Spray Theory and Applications
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Lecture – 05
Statistical measures on spray

Good Morning again, what we are going today is build upon the last lecture, when we found out that somehow this idea of drop size distribution, velocity distribution and if I may even said temperature distribution. These are now becoming relevant quantities. I need to get some mathematical framework in which I can talk of these distributions.

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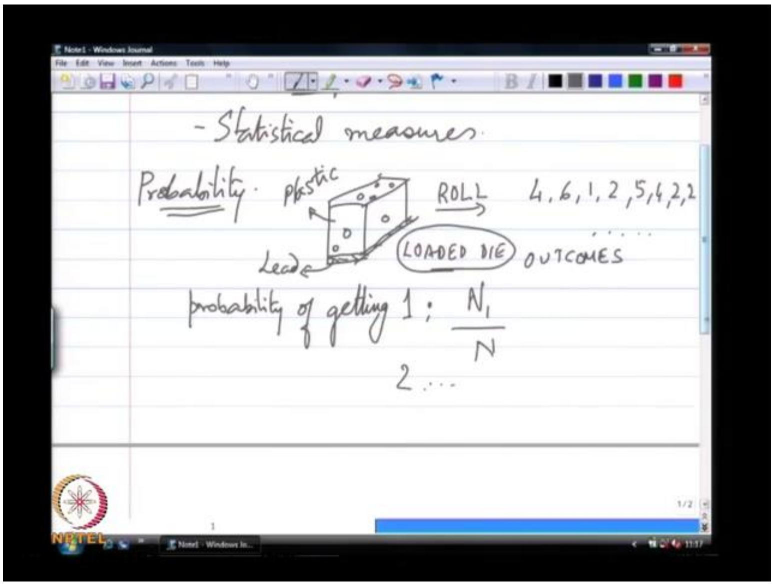

We will see the whole idea of this frame work is going to be that from knowing some information of all the drops that have sample that is far, whether it is in a special sense or in a temporal sense. I have no idea, what to expect from the next drop with a 100 percent surety. If I sampled half the frame in the special sense and I am about the step into the second half I have some estimates, but I cannot be 100 percent sure what to expect of the next drop I am going to sample. This is the idea that this is not deterministic, but I need some sort of a statistical measure because there is some sort of stochasticity there is some uncertainty.

Now, we will probably about third of the class later on, will see what are the possible sources of this uncertainty, but for now we will just say look I do not know, what the what the next drop is going to be. I need some statistical measures of these distributions. We are going to try and understand that those. And one of the fundamental concepts we need to understand is this idea of probability. We are going to high school probability for a moment and will sort of try and see if we can build upon it.

We will take a regular 6 sided dice. I have a 6 sided dice, I roll it and I get an outcome. That is say I get these as my outcomes. From these outcomes, I can build a probability of the number which is the number of times 1 occurred divided by the total number or I will use subscript to this. This is a fairly oblige, this is called oppose theory or e probability, that is I only can build this after I have done all these trials. But then you say that this dice has changed; the floor on which I am rolling has not changed. This can also become my a priori probability for the next roles. I can use this information to predict statistically, to predict, what could be the outcome for the next role.

These are some simple high school concept. What we want to do is? I have this probability of getting 1 2 dot, dot, dot. I have a perfectly fair dice I know that these are all going to be one-sixth. I have a loaded dice and let us say if I have this dice and this bottom part is all lead and this top part is all plastic. This is the case of a loaded dice. Then I am more likely to get the number 4 from the dice roll than other numbers I put. That does not mean, I will not get the other numbers at all, I will get, but I more likely to get the number 4.

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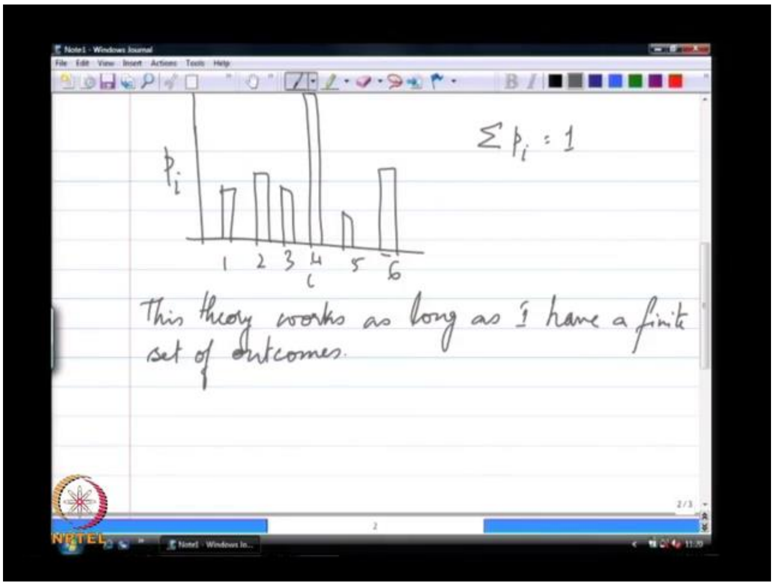

I can now write, probability of getting any number i, use the subscript notation as a function of i. The probability of 1 2 3 4 5 6, whatever these some probabilities associated with each of the 6 outcomes and the sum of all the probability has to be equal to 1.It is our basic you know high school understanding of probabilities because there is no outcome possible.

