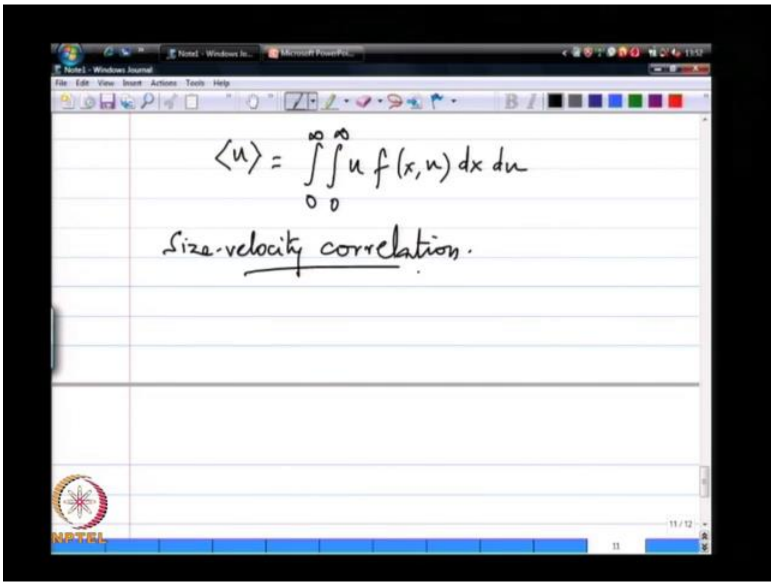

Now, the mean size in the spray at that point is simply the integral of this. So, if I take this x f of x comma u dx du this gives me the probability the f of x comma u time's dx du is the probability of finding a drop in this range that, in the set of values that I have defined. If I do this two variable, if I do this integration that gives me the mean value over all the sample of drops. This like, just like an average sizzling except since I have a second variable of integration u, do it I have to do a double integral. Likewise I can now define an average velocity and I can define all the moments that I described back then.

So, I can define like x 32, x 43 you know all of those that we defined earlier or also applicable except each one, each term the numerator and the denominator would now be a double integral. Now there is only one more additional physics that is introduced in going to two variables in the probability density function that is something we talked about in at the end of I think two classes ago, which is this idea of size velocity correlation. How do I understand this size velocity correlation? That is, are the larger drops moving faster than the smaller drops or vice versa or there is really no relation between the two, how do we go about that? The way to do it is to define an average velocity that is conditioned on the size.

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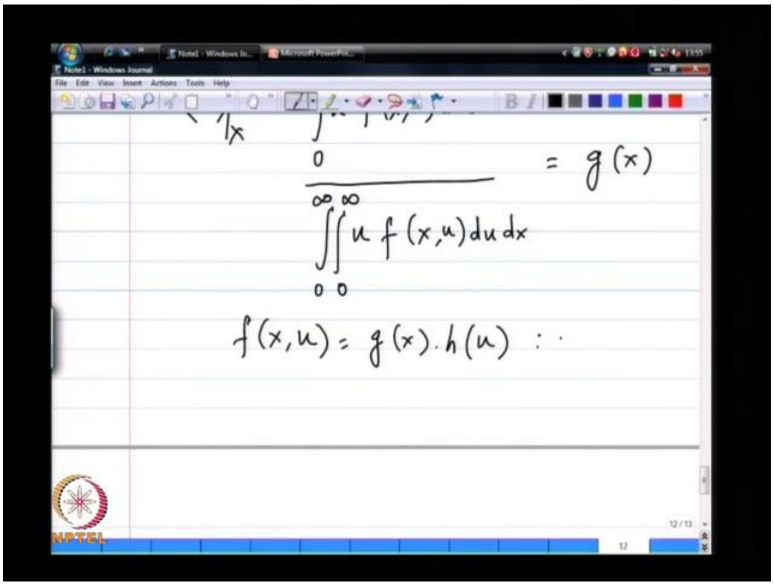

So, I will put this, that is if do this integral f of x comma u dx, sorry f of x comma u du, f of x comma u du is the probability density, it is still a probability density because I am not eliminated the other variable x, it is a probability density in the space in the x space. So, this gives me and that times u gives me mean value that is conditioned on the size. So, this tells me the velocity while x is still a function of the, so if I take this and divided by the average velocity this gives me a function of x because the numerator is not is a function of x.

So, numerator is the function of x, the denominator is just the mean velocity like we have defined in the past. The ratio of these two would remain a function of x and this tells me the average velocity of each x, of each set of drops in the size x to x plus dx. So, it is like an average velocity of that population of drops and as it turns out if this is a straight line, if this average velocity is not really a function of x, but is relatively the same for all x that tells me there is no size velocity correlation.

So, one way to another way to understand this is that I have this two dimensional functions, I want to look at two; I want to see if I can split it up into two independent if I can separate the variables out. So, we will talk about this in the next class. If I start of by saying I can do this; that means, the velocity distribution in the u coordinate is independent of the distribution in the size coordinate. So, this tells me that there is no size velocity correlation. So, both are equivalent if I through the first measure find that there is no size velocity correlation; that means, I am able to separate the function of f of x comma u in to this form.

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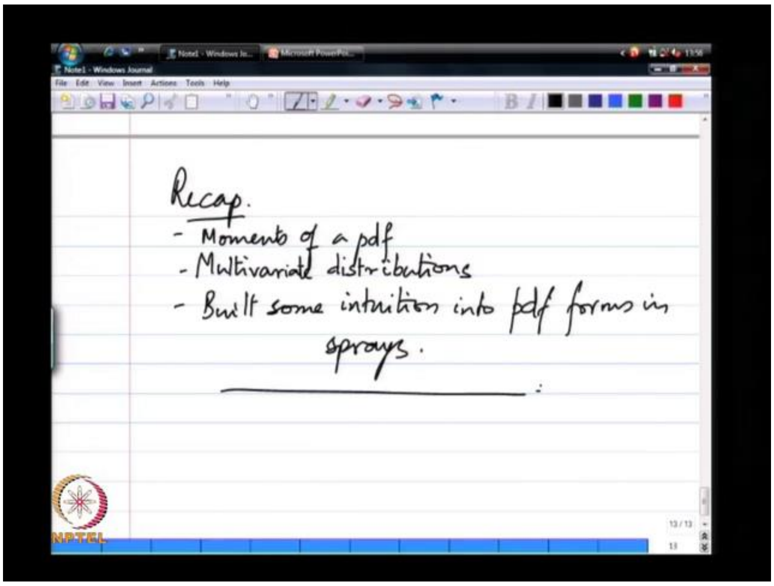

Let us quickly recap what we learn today and then we will stop here. We started with moments of a pdf and then looked at multivariate distributions and along the way we built some intuition, built some intuition into pdf forms in sprays. So, pretty much every pdf you see will have that up and down and asymptotically going to 0 kind of a trend.

We will stop here.