Fundamentals of Operations Research

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Lecture No. # 09

Primal Dual Relationships

Duality Theorems

In this lecture we will continue our discussion on the primal dual relationships.

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We look at three important theorems. They are called Weak duality theorem, Optimality criterion theorem and Main duality theorem. We will also look at a fourth one called Complimentary slackness condition. Let us look at each one of this in detail.

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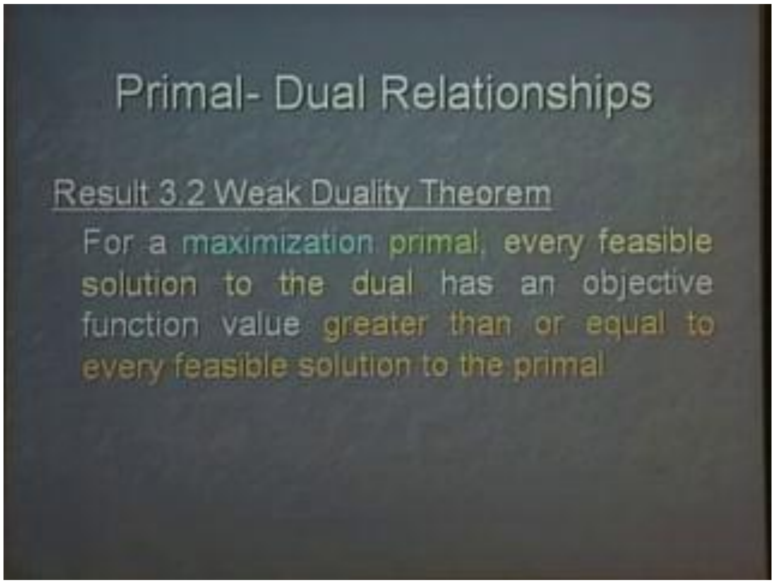

The weak duality theorem is as follows. If the primal is a maximization problem, every feasible solution to the dual has an objective function value greater than or equal to that of every feasible solution to the primal. Let us try to explain the weak duality theorem by considering the example.

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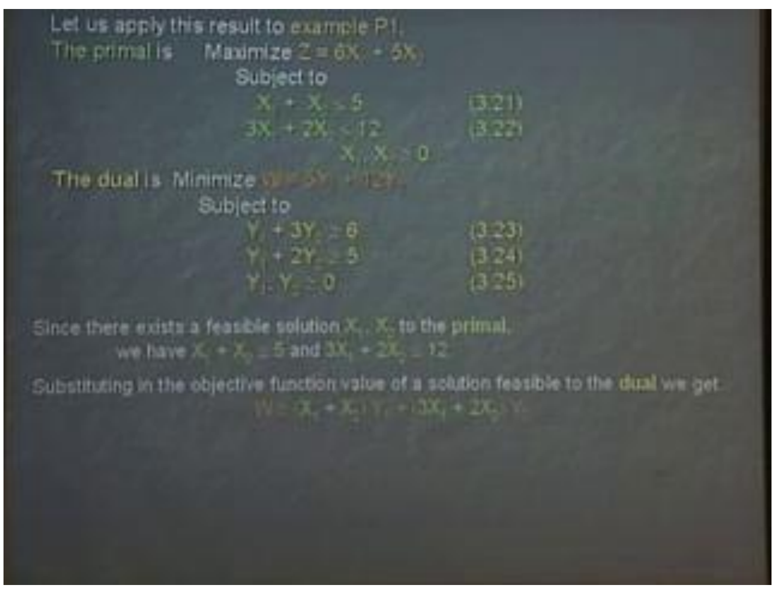

We address the same problem.

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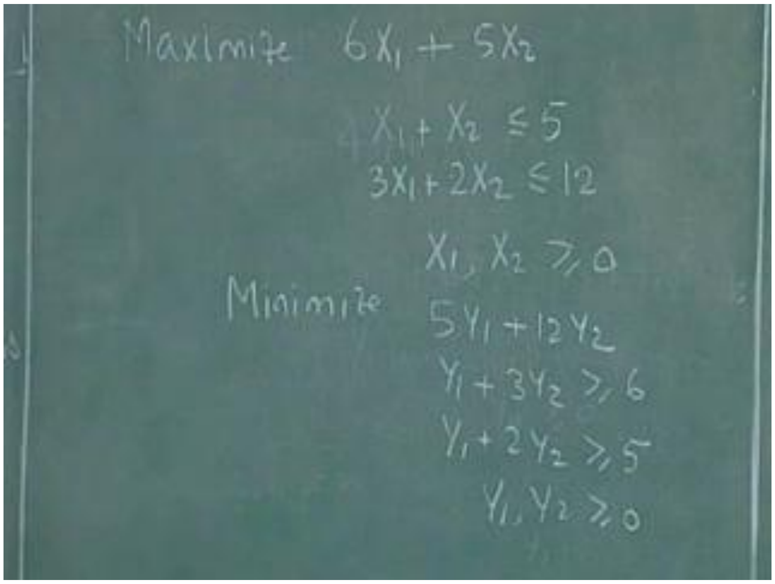

Maximize, $6X_1 + 5X_2$; $X_1 + X_2$ less than or equal to 5; $3X_1 + 2X_2$ less than or equal to 12; X_1 and X_2 greater than or equal to 0; Now the dual will have two variables Y_1 and Y_2 so dual will be to minimize $5Y_1 + 12Y_2$ subject to $Y_1 + 3Y_2$ greater than or equal to 6; $Y_1 + 2Y_2$ greater than or equal to 5; Y_1, Y_2 greater than or equal to 0. Now let us consider a couple of feasible solutions for the primal.

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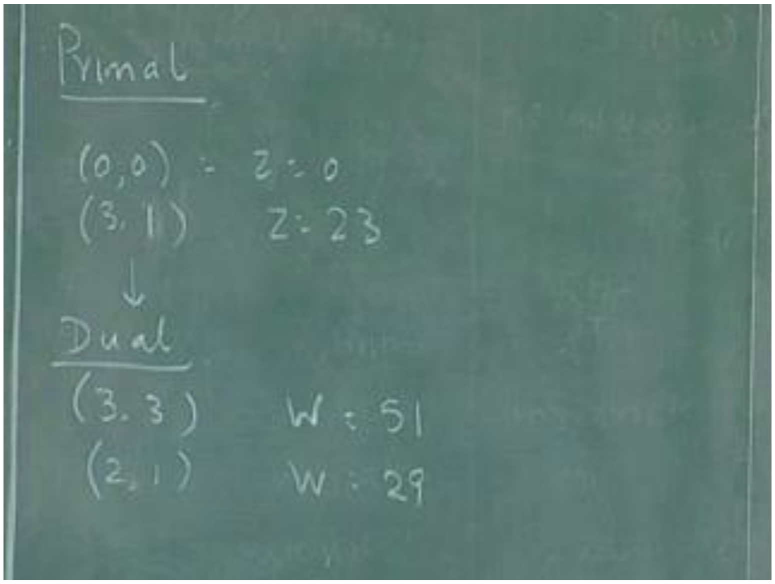

For example, (0, 0) is feasible with Z = 0. Let us consider a solution say (3, 1). It satisfies this, so it is feasible to the primal. This would give us an objective function value of Z = 6 into 3 = 18 + 5 = 23 and so on. Let us consider a couple of feasible solutions to the dual. For example a solution (3, 3) would be feasible to the dual, 3 + 9 is greater than 6; 3 + 6 greater than 5. This would have a value W = 15 + 36 is 51. If we consider another solution (2, 1), (2, 1) is feasible. 3 into 2 = 6 + 1; 4 + 1 so (2, 1) is feasible. Let W = 29 and so on.

So we could evaluate any number of feasible solutions to the primal and the dual. We can easily compute and show that the primal being maximization problem every feasible solution to the minimization problem dual will have an objective function value greater than that of a every feasible solution to the primal. Now the proof of the weak duality theorem is not very complicated but the way we have derived the dual, we have already assumed the weak duality theorem because the way we derived the dual we were only looking at whatever combinations that we saw when we were deriving the dual. They were all feasible solutions to the weak duality to the dual. They are upper bound to the objective function value of the primal at the maximum so the weak duality theorem follows from the way the dual has been derived. Let us go back and look at some more aspects of weak duality theorem.

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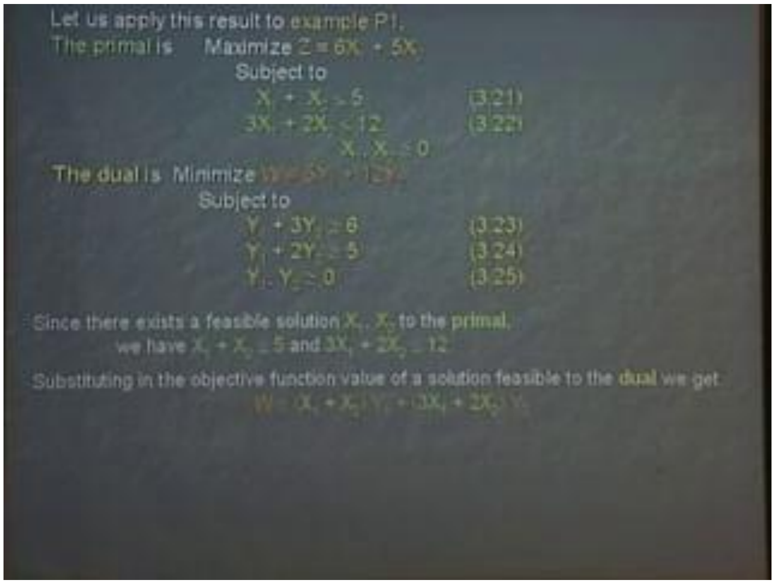

Now we have written the primal and dual to the problem and register feasible solutions X_1, X_2 to the primal, assuming that there exists a feasible solution. Now we have X_1 and X_2 less than or equal to 5; $3X_1$ and $2X_2$ less than or equal to 12. So every feasible solution to the primal should satisfy these two. If there is a feasible solution to the dual now substituting the objective function value of the solution feasible to the dual we will get W will be greater than or equal to, now if I have a feasible solution to the dual than that will be $5Y_1 + 12Y_2$ and if we have a feasible solution to the primal, X_1 X_2 then $X_1 + X_2$ is less than or equal to 5. This solution gets feasible to the dual which is $= 5Y_1 + 12Y_2$ will be greater than or equal to because 5 is greater than or equal to $X_1 + X_2$. This will be a greater than or equal to $X_1 + X_2$ into $Y_1 + 3X_1 + 2X_2$ into Y_2 and that is shown here.