Now, if I take this 6 sided dice and I make it 20 sided dice. I can create a 20 sided dice and it has got 20 flat facets. Every time I roll, 1 flat facet comes up to the top. I have 1 of 20 outcomes possible and for every 1 of those, I do a sufficiently large number of dice rolls, I can build in apposed area of probability from each of the different sets of count on count of the each outcomes. I can do this for 20 sided dice; can I do it for a 40 sided dice? Sure, I roll it 1 faces on top.

The only limitation I can go from 6 to 8 10 20 40 100, I can keep going, as long as I do not get into the point where there are an infinitely many outcomes. As long as I have somehow countably many outcomes from this dice role, this theory works. If I say that there is only 1 2 3 4 to 20000, those are the integer outcomes possible in this experiment, in this trial, I can do if there are 20000 outcomes, I have to do a very large number of trials, but I can build this with some patience, I can build this probabilities of each of the outcomes. But if there are infinitely many outcomes; now I want to talk of a case where there are infinitely many outcomes, what does this means. This works, first of all what do I mean infinitely many.

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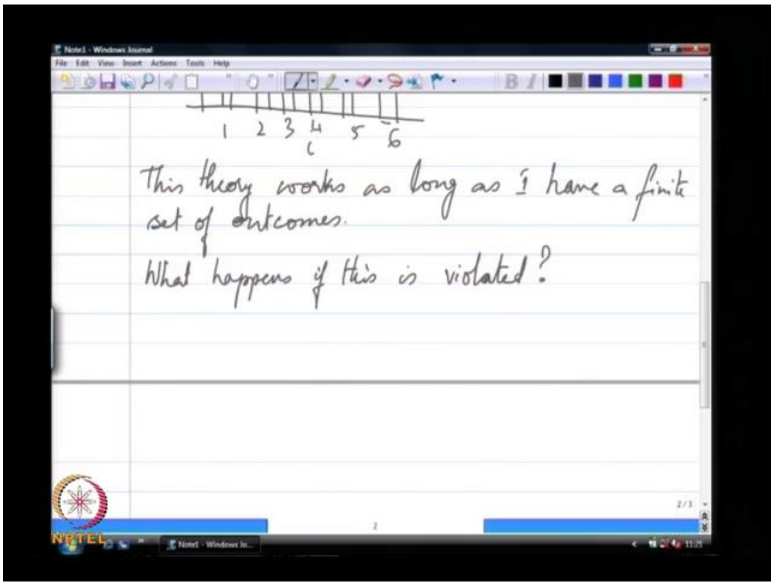

This works, this theory works. I want to understand what happens if this is violated, because if I have an infinitely many outcomes, the probability of any 1 outcome is nearly 0.

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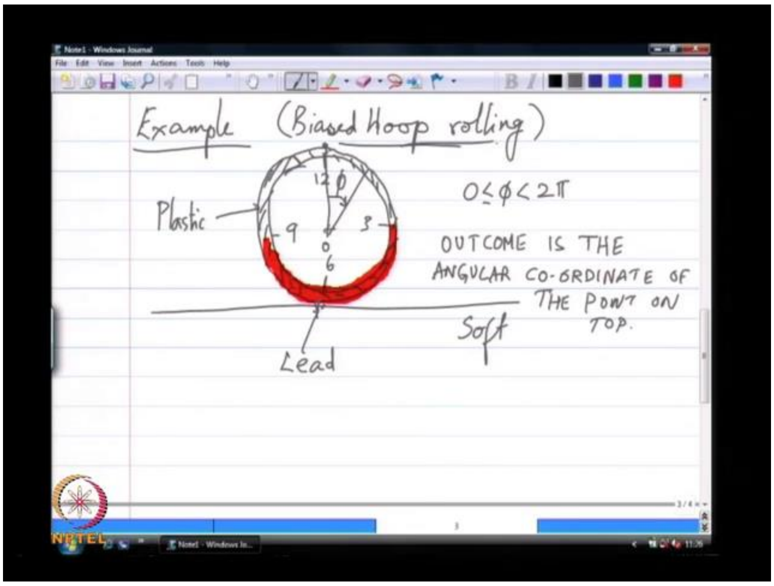

We will take a simple example of such an outcome case I will take a hoop. Instead of dice roll like we saw we are going to now roll a hoop; a hoop is a round bangle kind of things. I am going to mark it, let just like the dial on a clock. I am going to say, this is 12 o clock, this is 3 o clock, this is 6 o clock, next 9 o clock, this is the hoop I am going to roll this on a floor where the floor is soft.

