Fundamentals of Operations Research

Prof. G. Srinivasan

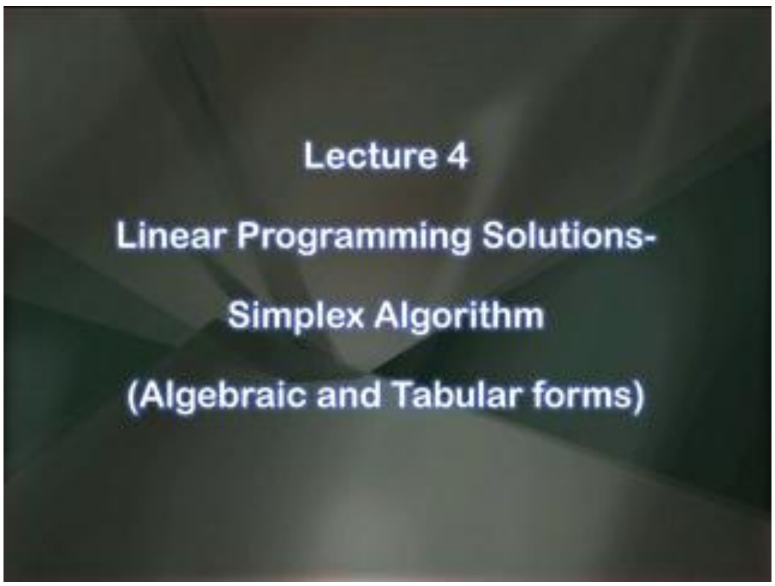
Department of Management Studies

Indian Institute of Technology Madras

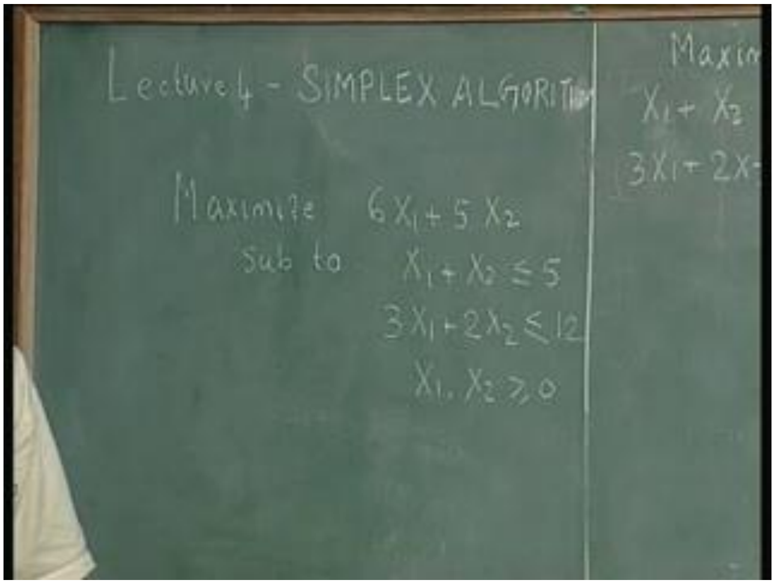
Lecture No # 04

Linear Programming Solutions Simplex Algorithm (Algebraic and Tabular forms)

(Refer Slide Time: 01:12)



(Refer Slide Time: 01:20)

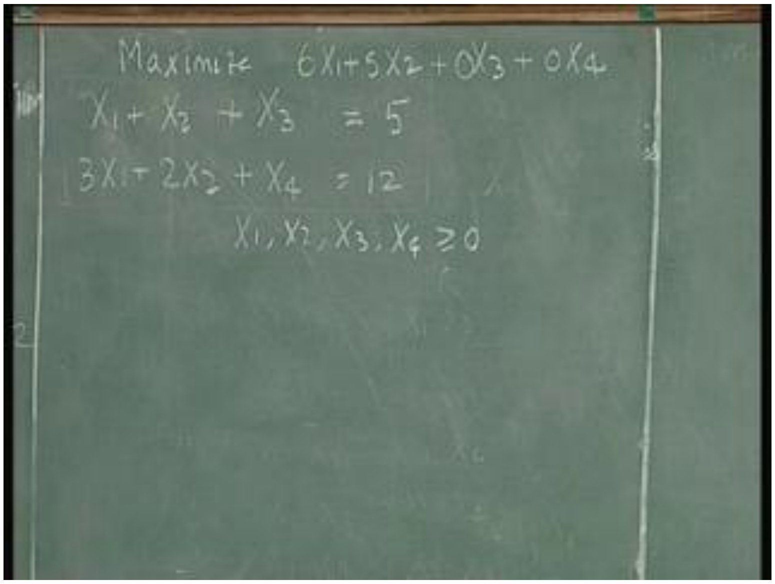


In this lecture will be looking at the simplex algorithm to solve linear programming problems. In the last lecture we looked at the algebraic method and we said that the algebraic method should have 3 important characteristics:

1. It should not evaluate any infeasible solutions
2. It should be capable of giving better and better basic feasible solutions
3. It should be able to identify the optimum when it has reached and terminated.

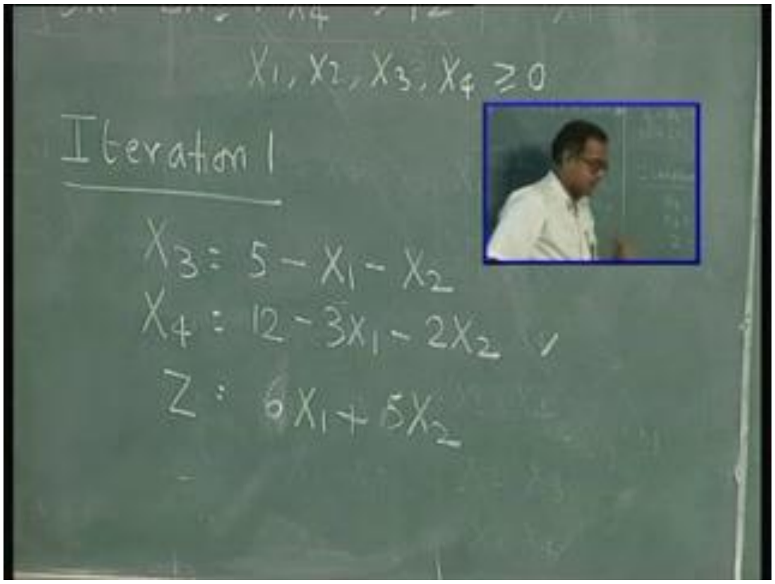
We will now see the simplex method which does all the 3. We will first see an algebraic form of the simplex method and then we see the tabular form of the simplex method in this lecture. Let us take the same example and explain how the simplex method works for this problem. So as usual we convert the inequalities to equations by adding a slack variable which we have done here. Slack variable has 0 contributions to the objective function.

