Fundamentals of Operations Research

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Lecture No. # 20

Dynamic Programming - Examples to Solve Linear and Integer Programming Problems

In the last lecture we were looking at this example of dynamic programming.

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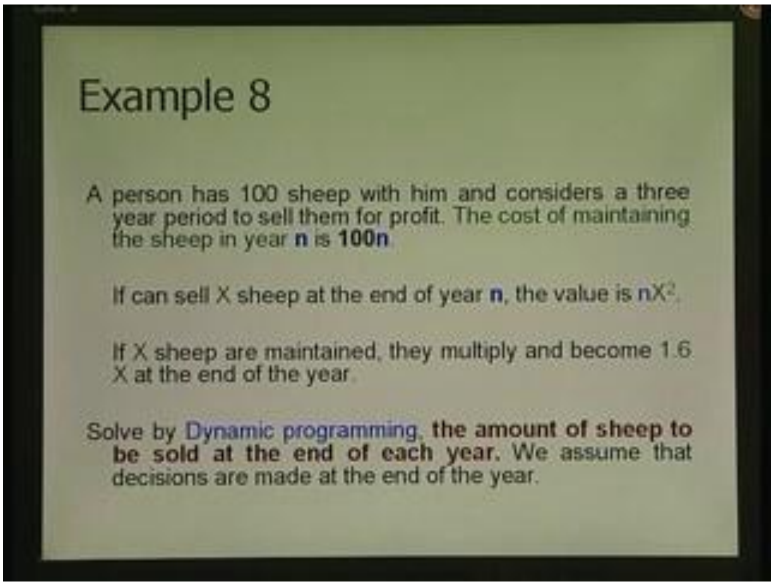

This example is as follows. A person has 100 sheeps with him and considers a 3 year period to sell them for profit. The cost of maintaining the sheep in year n is 100n. If the person can sell X sheep at the end of year n, the value or the profit is n into X square. If X sheep/s are maintained, they multiply and become 1.6 times X at the end of the year. Solve by dynamic programming the amount of sheep to be sold at the end of each year. We assume that decisions are made at the end of each year in this example.

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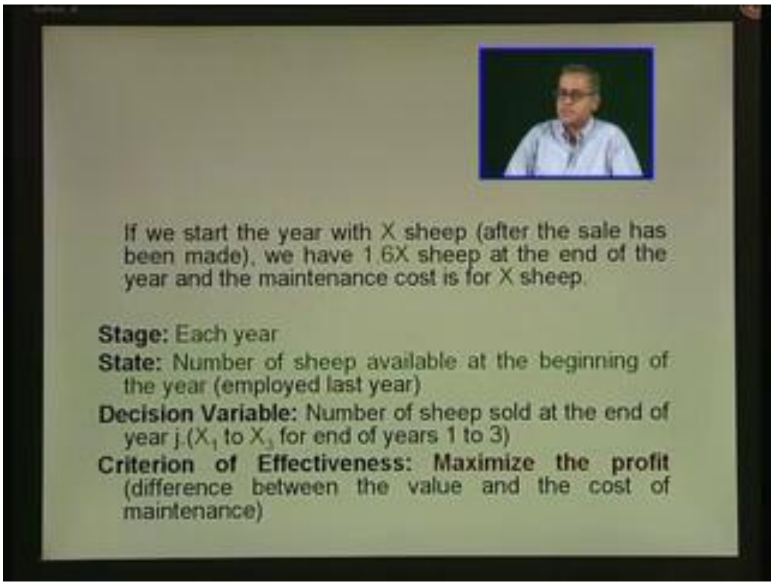

If we start a particular year with X sheep that is after the sale has been made we will have 1.6 X sheep at the end of the year but the cost of maintaining them is only for X sheep. In this problem each year is a stage because we make decisions at the end of each year. State is the number of sheeps available at the beginning of the year. Decision variable is the number of sheeps sold at the end of year j X_1 to X_3 for end of year S_1 to 3. The criterion of effectiveness is to maximize the profit which is the difference between the money value obtained by the sale and the cost of maintaining the sheeps.