In other words, if I roll a bangle it will never come to our rest as long as the floor does not provide in a friction, just a simple understanding of how to, I need the roll to stop for me to count the outcome. I roll this hoop and it comes to rest. Let us say, I am very lucky, a very first outcome is where 12 o clocks is on top. And just for the sake of argument, I will mark that as my 0 degrees and count an angle phi going 0 to 2 pi. My outcome is my outcome is the point on top. That is what I call their result of this trial.

I roll this and let us say, if the number of 3 was exactly on top, then my outcome is pi by 2. We will first just drop on our intuition briefly before going to the theoretical aspects. If I roll this hoop 10 times, I do not know initial condition, I am just sort of randomly rolling it just like I do with the dice. What is a chance that any 2 numbers in this would be exactly the same is almost 0. I only have to say almost 0, because I can never say 0. But I do know, I have an infinitely many outcomes because all the real numbers between 0 and 2 pi are possible choices of outcomes and between any 2 integers there are a set of real numbers.0 In this case between, then the real number is 0 and the real number 2 pi, there are a infinite number of real numbers.

I am now; I have to somehow construct the same apposed area probability for this hoop. Let us say this bottom part of the hoop has got some lead in it. This bottom part is lead and the top part is plastic, just like before. I have created by a biased hoop. I will now go back, biased hoop roll. Every time I roll, if I know that the bottom part is got lead then I know what to expect. But the idea of using statistical description, this is that I do actually do not know what part of the hoop has lead through my trials I want to understand that.

If I do this, I can sit and count number coming out from my outcomes. They will all be real numbers irrational, some rational, a few rational many irrational have to, if I have a way of actually counting the angular position. I will get as many numbers as the trial I do and chance of any 2 numbers being the same in a finite set is almost 0. How do I construct my apposed area probability or for that matters some information about the hoop is made of ultimately that sought we are after.

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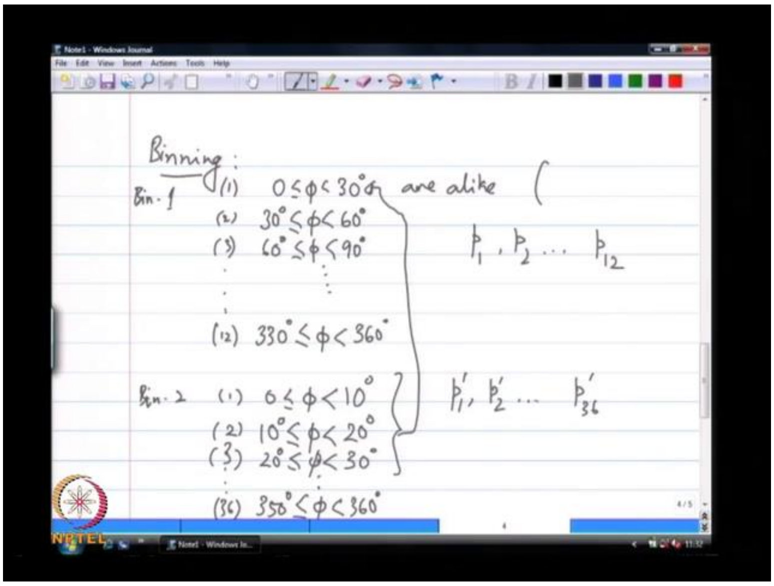

What do I do here? I create this leads and then I have to go to this histograms. I am going to say, instead of assuming that every individual real number outcome is the same, I am going to through a process called binning. I am going to now say or all outcomes from 0 to 30 degrees are alike. I am going to not distinguished between 29.9 and even 13, I am going to put all of them in one bin. Now I can do a count, because even if 2 outcomes are not exactly the same, since I have sort of created this equal the similarities between outcomes I can put them in the same bin, moment I put them in the same bin, they are all the same I just need a count.

This process, they are alike. Once I do this, I create this bin in this outcome. Now, I only have 12 bins. The way I have done this, the last bin is 330, this is my last bin. I have 1 2 3. I have taken a hoop and created a 12 sided dice that is essentially what I have done. I can now go back to my old theory and find the probability of bin 1, probability of bin 2 dot dot dot, create the same histogram just like I did with the 6 sided dice.

Now, remember that these divisions are entirely artificial. The hoop does not have any distinguishing feature between 29.9 and 30.1 correct, but I have created that distinguishing feature by putting this line at 30 degrees. I have an obligation to check if this distinguishing feature that I have put in there has a consequence. In other words, if I do this and I find the probability of 1 probability of 2 dot dot dot probability of 12. If I recreate a different set of probabilities following a different binning sequence, this is I will say, binning sequence 1. If I do it following a different way where this will be the 36th bin, is in the 35th or 36th, 36th.