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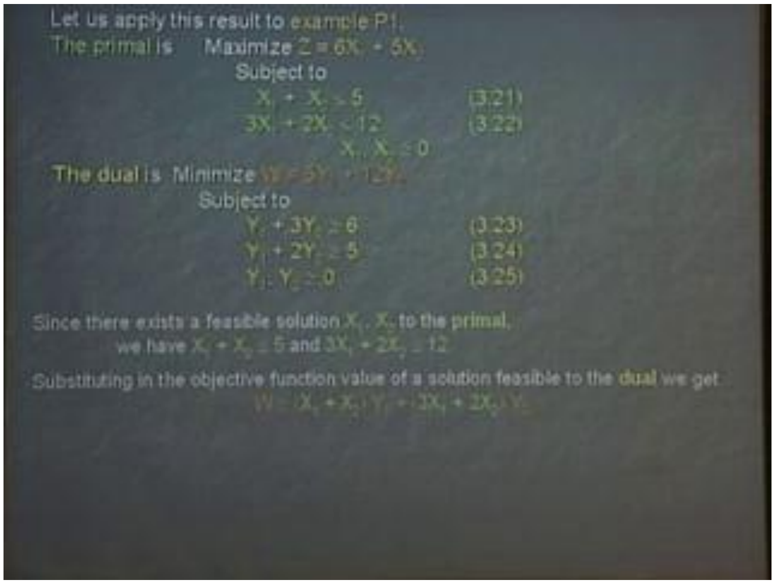

So this feasible solution to the dual W will be greater than or equal to $X_1 + X_2$ into $Y_1 + 3X_1 + 2X_2$ into Y_2.

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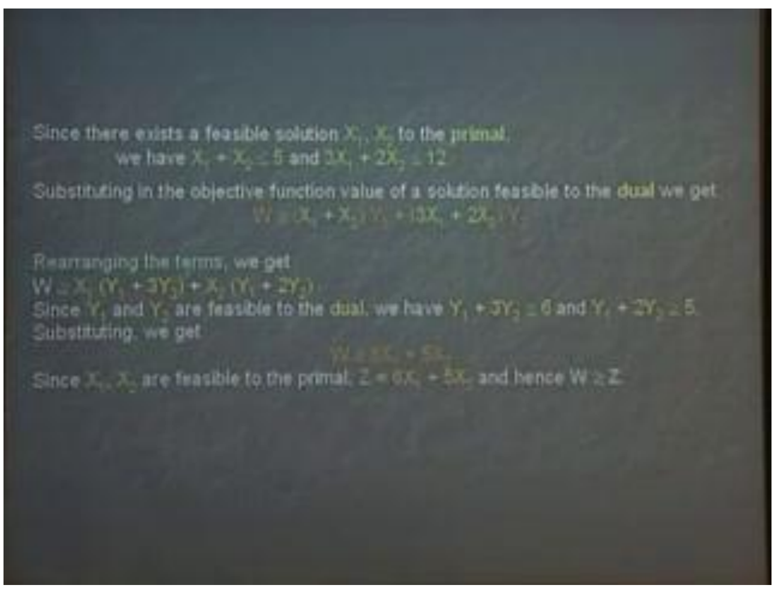

Now when we simplify this we get, we have $X_1 + X_2$ less than or equal to 5 and dual substituting we get this. Rearranging the terms you would have now, W is greater than or equal to X_1 into $Y_1 + 3Y_2$. Look at this expression and look at the next expression. We are now arranging them in terms of X_1 and X_2. We are grouping them in terms. For X_1 and X_2 we get $Y_1 + 3Y_2 + Y_1 + 2Y_2$ Since Y_1 and Y_2 are feasible to the dual. We realize that $Y_1 + 3Y_2$ is greater than or equal to $6Y_1 + 2Y_2$ is greater than or equal to 5. Substituting W will be greater than or equal to $6X_1$ and $5X_2$

and since X_1 X_2 are feasible to the primal Z is $6X_1 + 5X_2$ so W is greater than or equal to Z, therefore weak duality theorem is almost a very obvious result, the way we have derived the dual. If the primal and dual have feasible solutions or for every feasible solution to the primal and every feasible solution to the dual every feasible solution to the dual will have an objective function value higher than that of every feasible solution to the primal. We can look at it in a different way. If the dual is a minimization problem every feasible solution to the dual will have an objective function greater than that of the optimal.

Primal being a maximization problem, every feasible solution to the primal will be less than or equal to the optimum of the primal. The way we have derived the weak duality theorem, every feasible solution to the dual is an upper bound to the optimum value or to the objective function value at the optimum of the primal and hence weak duality theorem follows. Weak duality theorem is a very important result. We assume the primal to be dual minimization. Every feasible solution to the dual will have an objective function value greater than or equal to that of every feasible solution to the primal. Let us go back to the second theorem which is called the optimality criterion theorem and this is as follows.

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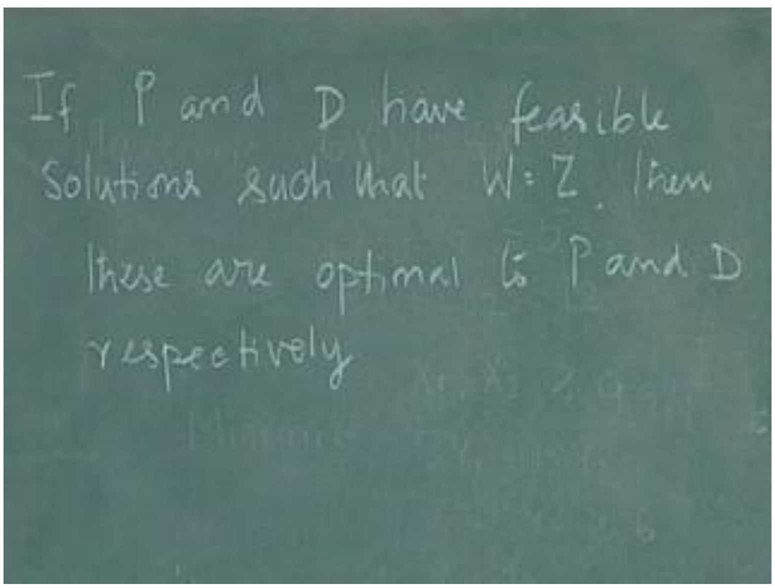

If primal and dual (we represent them as P and D) have feasible solutions such that W = Z then these are optimal to primal and dual respectively.

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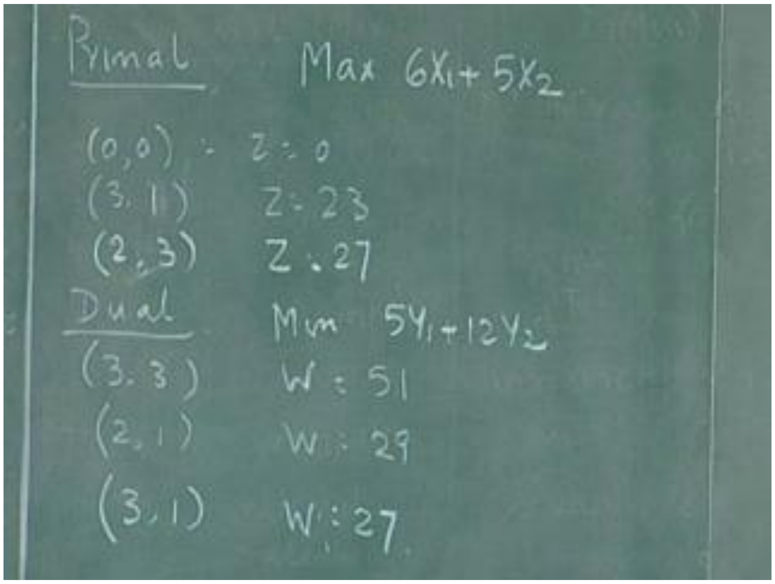

For example if we go back here and say that the primal has a feasible solution say (2, 3) with $6X_1$, the objective function was maximize $6X_1 + 5X_2$.

So if we have a feasible solution (2, 3) with Z = 27 now dual has a feasible solution. Dual was minimize $5Y_1 + 12Y_2$ so we say dual has a feasible solution with (3, 1) with W = 27.

Remember primal now has a feasible solution (2, 3), $6X_1 + 5X_2$ is = 27. Dual has a feasible solution (3, 1). Now let us go back to the problem.

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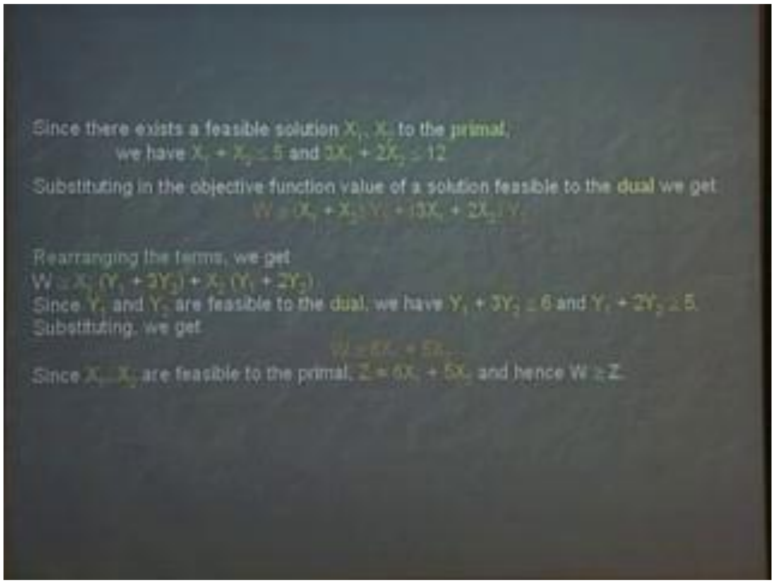

The primal constraints are $X_1 + X_2$ less than or equal to 5.