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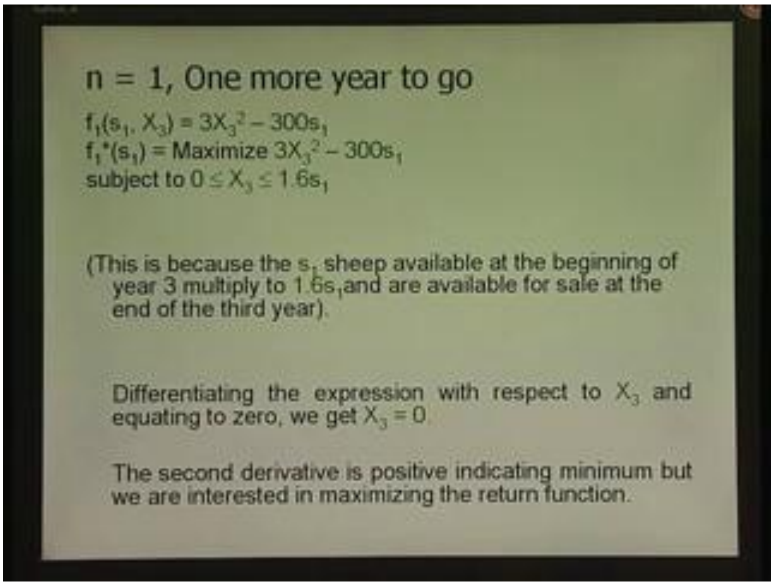

Now n = 1; 1 more year to go. f_1 of S_1 equals $3X_3$ square – $300S_1$ and the S_1 is the state variable which indicates the amount of sheep available at the beginning of the third year and X_3 is the decision variable which indicates the amount of sheep to be sold at the end of the third year and f_1 star of S_1 is to maximize $3X_3$ square – $300S_1$. $3X_3$ square comes by selling X_3 square. We get 3 square rupees. The $300S_1$ comes in to maintain S_1 sheep in year 3, we incur $300S_1$, subject to the condition 0 less than or equal to X_3 less than or equal to 1.6 times S_1. 1.6 S_1 comes because S_1 sheep available at the beginning of year 3 multiplied to 1.6 S_1 and is available for sale at the end of the third year. Differentiating the expression with reference to X_3 and equating it to 0 we get X_3 = 0. $3X_3$ square – 300 S_1, on differentiation with respect to X_3 would give us $6X_3 = 0$ from which X_3 is 0. Second derivative is positive which is 6 indicating a minimum but we are interested in maximizing the net return function.

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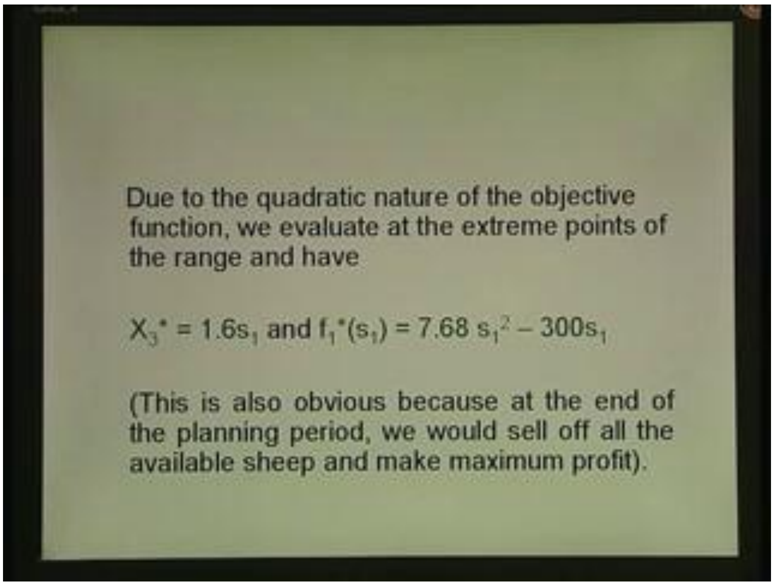