(Refer Slide Time: 02:16)



So solving this (Refer Slide Time: 02:38) problem is the same as solving this problem. Since the simplex method should not evaluate any infeasible solutions. We need to begin the simplex method by considering a basic feasible solution. Now that can be done easily. The way this problem is. By fixing initially X_1 and X_2 to 0 so that we get a value of $X_3 = 5$ and $X_4 = 12$ very easily. In fact one of the important things in any linear programming problems is that the constraints should not have a negative value on the right hand side. If a constraint has a negative value on the right hand side then we need to multiply this by – 1to make it non negative. In the process the sign of the inequality may be reversed. So we will make an assumption that in all linear programming problems that we solve, the constraints have a non negative value on right hand side. It can have a 0 but it should not have a negative. Since these constraints have non negative value on the right hand side and since every slack variable appears only in one equation it is not very easy for us to fix the rest of the variables to 0 and have a starting solution with the $X_3 = 5$ and $X_4 = 12$ which is basic feasible. It is basic because the variables X_1 and X_2 are fixed to 0. It is feasible because X_3 and X_4 as variables appear only in one of the constraints and they have a non negative right hand side value. So the solution with fixing $X_1$and X_2 to 0 and evaluating $X_3 = 5$and $X_4 = 12$ is basic feasible and that is the first solution that we will look at.

(Refer Slide Time: 04:09)



So the first two sets of basic variables that we will be solving are these. We call this first iteration. We are going to solve for X_3 and X_4 and we are going to fix X_1 and $X_2$to 0. So we rewrite this as

$X_3 = 5 – X_1 – X_2$;

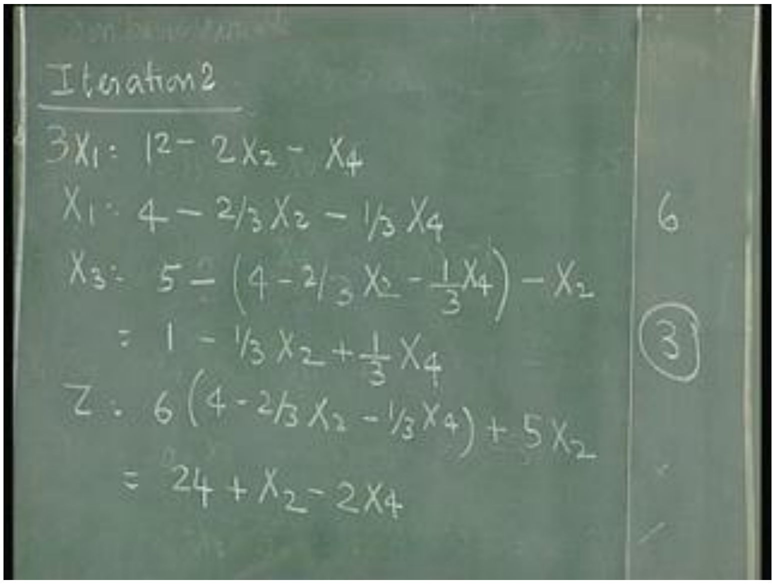
$X_4 =12 – 3X_1 – 2X_2$;

$Z = 6X_1 + 5X_2$;

What we have done here is we have identified the basic variables. We have now written the basic variables in terms of the non basic variables and we have also written the objective function in terms of the non basic variables. X_1 and X_2 are the non basic variables X_3 and X_4 are the basic variables. Now we can put X_1= 0; X_2 = 0 to have a solution X_3 = 5; X_4 = 12 and Z = 0. Now what are we interested in is we are interested in maximizing the Z. Now the Z is at 0 because X_1 and X_2 are non basic with 0. Now we want to increase Z we can do that either by increasing X_1 or by increasing X_2. Both have a strictly positive coefficient and both are at 0. So we can increase either X_1 or X_2 or both. In order to increase that, what we do in this simplex method, is we try to increase only one variable at a time. So we can choose to either increase X_1 or to increase X_2. Now between the two we would prefer X_1 because the rate of increase is higher because it has a bigger or higher coefficient. So we try to increase X_1. X_1 is presently at 0.

Now there will be a limit on the value that this X_1 can take because as we increase X_1 we realize that X_3 and X_4 are going to reduce. They are right now at 5 and 12 respectively. For example if X_1 becomes 1then X_3 will become 4; X_4 will become 9. As X_1 increases, X_3 and X_4 starts reducing and comes closer to 0. So X_1 can increase up to the point where one of them becomes 0 because increasing X_1 beyond that would end up making either X_3 or X_4 negative which we do not want. We will be violating this. This equation will allow X_1 to go up to 5. This (Refer Slide Time 07:26) equation will allow X_1 to go up to 4. The limiting value that you can increase X_1 up to is 4 which is a minimum of 5 and 4 and increasing X_1 beyond 4 will make this X_4 negative and hence violate this. So now this constraint or equation becomes the binding equation which is going to determine the value that X_1 can take.

(Refer Slide Time: 07:23)



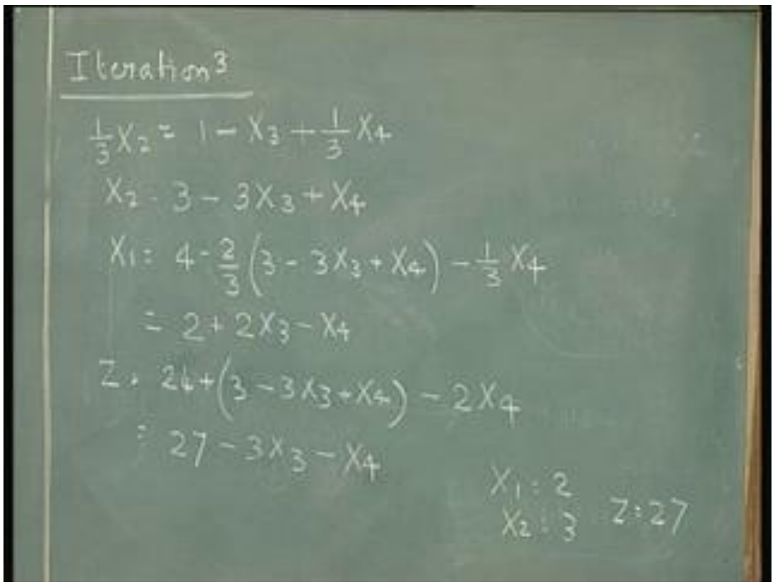
So we write this again as iteration 2. We now rewrite as
$3X_1 = 12 – 2X_2 – X_4$
$X_1 = 4 – 2/3X_2 – 1/3X_4$
Now we write this again by saying $X_3 = 5 – X_1$.
We go back and substitute $5 – (4 –2/3X_2 –1/3X_4) – X_2$. This on substitution would give $(1 + 2/3X_2 – X_2)$ and this $–1/3X_2 + 1/ 3X_4$. Now the objective function Z becomes $(6X_1 + 5X_2)$, 6 into $(4 – 2/3X_2 – 1/3X_4) + 5 X_2$ which on substitution would give us 24; (this is $– 4X_2 + 5X_2$) and therefore we get this $+ X_2 – 2X_4$.