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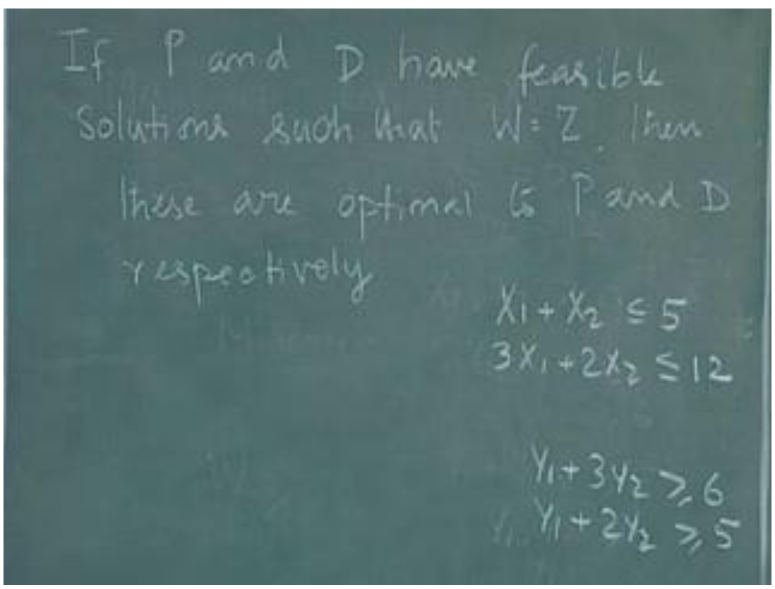

$X_1 + 2X_2$ less than or equal to 5 and $3X_1 + 2X_2$ less than or equal to 12, (2, 3) satisfies 2 + 3 = 5; 6 + 6 = 12 and has an objective function value of 27

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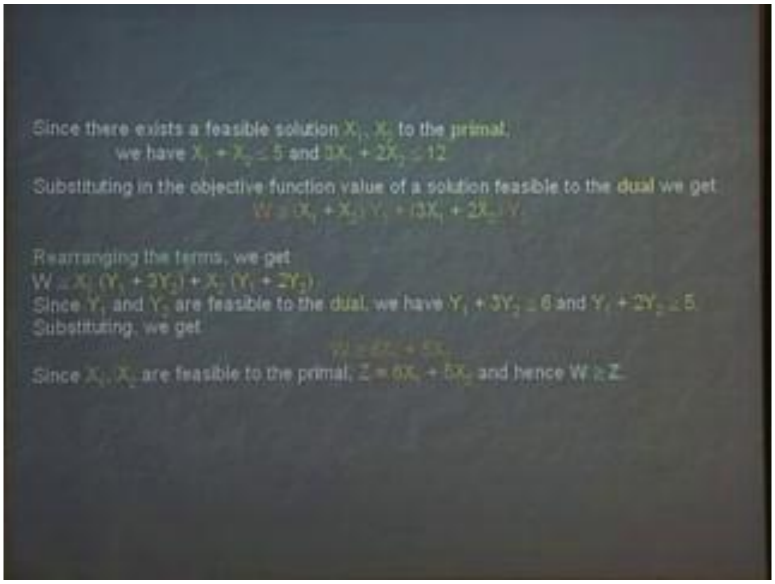

The dual constraints are Y_1+ $3Y_2$ greater than or equal to 6 and Y_1 + $2Y_2$ greater than or equal to 5. Let us look at (3, 1), 3 + 3 into 1 = is 6; 3 + 2 = 5 so (3, 1) is feasible to the dual. It has objective function value 27. (2, 3) is feasible to the primal has objective function value 27 based on this theorem. We can go back and say that this is optimal to the primal this is optimal to the dual. If you look at it very carefully the optimality criterion theorem is only an extension of the weak duality theorem. If for example these two are not optimal to primal and dual respectively then this should have a feasible solution with greater than 27 then it will straight away violate the weak duality theorem because weak duality theorem says every feasible solution to the dual will have an objective function value greater than or equal that of the primal. By the same logic we can prove that this is optimal to the dual so the optimality criterion theorem says that if primal and dual have feasible solutions with the same value of objective function then those solutions are optimal to the primal and dual respectively.

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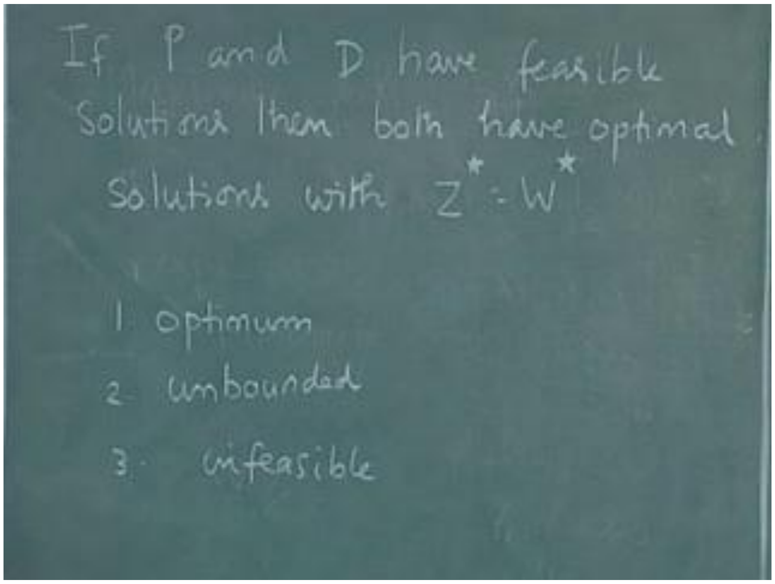

Now the third one is called the main duality theorem which is this. If primal and dual have feasible solutions then both have optimal solutions with Z star = W star with the same value of the objective. These are more general statements because it assumes that P and D have feasible solutions. The first part of it can be looked at more from the fundamental theorem of linear programming. That is the reason for the fundamental theorem. Every linear programming is either feasible or has an optimum or unbounded or infeasible. So there are three things. It has an optimum; it can be unbounded and infeasible. Now if we assume that P and D have feasible solutions it means that neither of them is or we have to make sure that neither of them is infeasible because if if either the primal or the dual are infeasible then if they will not have feasible solutions therefore this is taken care of.

It simply means primal and dual have an optimum or are unbounded. Now one of them is unbounded. Then the weak duality theorem will give us some results. It will make sure that some where it will be violating some of the weak duality conditions because one of them will be having infinity. So the weak duality theorem will be violated. We therefore look the situation where both will have feasible solutions then both should have optimum solution.

Then for example if both are feasible solutions both cannot be unbounded or infeasible so both will have only the optimal solutions and the moment we agree that both will have optimal solution then automatically the optimal criterion theorem takes over and says that the optimal values will have to be equal otherwise the weak duality theorem will be violated. In any case we are not going very deep into these theorems but we just state them so what we understand is, that these theorems exist and these theorems define the relationships between the primal and the dual.

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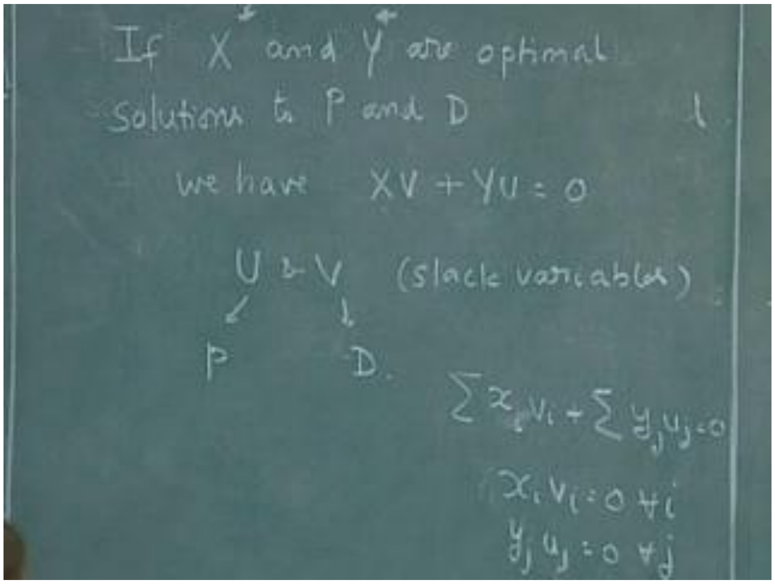

We now move into another one called complimentary slackness conditions which is this. If X star and y star are optimal solutions to primal and dual, in fact, we would define something more, here we have Xv + Yu = 0. This is called complimentary slackness condition. We have to define what these v and u are. X and Y are the solution vectors corresponding to primal and dual respectively. u and v are also vectors which are slack variables corresponding to primal corresponding to dual respectively. So if the primal has X, it is the vector of decision variables, u is the vector of slack variables for the primal. Y is the vector of decision variables, for the dual, V is the vector of slack variables for the dual then at the optimum X into v +Y into u = 0; Xv Yu are all vectors and dot product. Therefore this can be generalized as sigma Xi vi + sigma y_j u_j = 0 because it is a dot product. Since in the given problem XY and vu are all greater than or equal to 0 then for every individual xi corresponding Xi vi = 0 for all I, and y_j u_j = 0 for all j; i and j are used because if the primal may have different number of constraints and different number of variables.

X is the number of decision variables in the primal. This is the number of decision variables in the primal so, this many constraints the dual will have, this many slack variables the dual will have so X_i, Y_i, X_i, and v_i is taken care of. Similarly y is the number of decision variables in the dual which is the number of constraints in the primal and each constraint in the primal will have a slack. j essentially represents the number of variables in the dual which is the number of constraints in the primal and i represents the number of variables in the primal and the number of constraints in the dual. Because X Y u v are all greater than or equal to 0 in the linear programming problem for every i Xi vi = 0 Yj uj = 0.

Now what does it tell us? It tells us that at the optimum, if a particular variable X is a basic variable and takes a positive or a non-negative value; now let us leave out things like degeneracy and alternate optima for a moment. If a particular X is a basic variable at the optimum then the corresponding dual slack will be = 0.

If a particular dual variable is basic at the optimum then the corresponding primal slack will be equal to 0. If a particular dual slack is 0 or if a particular dual slack variable is non basic at the optimum then the corresponding or if this is basic at the optimum and takes a value then the corresponding primal decision variable will be non basic and so on. This is what the complimentary slackness tells. Let us go back to the complimentary slackness condition and try to learn a few more things about it.