Now due to the quadratic nature of the objective function we evaluate the objective function at the extreme points which is 0 and the 1.6 S_1 and X_3 star is = 1.6 S_1. f_1 star of S_1 is 7.68 S_1 square – $300S_1$. As X_3 star = 0 we would get – 300 S_1. So it is optimal that S_3 star is = $1.6S_1$. This is an obvious result because at the end of the planning period we would sell off all the available sheeps and try to make as much profit as we can.

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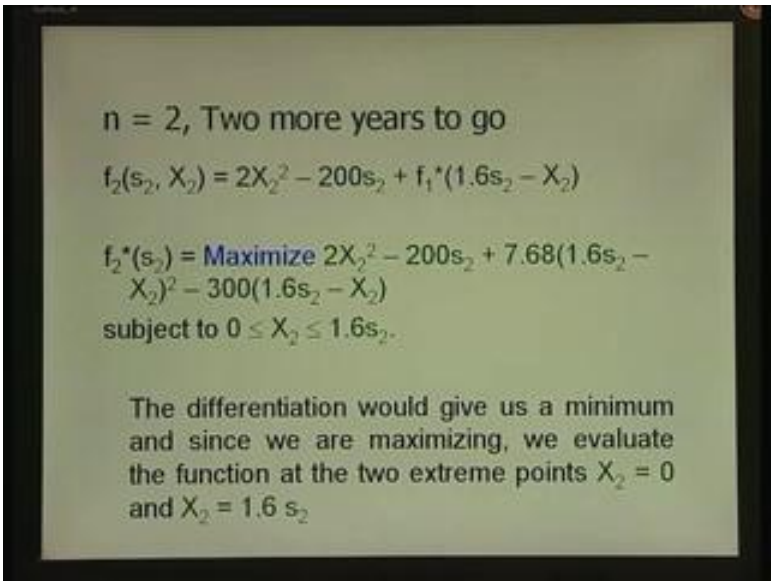

Now n = 2 more years to go or 2 more stages to go. f_2 of S_2, X_2 is equals $2X_2$ square – $200S_2$ + f_1 star of 1.6 S_2 – X_2. S_2 is the state variable which tells us the amount of sheep available at the beginning of year 2. X_2 is a decision variable which is the amount of sheep sold at the end of year 2. $2X_2$ square is the money realized by the sale of X_2 amount of sheep at the end of year 2. 200 S_2 is the cost of maintaining S_2 sheep during the second year. Now these S_2 sheep, on maintaining becomes 1.6 times S_2 out of which an X_2 is sold and the balance 1.6 S_2 – X_2 is carried to the next stage as state variable as S_1. Now f_2 star of S_2 which is the optimum value is to maximize $2X_2$ square – $200S_2$ + 7.6 8 times 1.6 S_2 – X_2 the whole square – 300 times $1.6S_2$ – X_2. The last 2 terms come from the earlier value of f_1 star of S_1 equal 7.68 S_1 square – 300 S_1 subject to the condition 0 less than or equal toX_2 less than or equal to 1.6 S_2. $1.6S_2$ is the maximum amount of sheep that is available that can be sold. Once again differentiation would give us a minimum second derivative would be positive since we are maximizing we evaluate the function at 2 extreme points X_2 = 0 and X_2 = 1.6 S_2