Now we have iteration where your X_1 and X_3 are basic and X_2 and X_4 are non basic. The solution is given by substituting $X_2 = X_4 = 0$ getting $X_1 = 4$; $X_3 = 5$; $Z = 24$.So this is another basic feasible solution that we have obtained. Now this is basic because we are fixing X_2 and X_4 to 0. It is feasible because of the way we limited X_1. We made sure that X_1 is getting a non negative value. We also made sure that neither X_3 nor X_4 goes to a value below 0. So we ensure that X_1 is feasible X_3 and X_4 also are feasible one of them becomes 0. The other remains at 0. So this is another basic feasible solution that we have evaluated.

Now this basic feasible solution has $X_1 = 4$; $X_3 = 5$; X_2 and $X_4 = 0$ with $Z = 24$. Our objective is to try and increase this Z further. Now this Z can be increased further by either increasing X_2 or decreasing X_4 because X_4 has a negative coefficient. Now X_4 is already at 0. It is non basic. It is at 0 and we cannot decrease X_4 because decreasing X_4 would make it negative which would violate this. So we do not explore the possibility of trying to increase Z by decreasing X_4. On the other hand we can try and increase the Z by increasing X_2. X_2 is already at 0 so by increasing X_2 we can try an increase Z further. So we try to see whether we can increase Z further by increasing this X_2 or we try to find out what is the limiting value or the maximum value up to which this X_2 can be increased.

Now in this equation the first one would allow X_2 to go up to 6 because, 6 into 2/3 = 4 and 4 – 4 is 0. So when X_2 takes a value 6, this would be 0. Increasing X_2 beyond 6 would make this negative. We do not want to do that. This would make X_2 go up to 3 because increasing X_2 beyond 3 would make X_3 negative which we do not want. So the limiting values are 6 and 3 and we take the minimum one.
This is the minimum value. 3 is the maximum value that X_2 can take beyond which this one would become negative and therefore this is the limiting value. This is the equation which is going to make X_2 come into the basis and take a non negative value.

(Refer Slide Time: 11:50)



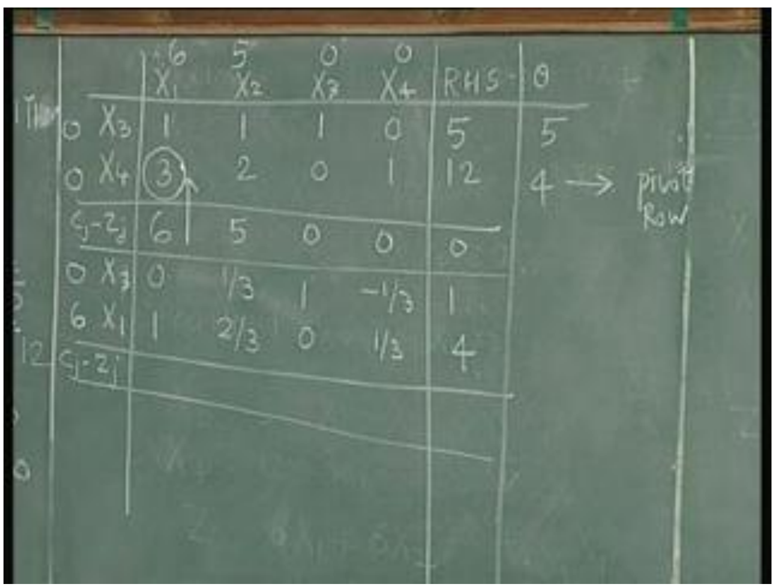
This is now rewritten here as iteration 3. This now becomes,
$1/3X_2 = 1 – X_3 + 1/ 3X_4$;
$X_2 = 3 – 3X_3 + X_4$. Now the next equation is this,
$X_1 = 4 - 2/3$ into $(3 – 3X_3 + X_4) – 1/3X_4$ This on simplification, 2/ 3 into 3 is 2, so you get 2 + $2X_3$ – 2/ $3X_4$ – 1/ $3X_4$ is = – X_4. Substituting Z is = 24 + X_2 +
$(3 – 3X_3 + X_4) – 2X_4$;
$27 – 3X_3 – X_4$. You get a + X_4 – $2X_4$ which will give you – X_4.
So now you have a solution. In this case X_1 and X_2 are basic, X_3 and X_4 are non basic at 0. So the solution is you substitute X_3 and X_4 = 0 in all this to get X_1 = 2; X_2 = 3 and Z = 27. This solution is also basic feasible. It is basic because X_3 and X_4 are fixed at 0 and it is feasible because of the way we define the limiting value that X_2 can take. This is the solution at the end of the iteration.

Now we want to see whether we can increase Z further now. Increasing Z further based on this expression can be done by decreasing X_3 or by decreasing X_4 because both have negative coefficients. Both of these are not possible. Decreasing X_3 and decreasing X_4 is not possible because both of them are already at 0 and decreasing them would make them infeasible. So we are unable to proceed from this point to try and increase Z further and since we are unable to increase Z further we stop here and we say that the best solution has been obtained which is X_1 = 2 ; X_2 = 3; Z = 27. Now this is the same solution that we got by the graphical method as well as by the algebraic method. If we look at this very carefully, this method has done exactly the three things that we wanted it to do.

1. It will not evaluate any infeasible solution because we put in that extra effort to find out the limiting value that this variable can take therefore none of the variables will become negative so we will not get any infeasible situation.
2. It will evaluate progressively better solutions because every time we are only trying to increasing objective function for maximization problem from where it is so 0 became 24 and 24 became 27.