Essentially, I have now p 1 p 2, I will call this primes just to distinguish between the other once. I have these 36th p primes; it is a same exact data set. Let us say, I said did this dice roll 20000 times, my biased hoop roll 20000 times and I got 20000 real numbers and I did binning in 1 way in the first part binning in another way in the second part, the histogram is looks completely different, but clearly it is a same data set, I have to convey the same statistical information in both, how do I do this. In order to make a equivalence between these 2, I have to define what is called up probability density.

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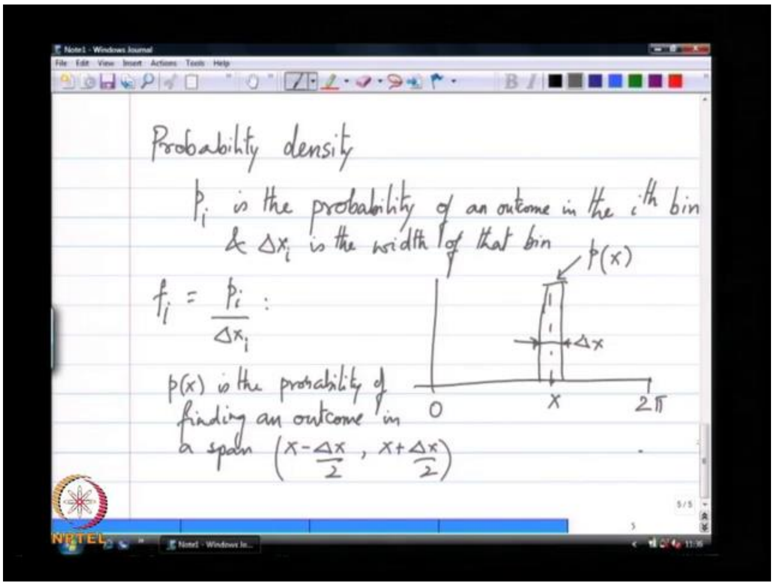

If I take this probability density say for example, in the first case of binning example I take 3 of these bins here, I will write down the third 1 also just to; 3 bins in case 2 map to 1 bin in case 1. All the count that goes into 3 individual bins in this case 2 map 2, only 1 bin in case 1. I can clearly say that the count in the 3 bins, so 0 to 10, 10 to 20 and 20 to 30, each 1 individually will surely be less than the count in the 0 to 30. Which also leads us to sort of an intuition that the wider I take the bin, the more count I am going to capture in that bin, the narrow or I make the bin the less count I am going to capture in that bin. If I did the 0 to 1, 1 to 2 degrees, the count in each bin for the same set of trial should be smaller.

I have the have some way a finding not the probability or going beyond probability and finding a probability density around a given value. If I take, if p i is the probability of the i-th bin probability of finding an outcome in the i-th bin, I can now define another f i which is given by this p i divided by delta x i. If I take all the range of outcomes say 0 to 360 degrees or 2 pi, if I take 1 value x and if I take all the outcomes in the bin that is delta x wide around that.

This is the probability of finding an outcome in this delta x width I will call this p of x because I now depending on the value of x, I choose the probability could be different because that is my idea of a biased hoop if x is closer to the 12 o clock position, p of x is going to be higher. If I take this p of x as the probability of finding a drop in a bin, that is width delta x around the value x. P of x, just to finish the discussion here, f of i is what we will call is leads just to the idea of a probability density. Regardlessly we will talk, what it is p of x is the probability f of f x is not equal to p of x divided by delta x.

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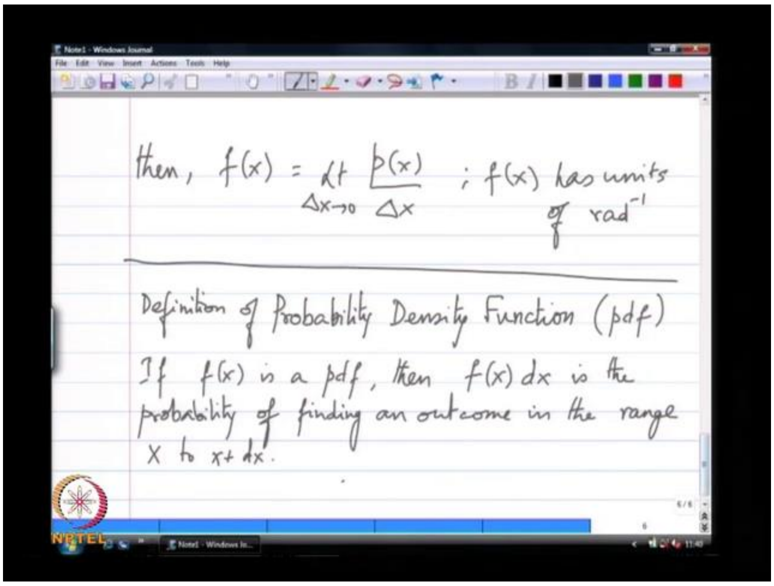

But really speaking, it is the limit sorry rewrite that as I come closer and closer and closer to that point x, the value of the probability of finding an outcome in a x minus delta x by 2 to x plus delta x by 2 becomes smaller and smaller. Actual probability and the width itself is becoming smaller, but the limit is a finite value, that limit is what we will define as our probability density. The idea you have to sort of understand these idea of density it is like I know the probability of any 1 number is 0, but how dense is the outcome is the outcome distribution around that point, that is all I care about.