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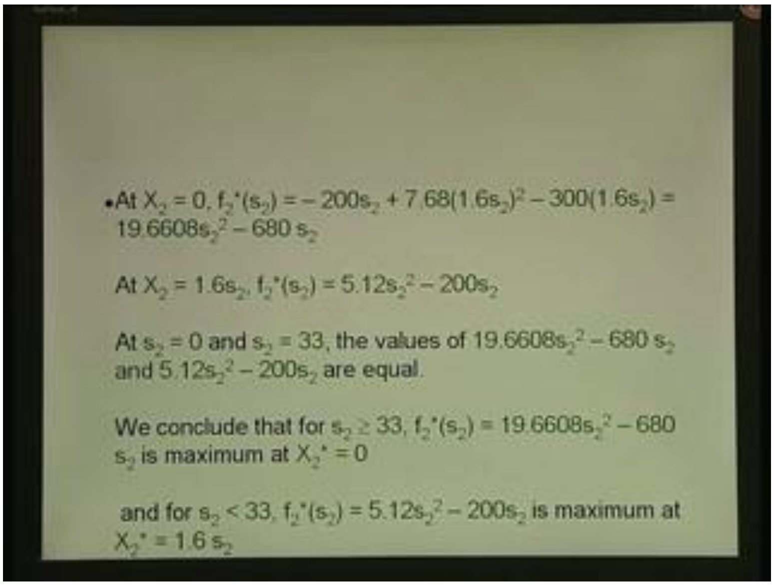

At $X_2 = 0$, f_2 star of S_2 becomes $-200S_2 + 7.68$ into $1.6S_2$ the whole square $- 300$ into $1.6S_2$ which is 19.6608 S_2 square $- 680S_2$. At $S_2 = 1.6S_2$, f_2 star of S_2 becomes 5.12 S_2 square $- 200$ S_2. Now we have to find out the value of S_2 at which both these become equal. Now that happens at S_2 equals to 0 and at $S_2 = 33$. In these 2 values of S_2 the values of 19.6608 S square $- 680S_2$ and the value of 5.12 S_2 square $- 200S_2$ are equal. Therefore we say that for S_2 greater than or equal to 33, f_2 star of S_2 is = 19.6608 S_2 square $- 680S_2$ is maximum at X_2 star is = 0 and for S_2 less than 33, f_2 star of S_2 = 5.12 S_2 the square $- 200$ S_2 is maximum at X_2 star = 1.6 S_2.

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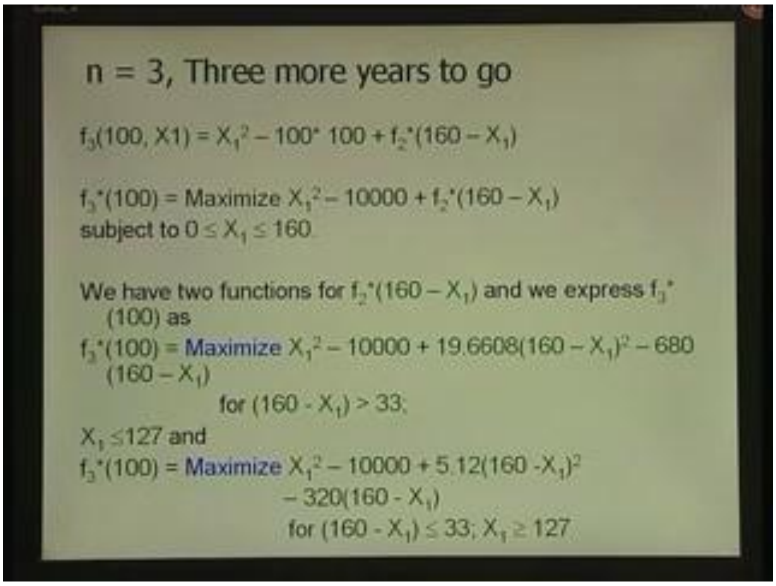