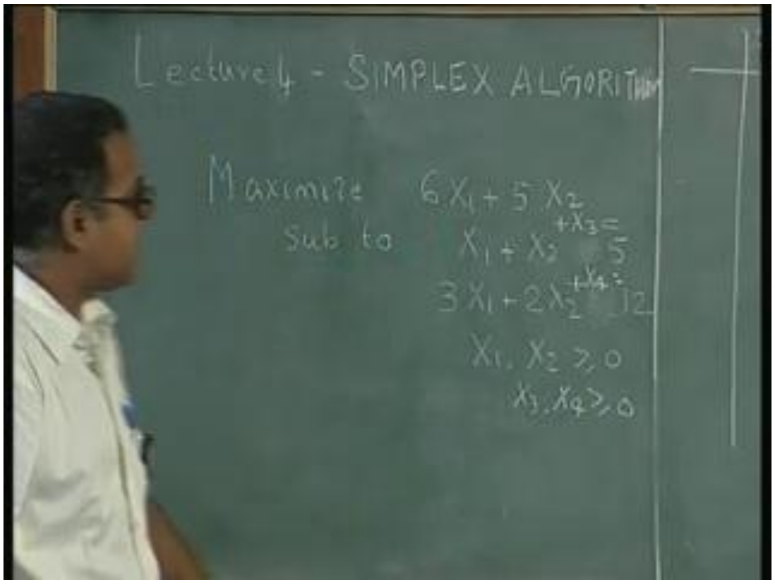
The moment the optimum is reached or the best solution is reached it will have a way to terminate and say that further increase is not possible. So you need not evaluate all the four basic feasible solutions to get here. Within three, we were able to reach the third one and as told as already here it is optimal. So this method is the ideal method that we were looking for with respect to the 3 requirements that we had. This is the simplex method represented in an algebraic form. We now see the simplex method in a tabular form.

(Refer Slide Time: 16:06)



We create a simplex table like this. There are 4 variables that we have and we write them as X_1, X_2, X_3 and X_4. The objective function coefficients are 6 and 5 and we also said slack variables X_3 and X_4 have a 0 objective function coefficient so we have 0 and 0. The previous method we started with X_3 and X_4 as the basic variables. We said we can fix X_1 and X_2 to 0 to get $X_3 = 5$.

(Refer Slide Time: 16:58)



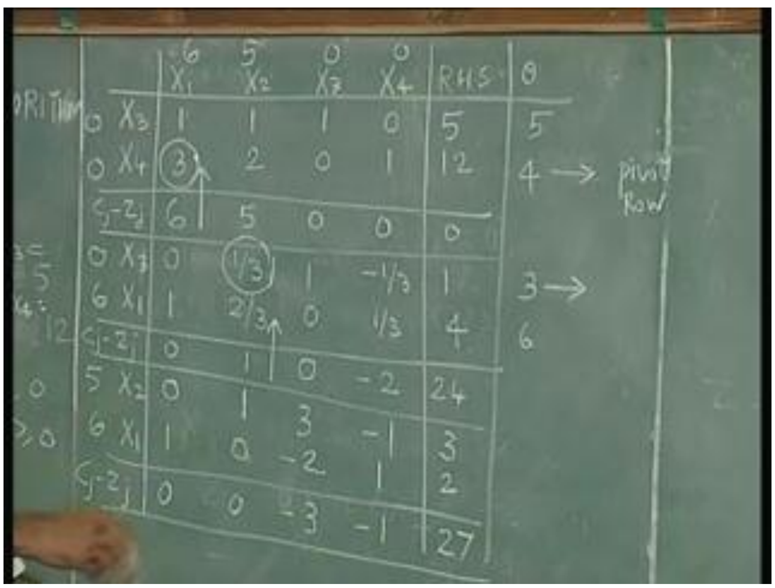
Remember that this was written this way and X_3, X_4 greater than or equal to 0 we started with fixing $X_1$and X_2 to 0 as non basic and started with X_3 and X_4 as basic variables. So the basic variables that we began are written here, X_3 and X_4. We also have the coefficients corresponding to the basic variables written here (Refer Slide Time: 17:03). The equations are written as they are so you have a $1X_1 + 1X_2 + 1X_3 + 0X_4 = 5$; $3X_1 + 2X_2 + 0X_3 + 1X_4 = 12$. This is called right hand side. We introduced another row which we call as $C_j – Z_j$ and we explain this now.
What is what we compute as Z_j is that, for every variable j, we compute the dot product of this and this. For example Z_1 will be 0 into 1 + 0 into 3 which is 0 C_1 is 6.C_1 is the coefficient in the objective function for variable X_1. So C_1 is 6 so $C_1 – Z_1$, the number that is going to appear here will be 6 – (0 into 1) + (0 into 3) = 0 so 6 – 0 is 6. In the very first iteration because this is 0, all the Z_j’s will become 0 so $C_j – Z_j$ will be the same as this value. Now for this it will be (0 into 1) + (0 into 2) which is 0. (5 – 0 = 5), Here it is (0 into 1) + (0 into 0 = 0) and (0 – 0 is 0) (0 into 0 is 0) + (0 into 1 is 0). (0 – 0 is 0). When it comes to the right hand side we do not have any value here so we simply multiply (0 into 5 + 0 into 12 = 0). Now we look at the variable which has the largest positive $C_j – Z_j$ which happens for variable X_1. So now we say that the variable with the largest $C_j – Z_j$ enters the basis. So X_1 enters the basis and we put an upward arrow indicating that X_1 is going to come into the solution. When X_1 will come into the solution and it will replace either X_3 or X. We need to find out which one it replaces. So to do that you create another column called theta. That is 5 divided by 1= 5. The right hand side divided by the corresponding element of the entering column so 5 divided by 1 = 5. 12 divided by 3 is 4.

Now choose the one with the minimum value of this theta so this is the variable that goes. Now variable X_4 will leave and variable X_1 will enter now. The next iteration will now have X_1 replacing variable X_4. So you will have X_3 and X_1 which will come here. Variable X_1 replaces variable X_4 so you have X_3 and X_1 coming in. X_3 has a coefficient 0; X_1 has a coefficient 6. What we need to do is, if we look at this table these two were the basic variables or variables which we were trying to solve. Now you need to have an identity matrix corresponding to these variables. X_3 X_4 in the order of appearance has a 1 0 0 1 which is an identity matrix. Now here we have X_3 and X_1 so in the order of appearance we want an identity matrix .So X_3 will have (1 0); X_4 and X_1 will have (0, 1) in the order in which the variables appear. In order to make this happen, we perform some row operations on this table. In order to do that we explain the row operations as follows:

If you look at the previous table, this is the entering variable. This is the leaving variable. The row corresponding to the leaving variable is called a pivot row and this particular number which is the intersection of the entering column and the leaving row is called the pivot element is usually marked with the circle. Now first step would be to divide the pivot row by the pivot element as it is. So I get 3 divided by 3 = 1, 2/3, 0, 1/3 and 4.