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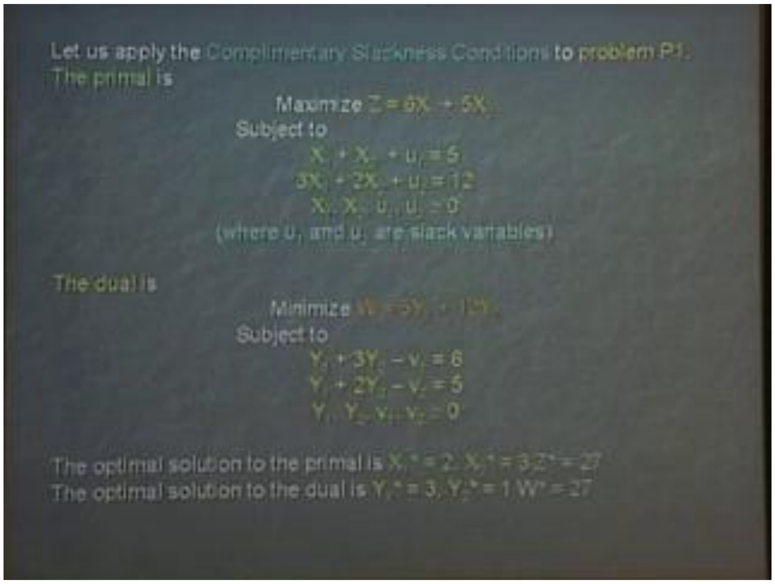

Considering the same example that we have now, we maximize $6X_1 + 5X_2$. So let us write the same example here.

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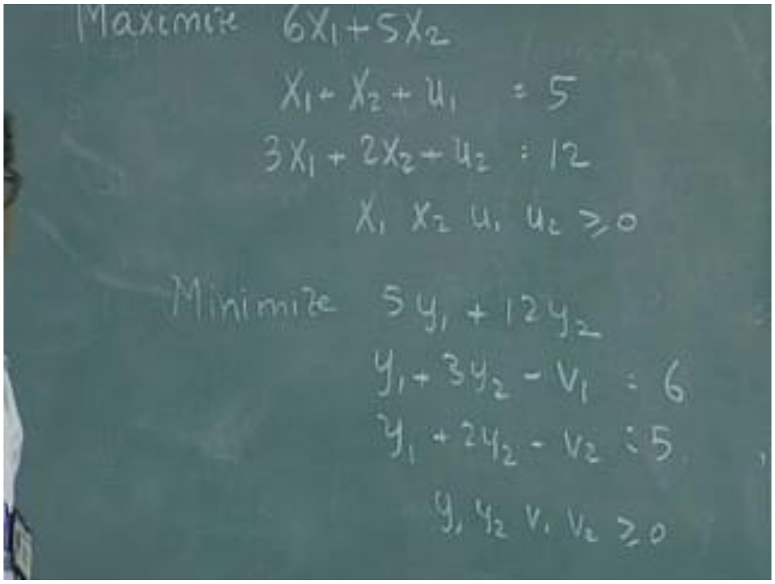

Maximize $6X_1 + 5X_2$ subject to $X_1 + X_2 + u_1 = 5$. Remember the slack X_3 has now become u_1 and $3X_1 + 2X_2 + u_2 = 12$; slack X_4 has become u_2; X_1, X_2, u_1, u_2 greater than or equal to 0.
Now dual is minimize $5Y_1 + 12Y_2$; $Y_1 + 3Y_2 – v_1 = 6$; $Y_1 + 2Y_2 – v_2 = 5$; Y_1, Y_2, $2v_1$, v_2 greater than or equal to 0. First let us look at the solutions and then verify the complimentary slackness and then let us go back and do something more with the complimentary slackness.

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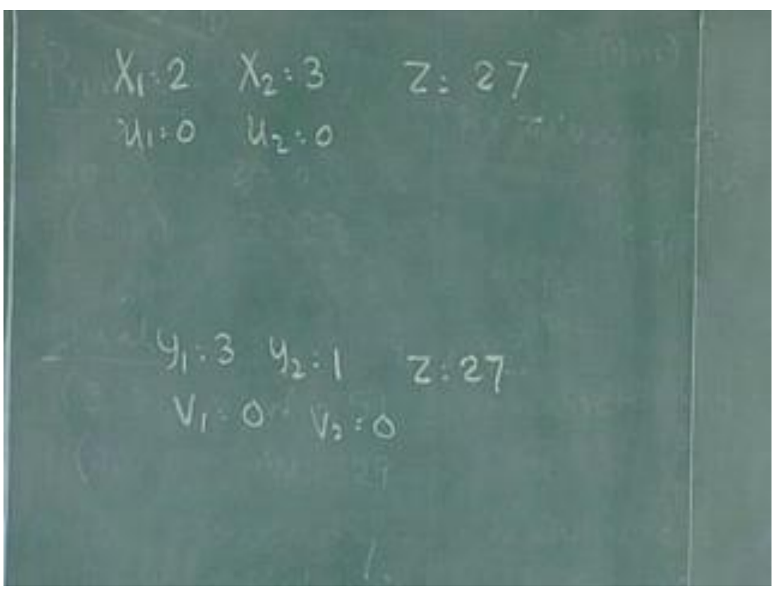

Primal has a solution which is
$X_1 = 2$; $X_2 = 3$; $Z = 27$
$X_1 = 2$; $X_2 = 3$ makes $u_1 = 0$; $u_2 = 0$; $Z = 27$

Let us assume we know the optimal solution to the dual is $Y_1 = 3$; $Y_2 = 1$; $Z = 27$. Primal and dual will have the same value of the objective function at the optimum. $Y_1 = 3$; $Y_2 = 1$ would make $V_1 = 0$, $Y_1 = 3$; $Y_2 = 1$ would make V_2 also = 0. Now let us go back and apply the complimentary slackness condition $Xv = 0$; $Yu = 0$; $X_1 v_1$ is 0. Product $X_2 V_2$ is = 0; $Y_1 u_1$ is = 0; $y_2 u_2$ is 0, so complimentary slackness is now satisfied at the optimum.

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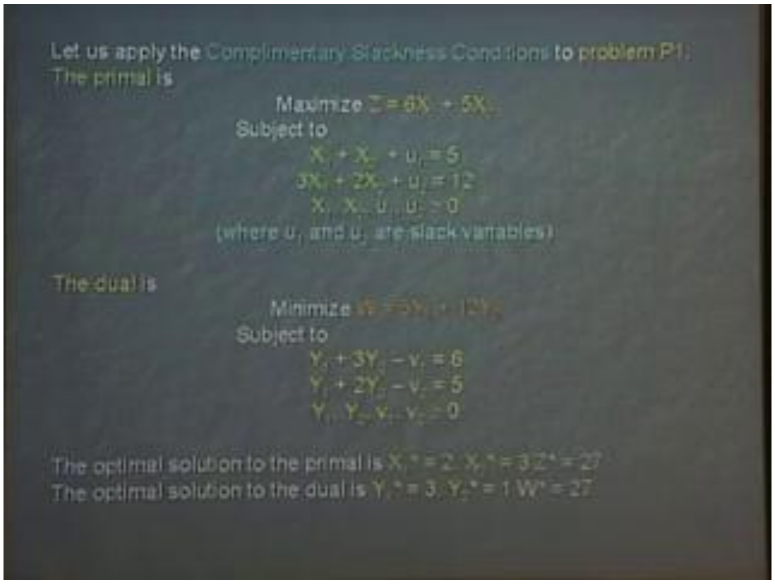

Let us go back and see that the same thing is shown here on the power point slide. In fact that we have written the primal and the dual and we can verify the complimentary slackness from that so next slide.

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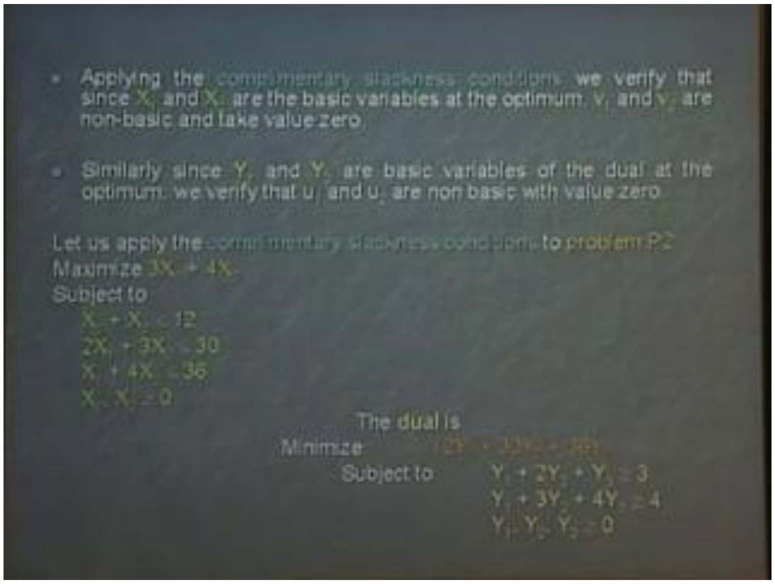

Let us go back and apply the complimentary slackness to the next problem. The same problem that we have worked out earlier is the problem maximizing $3X_1 + 4X_2$ subject to all these.
The dual is also written here $12Y_1 + 30Y_2 + 36Y_3$. We have solved this problem earlier in one of the early lectures.