If I go back to this f of x, now f of x has units of probability, has no units. This is the point that you have to understand f of x has units which is same as per radian or per degree, you look at what is in the denominator I should. Delta x with a, it has units of x basically in the denominator. For the case of a bias hoop roll, the probability density has units of per radian or per degree depending on what we choose to plot as the independent coordinate. Per degree, this is the density of outcomes possible. Let us find some simple argument. If I say, I will rewrite what I wrote clear; I will rewrite this statement in a more correct way. If f of x is a pdf, then f of x times dx is the probability of finding an outcome x plus dx. In an infinitesimal neighborhood around the value x or near the value x, you look at how many outcomes or what is the probability of outcomes that you have. Essentially let us draw this in a graphical sense.

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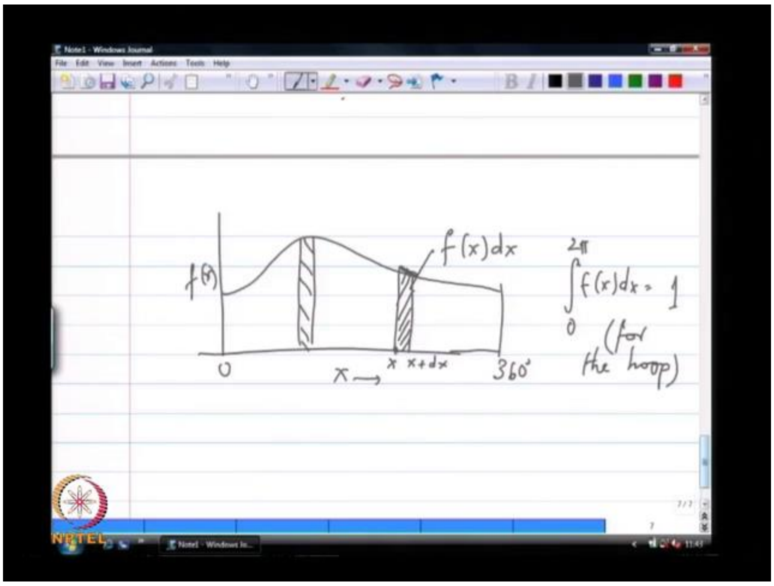

If I have a f of x is now a function. If I defined a probability density function, I can draw graph of it, something like that let us just say then at some value, x going from x to x plus dx f of x d, x f of x is a value at that point f of x. dx is essentially the area of that infinitesimal strip that infinitesimal strips, the area of that infinitesimal strips is the probability of finding probability density a probability of finding a value in the limit x to x plus dx in the range x 2 x plus dx, likewise you can see that probability of the same width here would be higher.

If I take all the range of values of my hoop going 0 to 360 degrees, the idea that no other outcome is possible, other than values between 0 to 2 pi or 0 to 360 tells me that integral 0 to 2 pi f of x dx sorry, whatever is my probability density function of f of x dx is a probability in a thin strip and that integral of that f of x the summation over all the areas of these thin strips which is what we call integral, going from 0 to 2 pi has to be equal to 1 that tells me that is just simply coming from the criteria that no other outcome is possible other than values between 0 to 2 pi.

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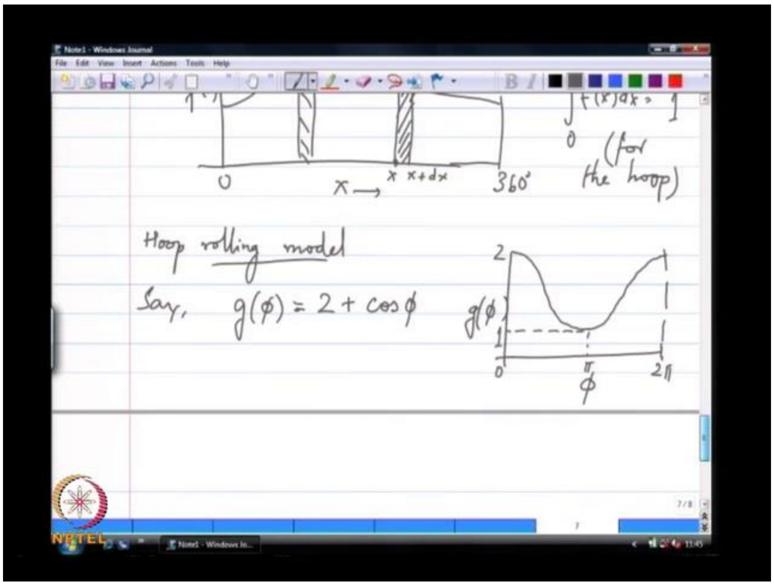

Let us see, how we can use this and let us apply this information to first let us continue our hoop discussion and will finish it. Let us say, I know it is a biased hoop around 12 o clock, I am seeing like more values come up near 12 o clock. I pustule at a model, I have not get done all these experiment. I have done like a 10 experiment, found that 12 o clock is coming up or point near 12 o clock is coming up more often than the points near 6 o clock, I have jump to a model. I say f of x is of the form 2 plus cos x.