Remember we wanted a (1, 0) and a (0, 1). We have we have got a 0 here and a 1 here. So we need to get a 0 here without disturbing this 1 and that is done by using row operations. This value is 1. This value is also 1.The row operation here would be if I subtract this from this I would get a 0 here. So 1 – 1= 0; 1 – 2/3 = 1/3; 1– 0 = 1; 0 – 1/3 = – 1/ 3; 5 – 4 = 1 and so the new row 1will be the old row 1 – the new pivot row that would give us this. Now this is the solution given by $X_3 = 1$ $X_1 = 4$. Now we compute the $C_j – Z_j$ again.

(Refer Slide Time: 23:34)

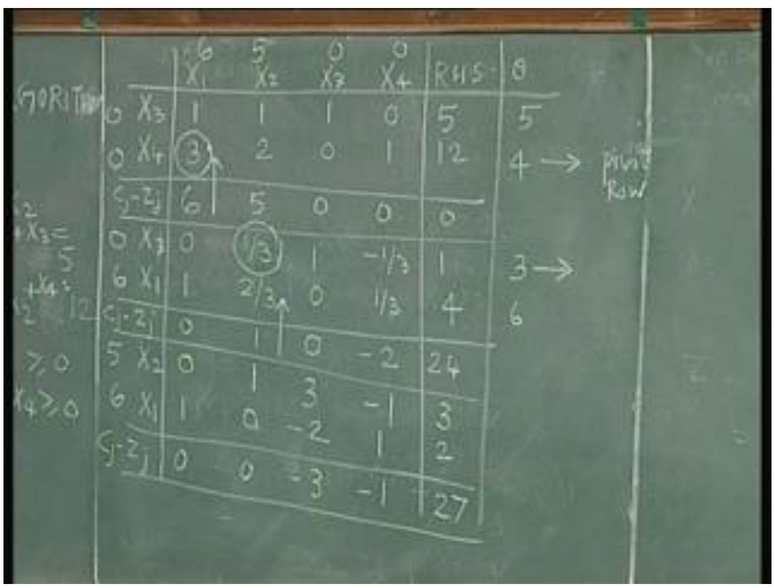


Now this value is the dot product of this and this (Refer Slide Time: 23:43) 6 – (0 into 0 + 6) so (0 into 0 + 6) into 1 is = 6; 6 – 6 is 0; 0 into 1/3 + 6 into 2/ 3 is 4; 5 – 4 =1 and 0 into 1/3 is 0 and 6 into 2/ 3 is 4. Sum is 4. 5 – 4 is 1, (0 into 1)+ (6 into 0) is = 0; (0 – 0 is 0); (0 into – 1/3)+ 6 into (1/3) i.e., 0 into – 1/ 3 is 0 and 6 into 1/3 is 2. (0 – 2) is = – 2 and here we simply multiply the values and write it (0 into 1) + (6 into 4) = 24. So we now have a solution $X_3 = 1$; $X_1 = 4$; Z = 24. Now once again we go back and look at the values of $C_j – Z_j$ and identify that variable which has the maximum positive value that happens for variable X_2. So X_2 now enters the basis and becomes the basic variable.

To find out the leaving variable we need once again to find out the value of theta which gives us the limiting value. That is now given by right hand side divided by the corresponding element in the entering variable column. 1 divided by 1/3 = 3; 4 divided by 2/3 = 6. Between the two the minimum, theta happens here. So this variable leaves the basis. In the next iteration X_2 will replace X_3 which is written here (Refer Slide Time: 25:23). X_2 and X_1 retains its position. Now this is the pivot row this is the entering column so this is your pivot element. Now once again as we did earlier we need to do row operation such that under X_2 we have a (1, 0) under X_1 I have a (0 1), so that the identity matrix appears in the order in which these variables appear here and that is done by following the same steps that we did in the previous iteration, divide the pivot row by the pivot element which is the first thing to do => 0 divided by 1/ 3 is 0; 1/ 3 divided by 1/ 3 is 1.1 divided by 1/3 is 3; – 1/3 divided by 1/3 is – 1and 1 divided by 1/ 3 is 3. Now I need to bring a 0 here by doing the row operation. Now this (Refer Slide Time: 26:12) is the element. So this row minus 2/ 3 times would give me a 0 here. So 1 – 2/ 3 (2/ 3 comes from this). 1 – 2/ 3 into 0 is = 1; 2/ 3 – (2/ 3 times 1) is = 0; 0 – 2/ 3 into 3 is – 2. 1/ 3 – (– 1 into 2/ 3) = 1/3 + 2/3 which is 1; 4 – 2/3 into 3; 2/3 into 3 is 2; 4 – 2 = 2, so we have completed this.

Here we write the objective function coefficients corresponding to X_2 and X_1 respectively which are 5 and 6. Now the $C_j – Z_j$ values are (5 into 0) + (6 into 1) which is 6; 6 – 6 = 0.
(5 into 1) + (6 into 0) = 5; 5 – 5 = 0; (5 into 3) + (6 into – 2) = 15 – 12 which is 3; 0 – 3 = – 3; (5 into – 1) + (6 into 1) i.e., – 5 + 6 which is 1; 0 – 1 is – 1.

(Refer Slide Time: 16:06)



Right hand side values are (5 into 3 = 15) + (6 into 2 = 12) which is 27. Now we have a solution with X_1 = 2; X_2 = 3 with Z = 27. We check whether we can improve the solution further by looking at the $C_j – Z_j$ values. Now the maximum positive $C_j – Z_j$ is what we enter we do not have a positive $C_j – Z_j$ at all. So the algorithm is not able to identify an entering variable so the algorithm terminates with the solution X_1 = 2; X_2 = 3; Z = 27. This is called the tabular form of the simplex algorithm. Now if we go back and compare this solution with what we did in the algebraic form, you will realize that we are doing exactly the same three things in the algebraic form. When we began the previous algebraic method we started with the solution X_3 = 5; X_4 = 12; Z = 0.

We had an expression $6X_1 + 5X_2$ and we wanted to increase based on increasing X_1 or X_2. We increased X_1 because it had the largest coefficient or the largest rate of increase. The second solution that we obtained was this solution $X_3 = 1$; $X_1 = 4$ and the value of the objective function were $24X_1 + X_2 – 2X_4$.

So this is the solution that we got in the algebraic form. We then entered X_2 and then we finally got $27 – 3X_3 – X_4$. We said that we cannot enter X_3 or X_4 because they will become negative and we terminate. So whatever we have done in the algebraic method is exactly the same that we do in this tabular method. The only difference in the algebraic method is that these two equations would look like $X_2 = 3 – 3X_3 + X_4$. Now here we write it as $X_2 + 3X_3 – X_4 = 3$.