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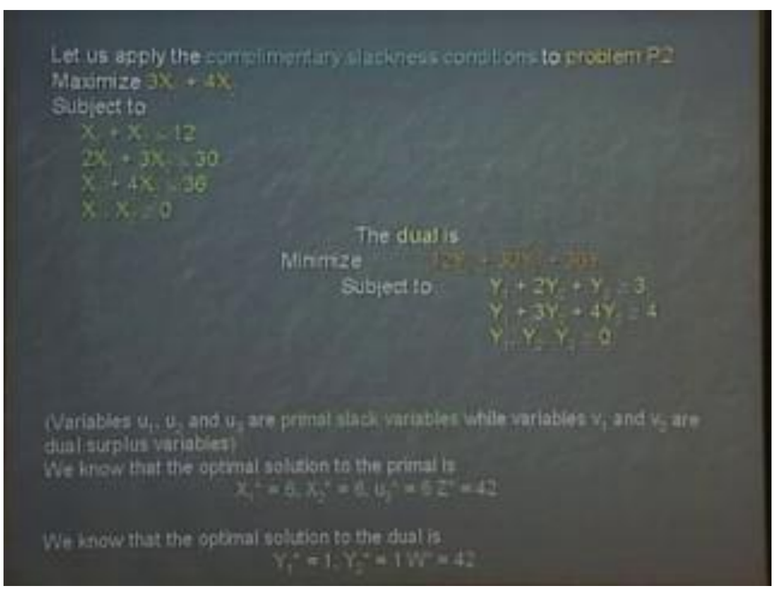

Now for this problem the optimal solution to the primal is X_1 star = 6; X_2 star = 6; u_3 star = 6; If you will remember X_5, $6X_5$ is u_3 Z = 42. The dual solution would be Y_1 star = 1; Y_2 star = 1; W = 42. Let us go back to that solution X_1 = 6; X_2 = 6

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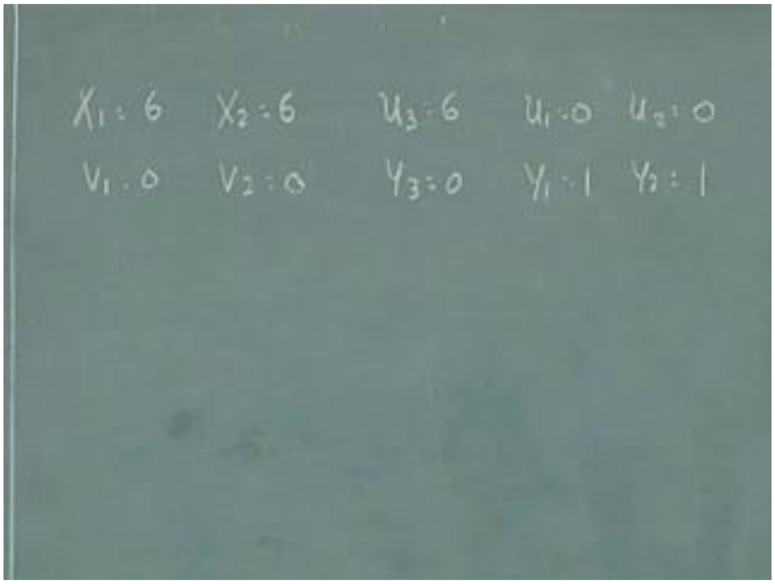

$u_3 = 6$; $u_1 = 0$; $u_2 = 0$ this is the optimal solution to the primal.
Optimal solution to the dual is $Y_1 = 1$; $Y_1 = 1$; $Y_2 = 1$ and the rest of them are 0. So you will have $Y_3 = 0$ we have $V_1 = 0$; $V_2 = 0$. The primal had three constraints so we have three basic variables, the primal has two variables so dual will have two constraints and it has two basic variables. Complimentary slackness is straight away satisfied.

What is also important is this there is a primal slack in the basis. Corresponding dual decision variable is non basic. There is a primal decision variable in the basis. There is a corresponding dual slack which is non basic with 0. This is the most important thing. There is a primal slack variable in the basis so the corresponding dual decision variable is non basic with 0 from the complimentary slackness conditions. Next we are explaining the same thing here slack. The next thing we can do is for this example what we have tried to do is we have written the complimentary slackness conditions and we have verified the complimentary slackness condition.

Now let us assume that we do not know the optimal solution to the dual. Let us assume that we know the optimal solution to the primal. The only thing that we know is the optimal solution to the dual will have W = 27. Let us assume we do not know which the basic variables at the optimum are and what values they take now. We can find that that out using the complimentary slackness.

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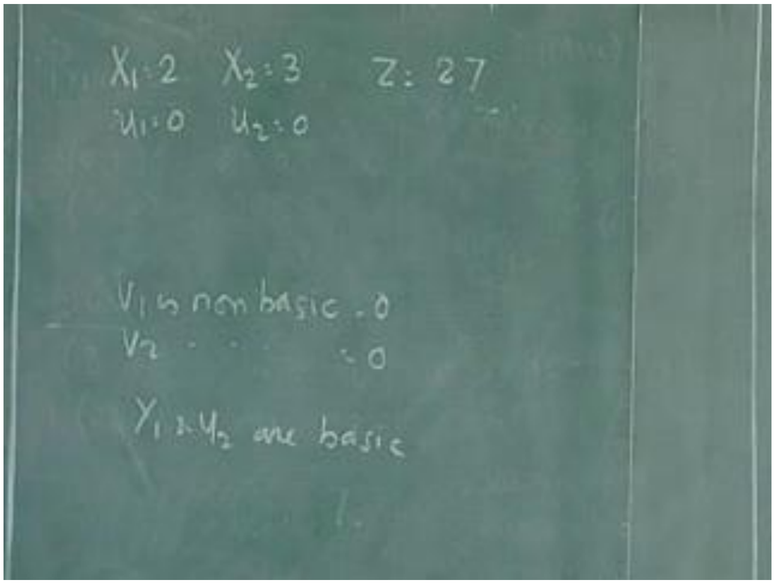

Now this is the optimal solution to the primal. So X_1 is basic, V_1 is non basic and = 0
X_2 is basic v_2 is non basic and = 0. $u_1 = 0$; $u_2 = 0$ implies Y_1 and Y_2 are basic.
All we need to do is to go back and substitute here, $v_1 = 0$; $v_2 = 0$; if we solve $Y_1 + 3Y_2 = 6$; $Y_1 + 2Y_2 = 5$ the values that Y_1 and Y_2 will take are optimal and they turn out to be (3, 1) with the value 27. Similarly you can go back to this example and then using the complimentary slackness conditions and the optimal solution on the primal you can compute the optimal solution to the dual.

The moment we solve the primal what we do is we know the solution to the primal i.e., the optimal solution to the primal. We also know the value of the objective function at the optimal
We do not have to solve the dual from the beginning. All we know is the moment we solve the primal and if the primal has an optimum assuming a finite unique optimum then the dual also has an optimum with the same value of the objective function. But we do not know what the solution of the dual is.

Now in order to get the solution of the dual, you need not solve the dual all over again. Separately as a linear programming problem by looking at the solution to the primal one can apply the complimentary slackness conditions and straight away get the solution to the dual so that is something which is now possible.

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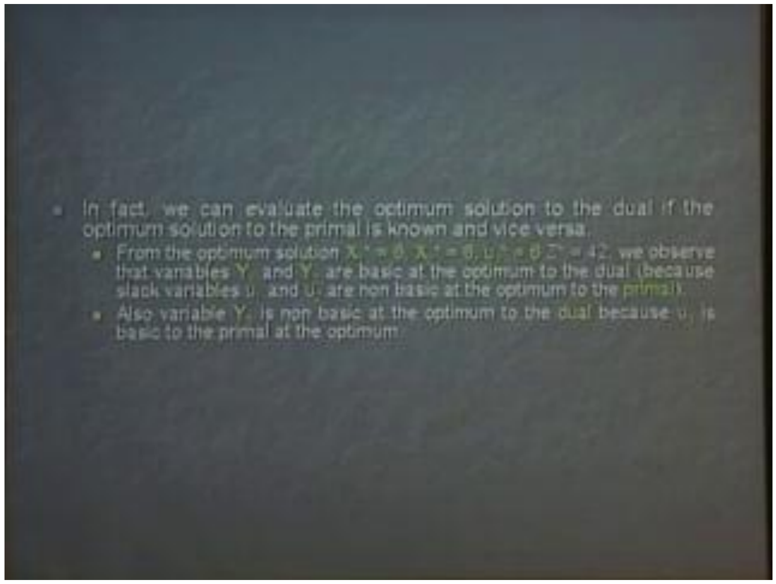

Let us go back to this. So we can evaluate the optimum solution to the dual. If the optimal solution to the primal is known and vice a versa extra then we look at all those conditions from the optimum solution. X_1 star = 6; X_2 star = 6; u_3 = 6 so in this example we would know that it is enough to solve only for Y_1 and Y_2. In this problem there are two variables and three constraints in the primal, two variables and three constraints and three variables and two constraints, so that we have two constraints in the dual. We just need to solve only for Y_1 and Y_2 rest of them will be 0.

Y_1, Y_2 is the optimum basis for the dual and we can solve it to get the solution to the next problem as well. Solve for $Y_1 + 2Y_2 = 3$; $Y_1 + 3Y_2 = 4$ which would should give us Y_1 star = 1; Y_2 star = 1 from here and $Z = 42$. So using this solution, if we know this solution you can once again apply the complimentary slackness condition and go back to the other one. That is also possible.

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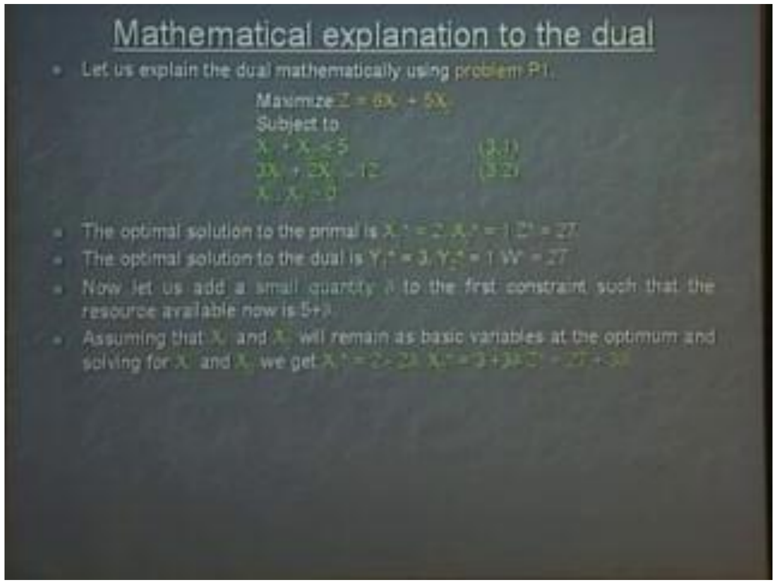

Let us look at another aspect now. What is this dual? Let us try to give a mathematical explanation to the dual. We have already seen some aspects of primal and dual relationships. We will try to see a few more and we will also try to understand where and how we see all those in this simplex table which we have looked at so far. Let us look at this mathematical explanation to the dual. We take the same example and do this. Right now we will note down the solution to the dual here.