If you go back, look at our definition of what I want to define a model for phi not x sorry, where did I come up with this function 2 plus cosine phi, let us because if I draw the graph of 2 plus cosine phi, I instead of using this f, I am going to use this function? I will call this function g just for the sake of differentiating it from f which I will use later on. This would be the graph of 2 plus cosine phi. At least graphically, it captures the idea that phi values near 0 which is also the same as near 2 pi are more probable then phi value is near pi, it is sort of graphically captures that information and I am happy to start with this model.

What do I know from here on is, g of phi of probability of density function no not yet all I have done is sort of postulated a function that seems to capture my imagination that in itself does not make it up probability density function.

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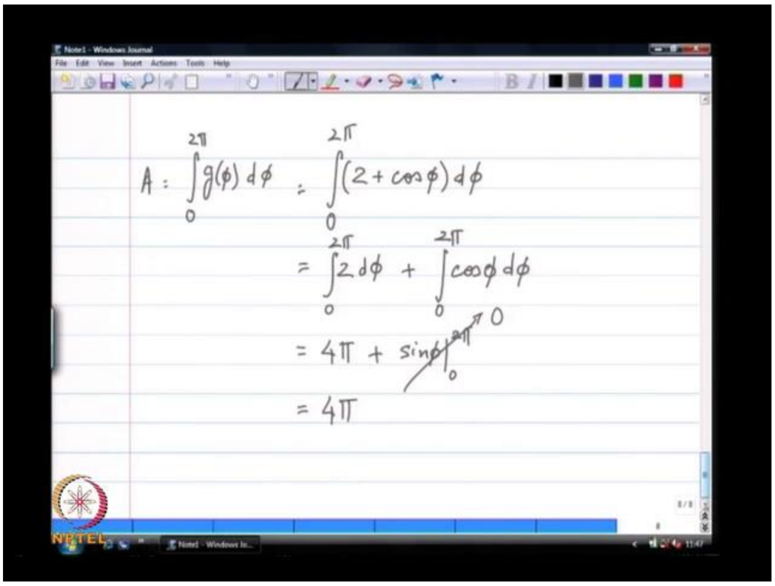

You need to make sure that integral g of phi d phi gives you the area under the curve. If I do this for this case integral of cosine phi is sin, but the limits are 0 to 2 pi. That becomes 0. The value of these integral is 4 pi.

The area under the curve g, the way I have drawn it is 4 pi. If I now define a new function f of phi equals 1 over 4 phi now this is the pdf this qualifies to be called probability density function because the area under the curve in the range of value is expected is actually to 1.

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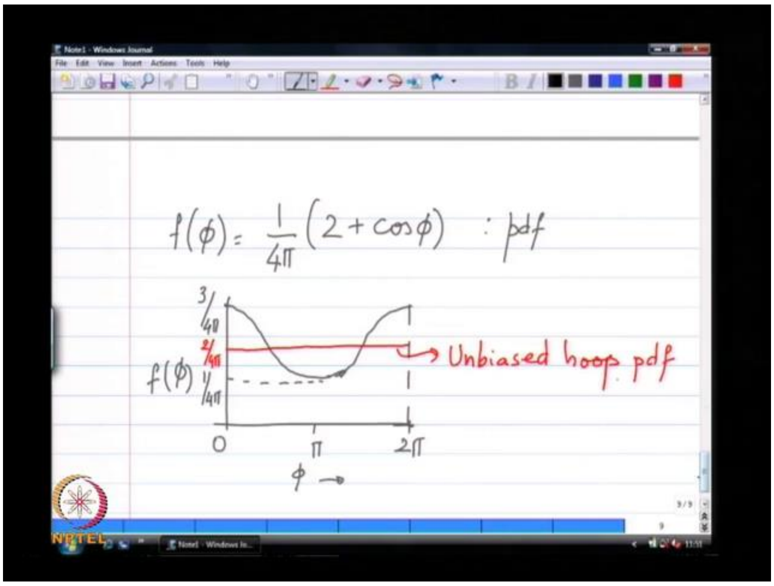

If I create a plot of this, the value of this is 2 over 4 pi and that would be the value in the previous case, also would be 3 for a maximum value for g. Same here would be from a maximum probability density the maximum probability density is 3 by 4 pi in the minimum probability density is 1 by 4 pi. This is the case of a biased hoop. If I had a perfectly fair hoop, where all outcomes are possible, you can see that the area under that curve would have to be equal to. Area of the curve going from 0 to 2 pi of some constant value has to be equal to 1 for that to be the case, the constant this would have to have a value 1 over I am sorry, 2 over 2 pi. 2 over 4 pi or 1 over 2 pi. I want to write it as 2 over 4 pi or 1 over 2 pi.