So the tabular form is exactly what we did as the algebraic form except that we do not write the variables every time as we did in the algebraic method. Tabular method is a very convenient way of representing all the numbers. So this tabular form of simplex method is more popular than the algebraic form because when we work out the problem by hand we do not have to repeat writing the names of the variables. So this would give a certain structure to this simplex method of solution.

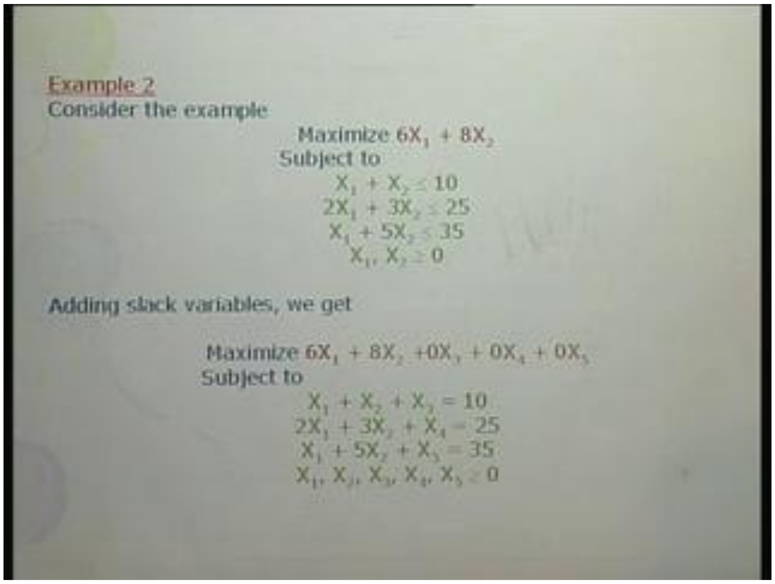
For a linear programming problem, many versions of the tabular form exist. Different books might give different versions. In some places you will find that you have another row here where you explicitly calculate this Z_j and then you have another row which does $C_j – Z_j$. There are versions where instead of $C_j – Z_j$, you calculate $Z_j – C_j$ and then you enter variables accordingly. But what we will do in this course is we will stick to this kind of a tabular format with only evaluating one additional row here which we call as a $C_j – Z_j$ row. Our entering rule will be any positive variable. Or any variable which has a positive value of $C_j – Z_j$ can enter, as X_1 entered here and X_2 entered here (Refer Slide Time: 31:28). Here you have all the $C_j – Z_j$'s are 0 or negative so we do not have an entering variable.

So our termination condition is when $C_j – Z_j$ is all less than or equal to 0. For a maximiZation problem the algorithm will terminate and the best value will give the optimum solution.

Now there are a few more things that we need to understand from the simplex table. The Simplex table tells us a few more things. As we progress in this course we will realiZe that there are many more things to be understood from the simplex table.

We will start with something like this. Now an X_3 and X_4 are basic variables. You will find that the $C_j – Z_j$ values for the basic variables is always 0 here. X_3 and X_1 had $C_j – Z_j$ values 0 and similarly X_2 and X_1. So $C_j – Z_j$ values will be 0. You do not have to explicitly compute the $C_j – Z_j$ for the basic variables. We need to compute the $C_j – Z_j$'s only for the non basic variables which is what we did in the algebraic method. The $C_j – Z_j$ only for the non basic variables need be computed and if they all of them become negative, the algorithm will terminate. If one of them is positive then that variable can enter. The limiting value which we found out in the earlier method is exactly the value of theta that we obtained in the very first iteration. If you go back you would realiZe that these were the limiting matrix which shows the minimum here. Also we put the limiting values. The only difference is, in the algebraic from you would see all these coefficients multiplied by – 1 simply because you have shifted the equation sign to the variable here. Now let us take another example for the simplex algorithm and then we work out a different problem.

(Refer Slide Time: 33:54)



The problem is to maximize:

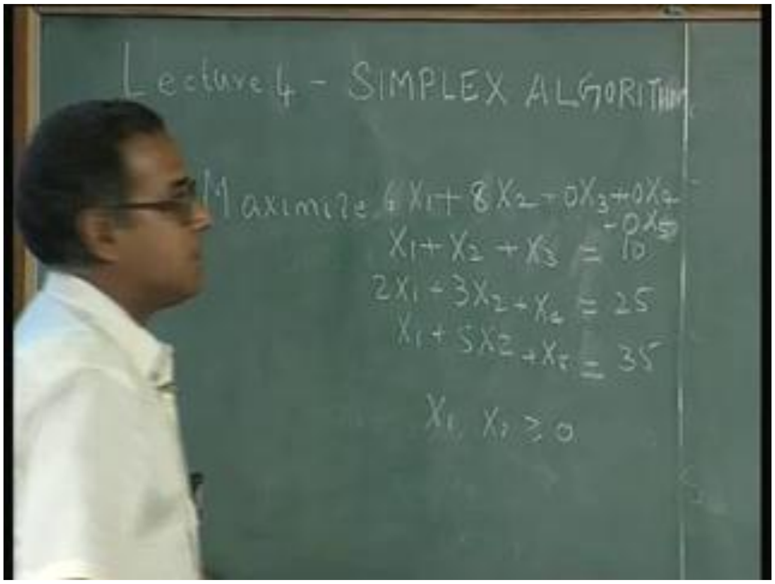
$6X_1 + 8X_2$

$X_1 + X_2$ less then are equal to 10

$2X_1 + 3X_2$ less than or equal 25

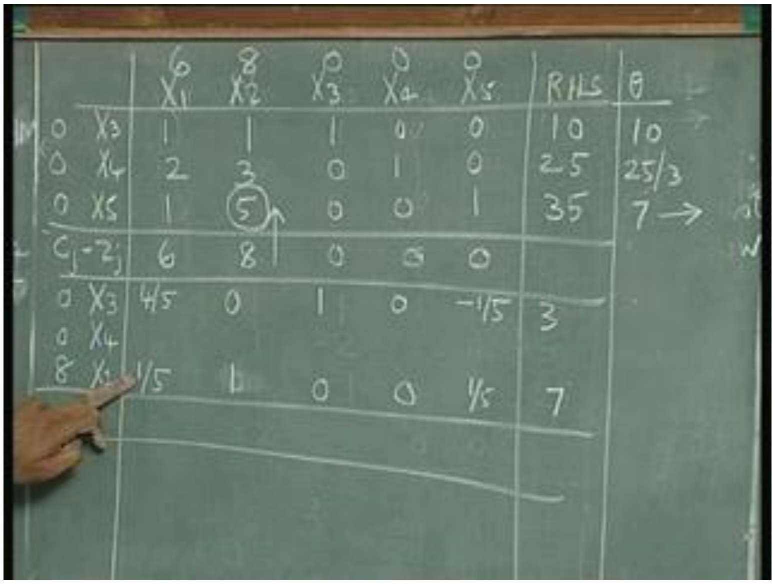
$X_1 + 5X_2$ less than are equal to 35.