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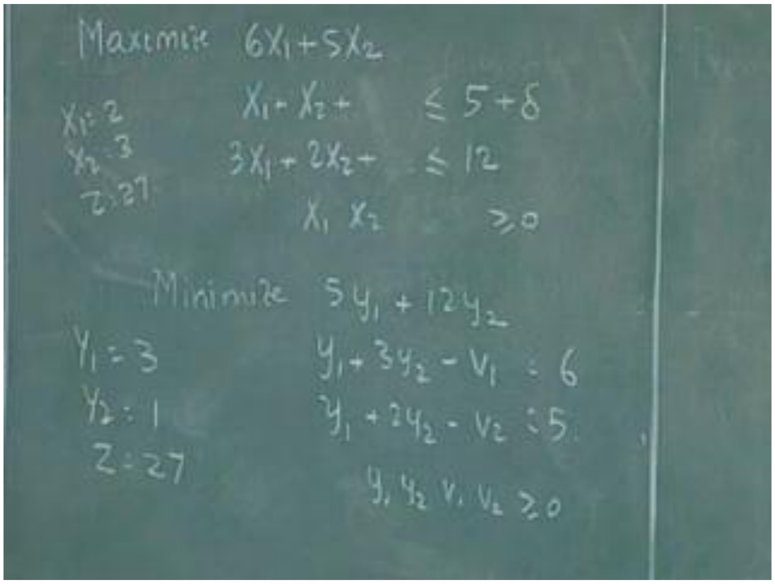

This is the dual. Solution to the dual is $Y_1 = 3$; $Y_2 = 1$ and Z = 27. Let us look at this primal problem again and let us write this as less than or equal to 5 less than or equal to 12 and let us leave this. Let us assume that we we know this solution to the problem and so on. Let us assume that this right hand side increases by a small quantity delta. We make this delta very small and we assume that when this becomes 5 + delta. Remember that X_1, X_2 are the basic variables here at the optimum which we know. Without this delta we know the solution $X_1 = 2$; $X_2 = 3$; Z = 27
So we assume that with the inclusion of this delta X_1 and X_2 would still continue to remain optimal.

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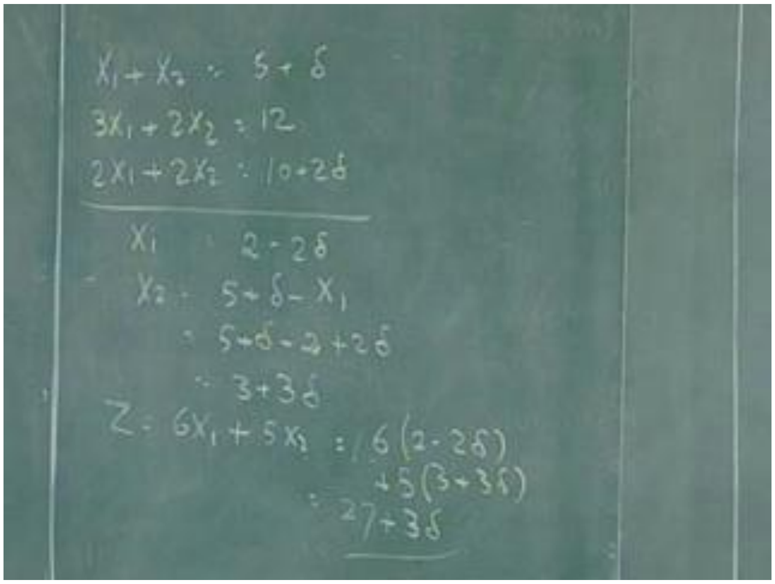

So let us try to solve this for $X_1 + X_2 = 5$ + delta and $3X_1 + 2X_2 = 12$; Assuming that the X_1, X_2 will be the basic variables at the optimum. Now multiply this by 2 to get $2X_1 + 2X_2 = 10 + 2$ delta so this would give us $X_1 = 2 – 2$ delta Substituting $X_2 = 5$ + delta – X_1, this is 5 + delta – 2 + 2 delta which is 3 + 3 delta and Z is $6X_1 + 5X_2 = 6$ into 2 – 2 delta + 5 into 3 + 3 delta which is = 27 – 12 delta + 15 delta + 3 delta.

So what happens is if we increase this right hand side resource by a very small value delta then the decision variables change. This is assuming that X_1, X_2 would continue to be the basic variables.

X_1 becomes 2 – 2 delta, X_2 becomes 3 + 3 delta and Z becomes 27 + 3 delta.

What is this 3 delta? Is this 3 the same as this 3? For example if we make this 5 – delta, would we get 27 – 3 delta here. If this 3 the same as this 3 then if we had retained this as 5 and if we had made this as 12 + delta then would this have become 27 + 1 delta?

The answer to all these questions is yes, provided X_1 and X_2 continue to be the basic variables at the optimum.

What we try to explain here is that the dual variables value is actually the the rate of increase of the objective function of the primal for a small increase in one of these results. If the first resource is increased by a small quantity delta, the objective function will now increase by a quantity 3 delta where this 3 is the value of the corresponding dual variable.

If the second resource is increased or reduced by a small quantity delta then we can go back and easily show that the objective of function will increase or reduce correspondingly by this.

What does it mean? It means the dual can now be explained as the the marginal value of the resource or the dual simply represents the extent. Mathematically the value of the dual variable represents the extent to which the objective function will increase or decrease for an incremental increase or decrease of the corresponding resource and that is the mathematical explanation to the dual variable. One can continue with this exercise by putting a minus delta and that will become 27 – 3 delta and so on. Now let us go back to this.

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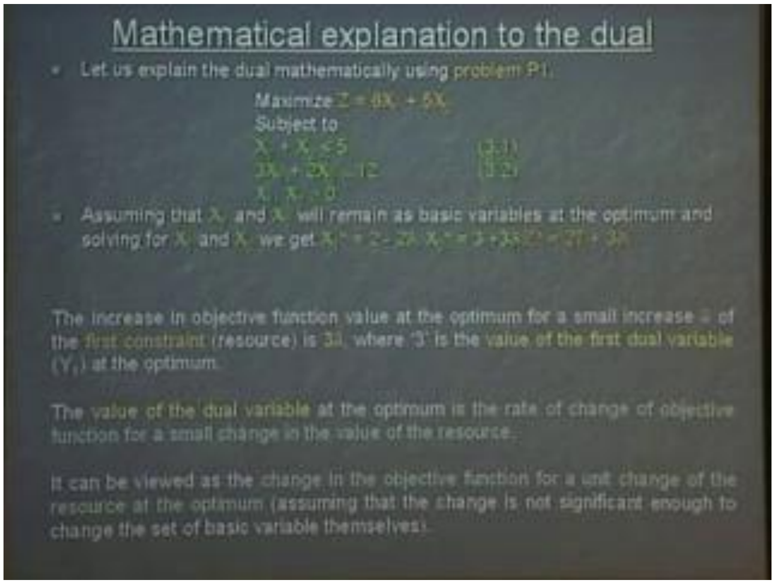

Now increase in objective function value at the optimum, (the most important thing is the term at the optimum), the dual has a meaning at the optimum. It actually does not have the meaning at places other than the optimum. It also has a good physical interpretation which we will see later. But right now we are considering all these at the optimum and how we are ensuring this in the optimum. In this calculation we are ensuring it by assuming that X_1 and X_2 will be the basic variables. So the optimum basis will remain intact.

At the optimum for a small increase of delta of the first constraint the actual value in the objective function is 3 delta where 3 is the value of the first dual variable at the optimum.
Now the value of the dual variable at the optimum is the rate of change of the objective function for a small change in the value of the resource. It can be viewed as a change in objective function, for a unit change at the optimum assuming that the change is not significant enough to to replace the basic combination. As long as the same basic variables are intact the dual can be interpreted as the rate of change of the objective function with respective to the resource.

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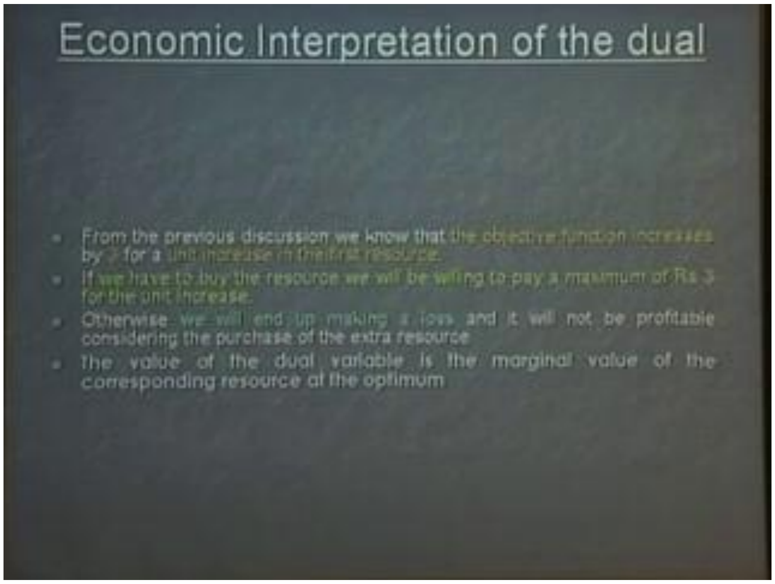

Let us go back to the economic interpretation of the dual. Let us do this.
Let us assume that this (Refer Slide Time: 35:01) is a primal that we are looking at and this is the dual. Let us go back to the problem formulation. Let us assume that these 5 and 12 are two resources which we call R_1 and R_2. X_1 and X_2 are the amount of product that is made. What does this tell us? It tells us that if instead of five quantities of the resource that I have here, if I had a (5 + delta), then it is possible for me to make 3 delta more than this. Interpreting it, physically if I have one more resource with me, if 5 had become 6 then that resource is capable of giving me three more rupees in terms of the objective function or the value of the additional resource from 5 to 6 is rupees three. If I have to go and buy this additional resource from the market then I will be willing to pay a price which is three rupees or less for this resource.

If someone is willing to sell me the 6th unit of this resource for three rupees or less then i know that by buying this resource for three rupees or less I will make three rupees more to my objective function and I will happily buy it. So it is the marginal value at the optimum of this resource or the price that i am willing to pay to get an additional unit of this resource. This is the interpretation with this dual variable. Similarly the next dual variable Y_2 will be the additional price that I am willing to pay for one more unit of this resource which is one rupee. Then do we have the same value at the optimum? For the primal and dual let us look at this. Let us assume that primal is the carpenter or the producer who makes these two products and let us assume that the person does not have any of these resources but has planned for 5 and 12 respectively. Let us also assume that they will go to the market or to a common place to buy these resources 5 and 12.