This idea of a probability density of unbiased hoop is also a number in just like a probability of an unbiased dice is a number like probability of any outcome is 1 by 1 over 6 probability density is also is a function probability density function, but the function takes on a constant value equal to 1 over 2 pi. If I know, I started to reconstruct this from some model like I said you know, have this model of a biased hoop that is biased towards the 12 o clock comparison, in comparison to the 6 o clock and this is where I ended up, if I want to start with just simply going through the process of doing multiple trials and then reconstructing these probabilities, what do I do.

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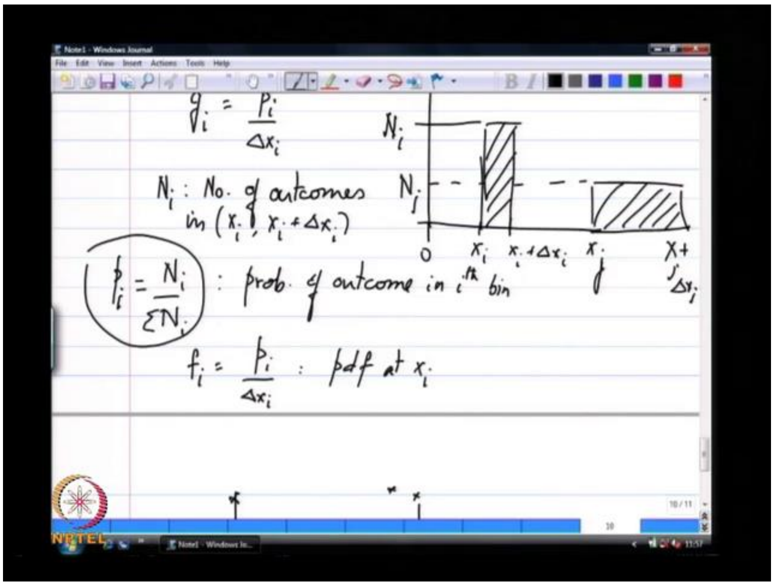

If I take to the experimental route and say I have done this trial like 30000 20000 times like large number of times, I have rolled this hoop over and over again, noted the real number that showed up on top. If I take that list of real numbers go through the binning process and create an f of x i, The first thing is to take is to create using the old terminology, find a g i which is equal to p i divided by delta x i. I take all the outcomes, find the set of outcomes that fall in a certain bin x i 2 x i plus delta x i. This is just the count. This is n I n I is the number of outcomes, I want you to also note 1 difference between what I have just written on the very first version that I wrote of this divided by delta x thing, where delta x itself can have a subscript i. In other words, I can now have the i-th bin, here b of 1 delta x and the x i x j to x j plus delta x j can be a different width. I now have a count n j associated with that j-th width.

I am not restricted to somehow delta x being uniformly spaced delta x being equal for all the values in this binning process. I can choose whatever bins I want and I can place the outcomes into these bins, depending on the actual value of the outcome. Once I do this n I divided by the total number which is essentially sigma, n i give me this number p i. p i is the probability of outcome in the i-th bin.

If I now define an f i which is equal to this p i divided by delta x I, this automatically gives me pdf of x i pdf at x i. It is like a it is the value of the function at x i. If I had a model, the problem with the experiment is only can compute these at discrete points of x i can do this at some at value 0 degrees 30 degrees 60 degrees etcetera. I take all the values in the range between let us say 60 and 61, put them in the bin and then find the probability of value in the range to 60 to 61 that gives me a and divided by that 1 degree that gives me a probability density.

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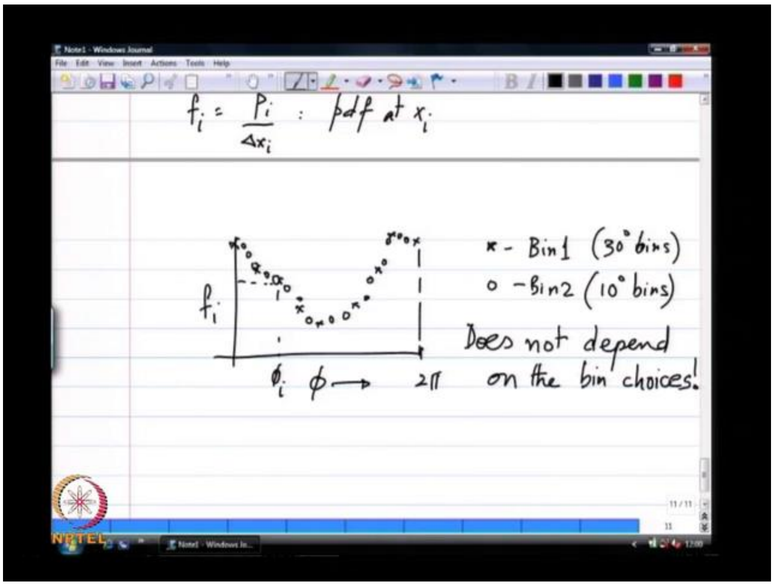