(Refer Slide Time: 34:08)



We convert these inequalities to equations by adding slack variables X_3 make it an equation, X_4 + X_5 = 35. The contribution of the slack variables to the objective function is 0. So we have $0X_3$ + $0X_4$ + $0X_5$.

(Refer Slide Time: 34:33)



Now we construct the simplex table here, with 5 variables X_1, X_2, X_3, X_4 and X_5 with the right hand side values given here. The coefficients are [6 8 0 0 0]. We start with X_3, X_4, and X_5. The corresponding objective function coefficients are 0. Now the equations are written as:

$X_1 + X_2 + X_3 + 0X_4 + 0X_5 = 10$

$2X_1 + 3X_2$ (0 1 0 = 25) and

(1 5 0 0 1 = 35)

Now you see the identity matrix under X_3, X_4 and X_5 in the order of appearance, i.e., (1 0 0); (0 1 0) and (0 0 1). We calculate $C_j – Z_j$. X_3, X_4 and X_5 are basic variables so they will have $C_j – Z_j$ = 0. For X_1 it is (0 into 1) + (0 into 2) + (0 into 1) which is = 0. So we get a 6 here. (0 into 1) + (0 into 3) + (0 into 5) is 0 so we get 8 here. 8 – 0. Now the non basic variable which has the largest $C_j – Z_j$ enters, so variable X_2 will enter now to find out the leaving variable. We need to find a theta. Theta values will be 10 divided by 1 which is10.

25 divided by 3 is, 25/3; 35 divided by 5 is 7. Now the minimum value is 7. So this is the leaving variable. X_5 is the leaving variable and this is our pivot element. Now in the next iteration, X_2 replaces X_5 so you have X_3, X_4 and X_2. The values are (0 0 and 8). Now the first step is to divide the pivot row by the pivot element, so that we get the identity matrix. Now we need a (1 0 0), (0 1 0) and (0 0 1) under $X_{2,}$ so we need a 1and we divide by the pivot element to get (1/5 1 0 0 1/5 and 7) now we need a 0 here (Refer Slide Time: 36:45) and a 0 here. So the first row of the previous iteration has a 1. So this row minus this row would give us a 0 here. So 1 – 1/5 = 4/5; 1 – 1 = 0; 1 – 0 = 1; 0 – 0 = 0; 0 – 1/5 = – 1/5; 10 – 7 is 3.

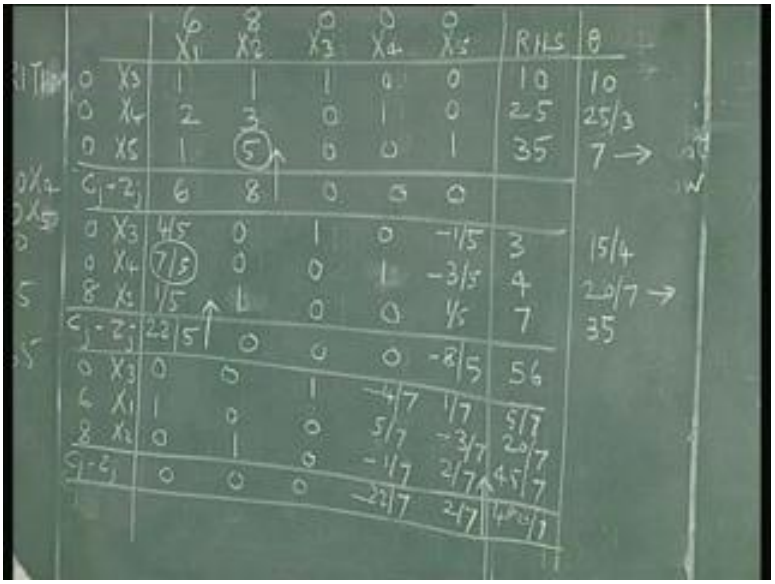
(Refer Slide Time: 37:52)

We need another 0 here so this row minus three times the pivot row would give us a 0. 3 – 3 times 1 = 0 and so the rule is this row – 3 times the pivot row would give us a 0. So 2 – 3/5 is 10/5 – (3/5) which is = 7/5. 3 – 3 times one is = 0; 0 – 3 times 0 is 0; 1 – 3 times 0 is 1. 0 – 3 times 1/5 is – 3/5; 25 – 3 times 7 is 4. So now we compute the C_j – Z_j values. X_3, X_4 and X_2 are the basic variables so (0 0 and 0). We have to compute Cj – Zj for the non basic variables X_1 and $X_{5.}$ Now this is 0 into 4/5 + 0 into 7/5 + 8 into 1/5 is = 8/5; 6 – 8/5 is = 30 – 8 = 22/5.
Now this is 0 into – 1/5 + 0 into – 3/5 + 8 into 1/5 is 8/5; 0 – 8/5 is = – 8/5;
0 into 3 + 0 into 4 + 8 into 7 = 56

Now among or between the two non basic variables X_1 and X_5, X_1 has a positive value. Only one has a positive value so X_1 enters. So X_1 enters to the basis. We need to find out the leaving variable. In order to do that, we need to compute this theta again.
3 divided by 4/5 is nothing but 3 into 5 by 4 = 15 by 4. 4 divided by 7/5 is = 4 into 5 /7 which is = 20/7; 7 divided by 1/5 is 35. Now we need to find out the smaller one between 15/ 4 and 20/7. So 15/4 will become 105/28 and 20/7 will become 8/28.

So 20/7 is smaller than 15/4 therefore this (Refer Slide Time: 39:41) is more than 3 and this is less than 3. So 20/7 is smaller and this becomes the pivot element. Now we need to get X_1 so we have X_3, X_1, and X_2. Now, X_1 comes here and so we need a (0 1 0) here. So the coefficients will be (0 6 and 8) divided by the pivot element to get (1 0 0 5/7 – 3/7 20/7). I now need a 0 here (Refer Slide Time: 40:37) and I need a 0 here. So this – 4/5 times this will give me a 0 so this becomes 0. 0 – 4/5 into 0 = 0; 1/4 by 5 into 0 is 1; 0 into 0 – 4/5 into 5/7 is 4/7 hence – 4/7.

(Refer Slide Time: 41:44)



(– 1/5 – 4/5) into – 3/7; so – 1/5 + 4/5 into 3/7; – 1/5 + 12/35; so – 1/5 – 4/5 into – 3/7 would give – 1/5 + 12/35.