Let us assume that there is some other entity or an individual who is selling exactly these two resources to this person. Now the person who is selling the resources is actually solving the dual. That is his problem. So the person is assumed to have already solved the dual knowing the requirements of this man and now will price these resources at 3 and 1 respectively.

This person will be looking at getting the first resource for three rupees or less and this person knows that the value of the resource is exactly 3 and this person will not sell it for less than 3 because if this person is selling it for less than 3, this man will happily buy and make more money. This person would like to sell it for 3 or more, knowing very well that the marginal value is 3 and if this person is selling it for more than 3, then this one will not buy because he would make a loss by buying it for more than 3. Both will agree to this. The value of the resource is exactly 3. Therefore at the optimum 5, 12 both of them know that the value of the resource is 27 which is the optimal value of the objective function of the dual. At the same time the person will also make 27 rupees. So this is called the economic interpretation of the dual.

The worth of the resource is equal to the money that this person makes. So physically every dual variable can be interpreted as the marginal value or the price that the person is willing to pay at the optimum to procure a one additional unit of the resource or any small value delta of that resource.

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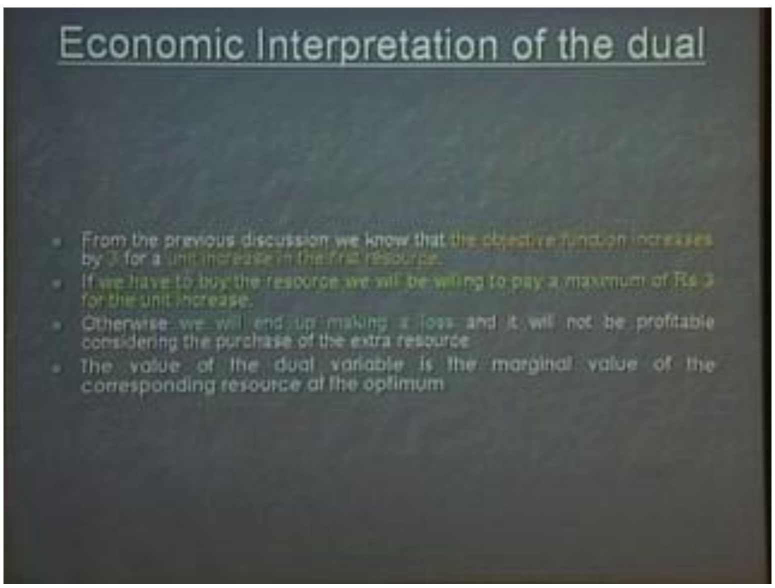

From the previous discussion we know that the objective function increases by 3. If we have to buy the resource we will be willing to pay a maximum of three rupees for the unit increase.
Otherwise we will end up making a loss and will not profit, considering the purchase of the extra resource. The value of the dual variable is the marginal value of the corresponding resource at the optimum.

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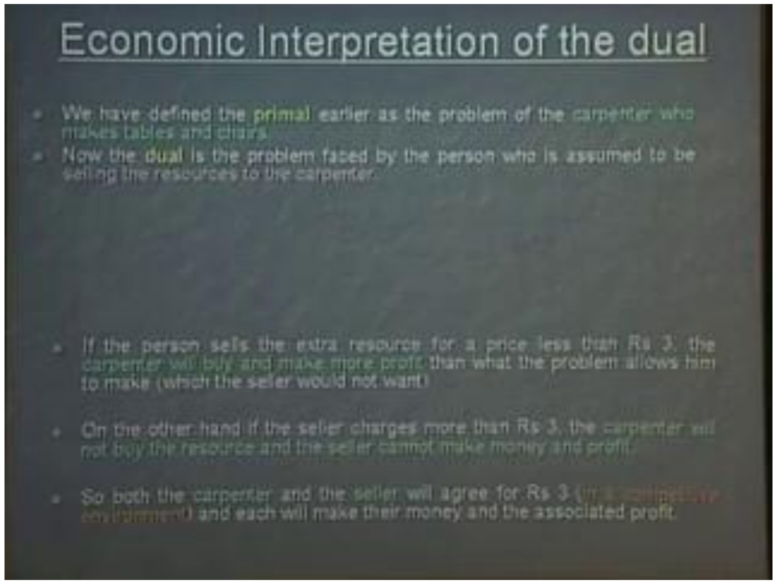

We go back and interpret it now. The same explanation that I gave here is now seen in the power point slide. So carpenter is the primal, the producer and the seller is the resource of the market will agree on rupees three in a competitive environment and each will be making their money and the associated profit.

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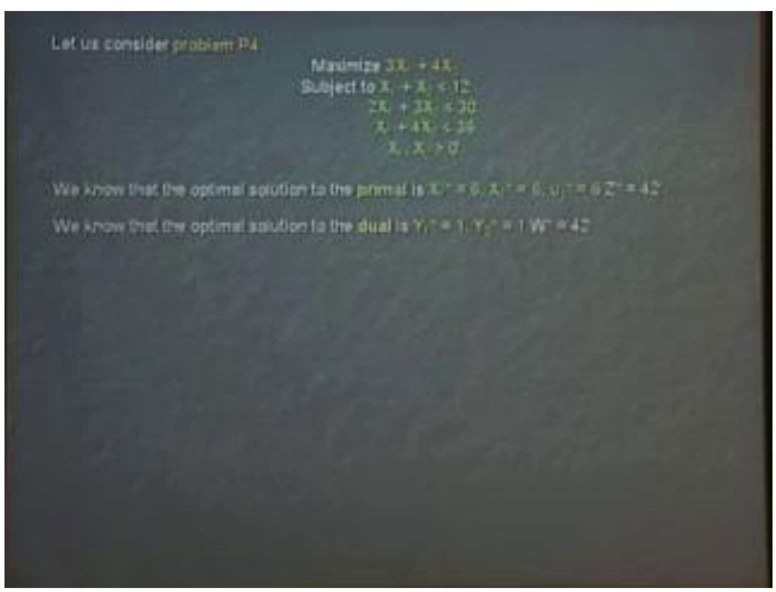

Now if we look at the next problem we get another interesting interpretation of the dual that we should be seeing. Now let us go back to this problem.

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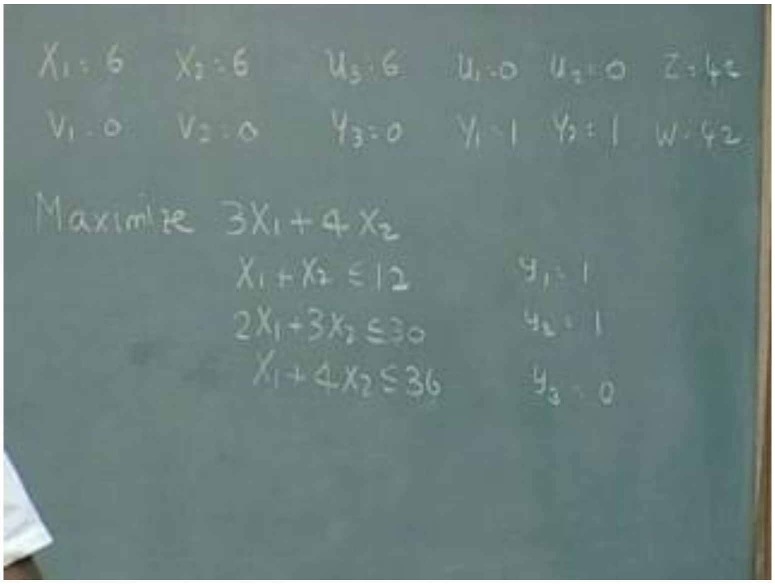

Now this problem is maximize $3X_1 + 4X_2$ subject to $X_1 + X_2$ less than or equal to 12; $2X_1 + 3X_2$ less than or equal to 30; $X_1 + 4X_2$ less than equal to 36. If you look at the optimum solution to this, you find the three dual variables. Y_1, Y_2, and Y_3 are the dual decision variables. From the optimum solution to the dual, we realize this is $Y_1 = 1$, this is $Y_2 = 1$; $Y_3 = 0$. It means that if I make this 12 as 13, if I make this increase it by delta, let us say the objective function will become 42 + delta now this is Z = 42 this is W = 42. This means I can and I am willing to pay one rupee more for this resource. I am willing to pay only 0 only for this resource. Why? The reason is, Let us go back to the solution $X_1 = 6$; $X_2 = 6$ which means I have 36 units of this resource. The requirement is 6 + 24 = 30.

I have not utilized this resource fully. I already have 6 resources with me so right now we do not have any marginal utility. I do not have to go to the market and pay for an extra resource. It does not cost me anything. So the marginal value of a resource that is not completely utilized is 0 because the resource is already available.

Now that is shown. If a slack variable is present in the basis which means the resource is not fully utilized then the corresponding decision variable will take a value 0. This is an explanation that we should understand. Sometimes the dual variable, dual decision variable can take value 0 indicating that in the primal some resource is available and not entirely utilized. That is shown as a slack variable in the basis of the primal. That I think is shown here. Now the slack variable is on the basis corresponding to dual decision variable which is non basic

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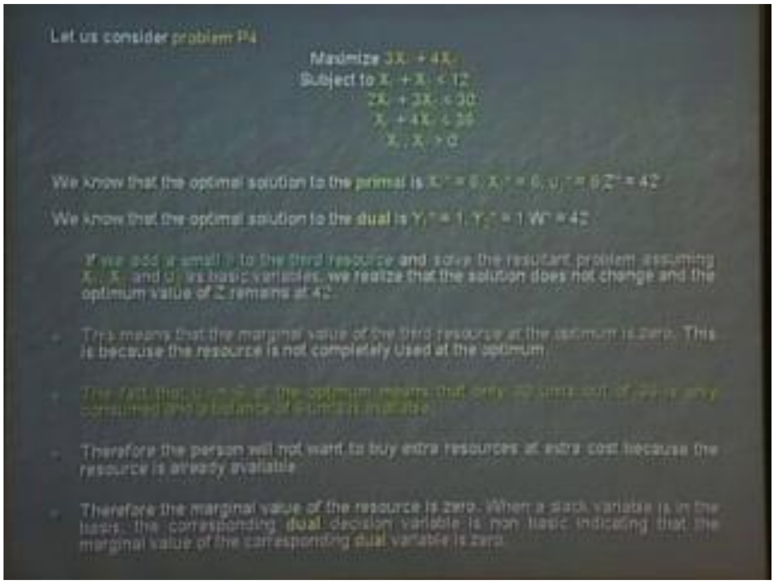

Now the slack variable is on the basis corresponding to dual decision variable which is non basic indicating that the marginal value of the corresponding resource is 0.