Now, if I do this and plot these functional these values f of f i and x i, I will do this for the case of a dice, where I have plotting, this going 0 to 2 pi, let us say if I do this with the 9 different bins, this these are the 9 values I get these are the actual values of f of i at each value of x i. At each value of phi i, this is phi axis at this value of phi i, this is the value of f i, how do I know that this is a pdf? Remember our condition that for probability density, the area under the curve has to be equal to 1 because we ensured by this definition of probability that the probabilities will add up to 1. The phi is nothing, but f i delta x i which is like the area of the strip around x i that is of width delta x i. In a sense of numerical integration we approximated the area to be equal to 1 as far as that as far within the accuracy of the binning. The smaller the delta x i values, the more accurately this number approaches the real probability density value.

If I use, if I get these x has through some bin 1, remember our using 30 degree increment or some using 30 degree increment, I get these x symbols if I do the same thing with the bin 2, which is 10 degree and if I do that and broad circles my expectations is that the circles would fall something like that. I have develop the better approximation of the actual function f of phi by taking finer bins and as I go towards finer and finer increments of these bins, I recover a better and better approximation of the actual analytical function, but at that point, once I reach this point, I need to do model, I need to find out the model. Like fit of an equation and find them model from dynamics and then come see how this data fits it, but our idea of pdf can be approached from both performing several trials and coming to a point where you can reconstruct this graph. This graph which does not depend on this is the key thing.

Remember that was the point, that was a problem we tried to solve, we had 2 different binning sequence that gave as to completely different set of probability numbers, how do I reconcile that to by essentially figuring out this idea of probability density and if you can plot the probability density at a given phi location phi I, which is f i, whether you do it in 1 binning fashion or another binning fashion, would only lead you towards the same answer. If I did this in 1 degree increments, just for sake of arguments if I did this in 1 degree increments, but took 1 degree on either side of my chosen value of phi i.

I go from 0.5 degrees to 1.5 degrees as the phi i for f i, to compute f i at 1 degree and 1.5 to 2.5, etcetera, etcetera or whether I do it in 1 degree increments, going to 0 to 1, 1 to 2 etcetera, they will give me exactly identically almost similarly the same answer. In other words, the bin width is the only parameter that determines how closely converged I am to the real value of the real probability density function this is going, starting with trial and trying to recover an analytical function if there is as though have done an infinite number of bins, that is the delta x tending to 0.

If I say take that 30 degrees, take the probability from, of finding an outcome from 25 to 35, 27 to 33, 29 to 31 and then come to smaller and smaller compute f of pis, this will converged to 1 value at that point. If I have a sufficiently large number of trials to start with that is the only problem.

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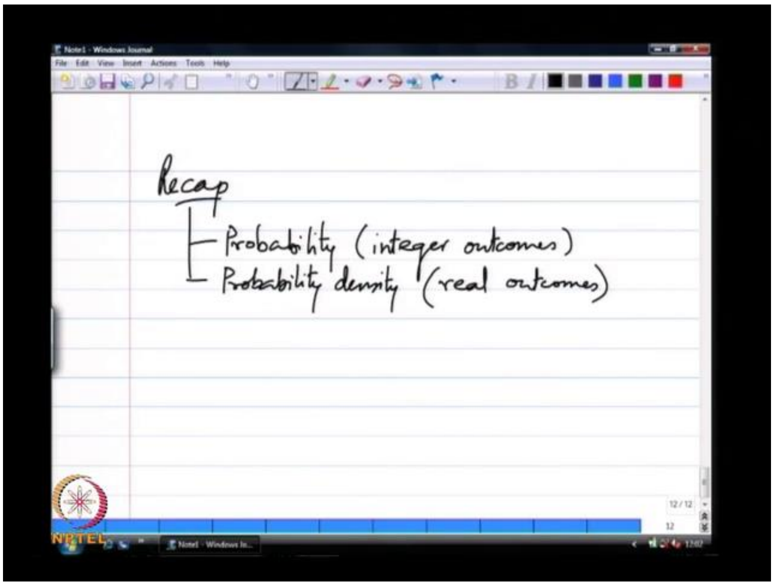

Let us quickly recap what we have done, we started with the idea of probability which is usually only valid for integer outcomes and then develop this concept of probability density which could be extended to real outcomes. If I go back to the spray, are my drops on integer's axis or a real axis? If I ask the question, clearly the answer is I just certainly not on integer axis which means I only have to assume that there are an infinitely many outcomes, therefore we have to go to pdf. Therefore, we need to understand this basic mathematical frame work.

Thank you.