So this will become 5/35 which is 1/7 and therefore I get a 1/7 here. 3 – 4/5into 20 / 7; 4/5into 20/7 is 16/7, so 3 – 16/7 is 5/7 so you get a 5/7 here and we need to get a 0 here, so this minus 1/5 times this and this will give me 0.

Now this is (0 1 0). This – (1/5 into 5/7) is = 0 – (1/5 into 5/7) is = – 1/7

(1/5 – 1/5 into – 3/7); (1/5 + 1/5 into 3/7) which is = (1/5 + 1/5 into 3/7) is = 3/35.

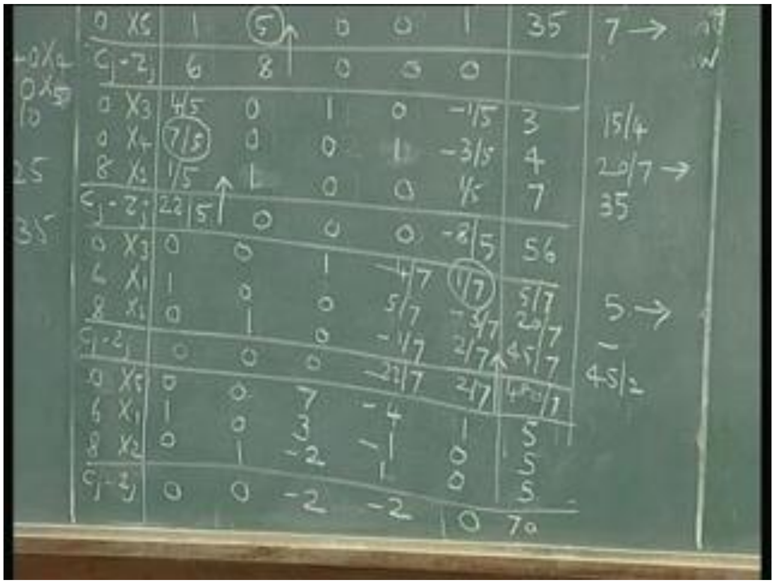
This is 10/35 which is = 2/7.

This – 1/5 times this so (7 – 1/5 into 20/7) so this will become 4/7. Therefore 7 – 4/7 is = 45/7. Now the $C_j – Z_j$ values will be (0 0 and 0).

We need to look at only the other two. This is 0. 6 into 5/7 is 30/7 – (8/7 is 22/7) Now the minus sign. This is 6 into – 3 – (18/7 +16/7) is = – 2/7 but 0 – 2/7 is + 2/7; 20 into 6 is = 120; 45 into 8 is = 360. So 360 + 120 is = 480. By now we realize that this variable can enter. That is the first thing. So first important thing is if we go back into the simplex table you realize that a variable X_5 at the left basis earlier can actually reenter which is what we are seeing in this example.

It is not absolutely necessary that once a variable leaves it cannot enter. It can enter. What we are interested in the simplex method is not whether an individual variable appears in the basis. What we are actually interested is the combination that appears in the basis. So variable X_5 which left the basis earlier can now enter again so that is what happens here. We have to find out which limiting value 5/7 divided by 1/7 is = 5, 20/7 divided by –3/7. This is the first time we are encountering a negative value here when we are divide till now we have we had all positive values so we could divide comfortably and get all the thetas. Now the rule is if you get a negative value or a 0 here you do not compute the theta. Theta value is computed only when the entering column has a strictly positive value. The right hand sides will always be non negative so theta cannot be negative theta cannot take indeterminate form theta cannot be 0 by 0 or something by 0. So we evaluate theta only when this number is strictly positive so that we get a non negative value for theta. Hence we do not evaluate. We put a dash 45/7 by 2/7 is 45/2. So this is the minimum. So this is the pivot element. Now we continue very quickly. Go to the last table now. X_5 replaces X_3. So you have X_5, X_1, X_2, 0, 6 and 8

(Refer Slide Time: 46:11)



Pivot element 1/7 divide the pivot row by the pivot element or multiply by 7 to get (0 0 7 – 4 1 5). Now we need a 0 and a 0 here (Refer Slide Time: 46:01). So this + 3/7 into this will give me 0, so 1 + 0 into 3/7 is 1, 0 and this + 3/7 is = 3 this + 3/7, so 5/7 + (3/7 into – 4) so (5/7 + 3/7 into – 4).

5 – 12 is = – 7 so I get a – 1 here. This is 0. This + 3/7; 20/7 + 15/7 3; 5/7 which is 5 and here I need a 0 so this – 2/7 times and this will give me a 0.
So I have 0 – 2/7 into 0 is 0; 1 – 2/7 into 0 is 1; 0 – 2/7 into 7 is – 2.
– 1/7 – 2/7 into 4 is – 1/7 + 8/7 which is 1. This is 0. Now 45/7 – 2/7 into 5 so 45/7 – 10/7 which is 35/7 which is = 5. Now $C_j – Z_j$ will be X_1, X_2 and X_5 are basic. We need to evaluate only for these two. 0 into 7, 6 into 3 = 18 – 16 is 2; 0 – 2 is – 2; – 6 + 8 is 2; 0 – 2 is – 2, the value is 6 into 5 + 8 into 5 which is 70.

Now both the non basic variables have a negative $C_j – Z_j$ the algorithm terminates.
Now what are the new things that we have learned?

1. In this example we realize that a variable that already left the basis which is X_5 can reenter the basis. The only uniqueness is, every iterations should have a different combination of basic variables which is what the simplex algorithm mentions. It is not unduly worried about whether particular variable leaves basis or enters the basis.

2. When we compute this theta (we do not compute the value of theta when you have a negative or a 0) here the theta value computed should be strictly non negative. Therefore when this is negative or 0, we do not compute the equivalent in the algebraic method. Theta is the limiting value that this entering variable can take such that beyond which this particular basic variable will become negative. Now a negative here implies that you can increase this X_5 to any value and

the present variable X_1 will not become negative therefore this 'dash' actually implies infinity but we do not indicate it as infinity we just leave it as a dash as a rule you do not compute theta when you have this.

3. Right through you would have observed that we have not represented any of these numbers in a decimal form and we have actually represented all of them in a fractional form. For example when we do this we might be tempted to write some of these as decimals and decimal form can create rounding off errors you realize that all these fraction finally became nice integers in the end.

   So it is a customary not to represent any of these numbers and decimal form but represent them only in fractional form. So there are other things that we need to learn. So far we have seen only a maximization problem with all constraint of less than or equal time.
   How do we address minimization problem? How do we address greater than or equal to time constraint?
   So those things will be seen with different examples in the next lecture.