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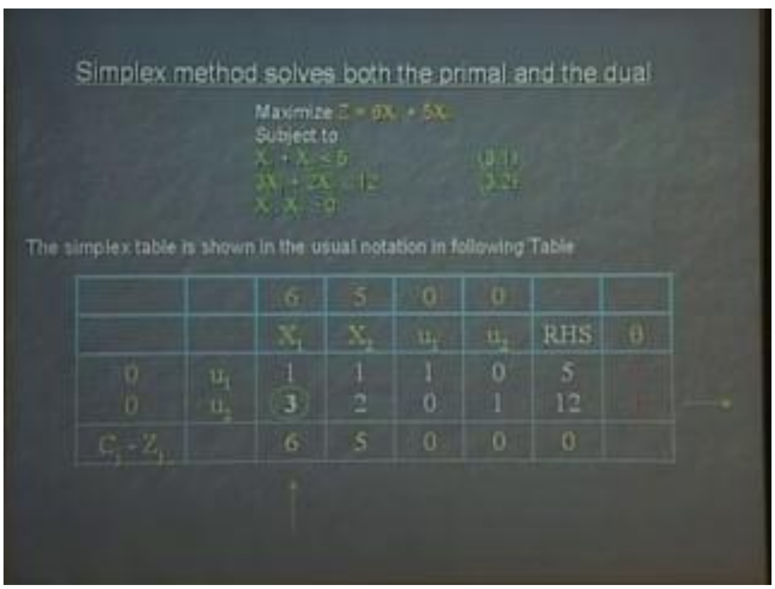

Let us look at the simplex method. What does the simplex method do? So far we have solved the given primal problem using the simplex method.
Does the simplex method give us any clue about the dual or does it explicitly look at the dual and try to solve the dual so that is something which we need to look at now.
Let us do that. Let us go back to the same example and look at the simplex table for this example.

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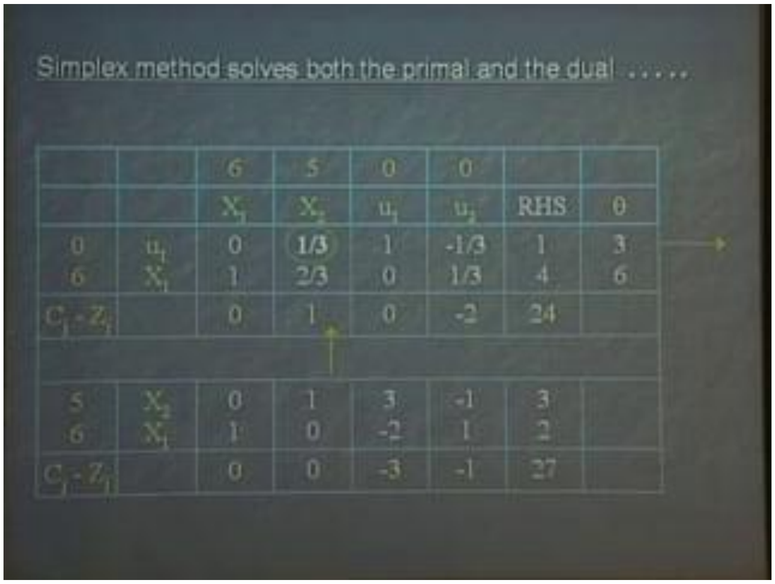

If we go back to this we are only showing that the last iteration of the simplex table. I am not showing all the intermediate iterations now.

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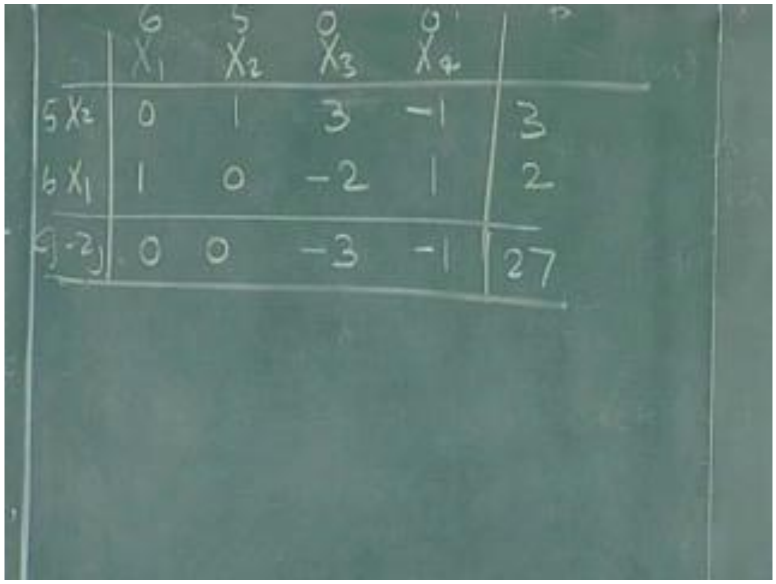

We are only looking at the final simplex table for this problem and the final table would be (6 5 0 0). We have X_2 and X_1. This is the optimum table in the simplex. We get 0 1, – 1, 0 1 3 – 1 3 1 0 – 2 1 2 $C_j – Z_j$. This is 5 and 6 0 0, 5 into 3 = 15, 6 into 2 = 12. I get a – 3, 5 into – 1 is = – 5; 6 into 1 is 6. I get a – 1 and I get Z = 27. The question is this 3 with the minus sign and is this 1 with a minus sign? The answer again is yes now simplex has a way to show the optimal solution to the dual also simplex not only solves the primal but simplex also solves the dual. Whatever we see here as $C_j – Z_j$ at the optimum is because we are solving a maximization problem and we are using $C_j – Z_j$. Our termination condition is that the non basic variables should have a negative value at the optimum. Now at the optimum for maximization problem, if you take the $C_j – Z_j$ and multiply it with a – 1 then you realize that you will have the values of the duals seen somewhere else and it is always seen here. It will be seen in the next couple of minutes.

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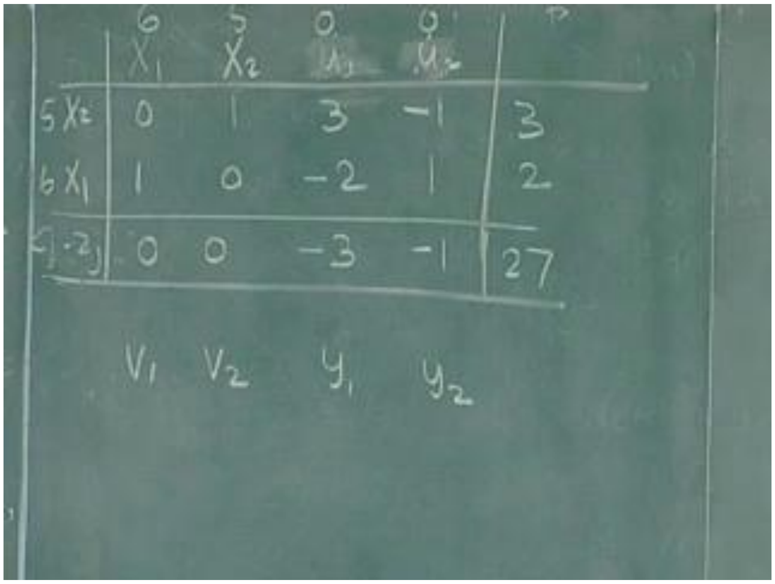

Now X_1 and X_2 are the primal decision variables so what we do here is, we write this as u_1 and u_2 which are the primal slack variables.

X_1 and X_2 are the primal decision variables from the complimentary slackness. Their corresponding dual slacks are V_1 and $V_{2.}$ These are the primal slack so they correspond to Y_1 and Y_2. What we see here as $C_j – Z_j$ is nothing but the solution to the dual for the dual decision variables as well as a dual slack variables. Here you can go back and see $Y_1 = 3$; $Y_2 = 1$ but with minus sign, of course that is because of $C_j – Z_j$ you need to multiply this with the minus one.

So you get $Y_1 = 3$; $Y_2 = 1$. Now $v_1 = 0$; $v_2 = 0$. This is the solution to the dual.

Solution to the primal $X_1 = 2$; $X_2 = 3$; u_1; u_2 which are not there are equal to 0. So at the optimum, the simplex algorithm not only solves the primal but also is capable of showing the solution to the dual.

In any case we know that the value of the objective function is the same 27. So the solution of the dual is seen as the $C_j – Z_j$ values with a negative sign under the corresponding slacks.

These are the primal slacks that correspond to dual decision variables. Primal decision variables correspond to the dual slacks. This $C_j – Z_j$ row can now be seen as a dual solution row and the right hand side is seen as a primal solution column. The right hand side will show only the basic variables of the primal with the corresponding values. The rest of the variables are non basic at 0 and the $C_j – Z_j$ row at the optimum multiplied with the minus sign because at the optimum, this row is entirely non negative. So you multiply this with the minus one. We will get the corresponding solution to the dual solution of both the decision variables as well as the slack variables. So simplex is capable of showing the solution both to the primal as well as to the dual in this example. Why simplex is doing this is, it is capable of showing why this, an optimum solution to the dual is. It will be seen a little later. In this course when we look at matrix methods for solving linear programming problems, we will show how exactly this method becomes or these values become the optimal solution to the dual.

The simplex also does a few interesting things. If the $C_j - Z_j$ in the optimum iteration is capable of showing as the solution to the dual then is there an interpretation for the $C_j - Z_j$ in an intermediate iteration. If you take the problem we solved again, we took three iterations to solve this. We had the first iteration which was initialization then we had intermediate iteration and then we had this as a final iteration. Now if the $C_j - Z_j$ in the final iteration can show as the dual solution, what is the explanation for the $C_j - C_j$ in the intermediate iteration? So that part of it as well as the complimentary slackness conditions will be seen in the next lecture.