When n = 3 and we have 3 more stages to go, f_3 of 100, $S_1 = X_1$ square $- 10,000 + f_2$ star of $160 - X_1$. 100 is the amount of sheep available at the beginning of the planning horizon.
X_1 is the amount of the sheep that is sold at the end of year 1, so the amount realized would be X_1 square. 10,000is the cost of maintaining 100 sheep and these 100 sheep at the end of

year 1 will become 160. So f_3 star of 100 is to maximize X_1 square – 10,000 + f_2 star of 160 – X_1 subject to the condition 0 less than or equal toX_1 less than or equal to 160. 160 again comes because the 100 sheep available at the beginning of the year 1, will multiply and become 160 at the end of year 1. Now we already have 2 functions for f_2 star of 160 – X_1 and therefore we represent f_3 star of 100 as f_3 star of 100 equals maximize X_1 square – 10,000 + 19.6608 into 160 – X_1 the whole square – 680 into 160 – X_1 for 160 – X_1 greater than 33. This comes from the earlier slide value 19.6608 S_2 the square – 680S_2 is minimum at X_2 star = 0. So for 160 – X_1 greater than 33 we have the first function. This implies X_1 less than or equal to 127 and f_3 star of 100 will be maximize X_1 square – 10,000 + 5.12 into 160 – X_1 the whole square – 320 into 160 – X_1 for 160 – X_1 less than or equal to 33 or X_1 greater than or equal to127. Second term again comes from here where we have S_2 less than 33 f_2 star of S_2 is 5.12S_2 the square – 200S_2 is maximum at 1.6 $S_{2.}$ Now we have these 2 functions for f_3 star of 100 and we have to find out the value of X_1 which maximizes each of them

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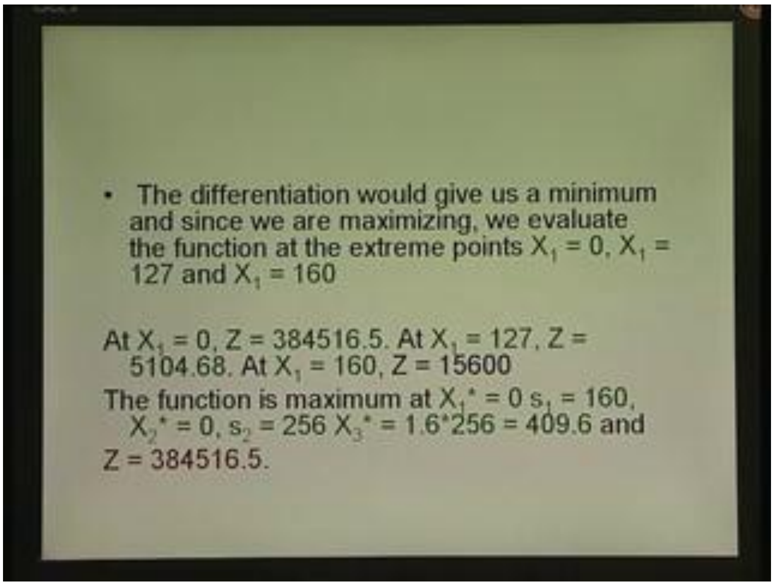

Now once again the differentiation would give us a minimum the second derivative. We have an X_1 square term which appears with the positive coefficient in both the expressions. So second derivative would give us positive indicating a minimum and since we are maximizing this, we evaluate the objective function at the 2 corner points. So in the first case we evaluate the objective function at the 2 corners, $X_1 = 0$ and $X_1 = 127$ and in the other case we evaluate between 127and 160 so at $X_1 = 0$ we find Z = 384516.5. At $X_1 = 127$ that becomes 514.68 and at $X_1 = 160$ the objective function takes the value 15,600. Now the maximum among them is 384516.5 at X_1 star = 0. When X_1 star = 0, S_1 becomes 160. Now this 160 is carried to the second year. We once again realize that X_2 star is 0, so 160 becomes 256 at the end of second year and at the end of the third year we have 1.6 into 256 which is 409.6 that is sold and we get Z = 384516.5. So the optimal decision would be X_1 star = 0. X_2 star = 0. X_3 star = 409.6. Z = 384516.5. The most important learning from this example is that the type of objective function is the quadratic objective function in this case with positive coefficients on X square would indicate a minimum while the objective function that we are looking at is maximum. In such cases we have to evaluate the objective function at the relevant points and then find out the optimum values of the decision variables.

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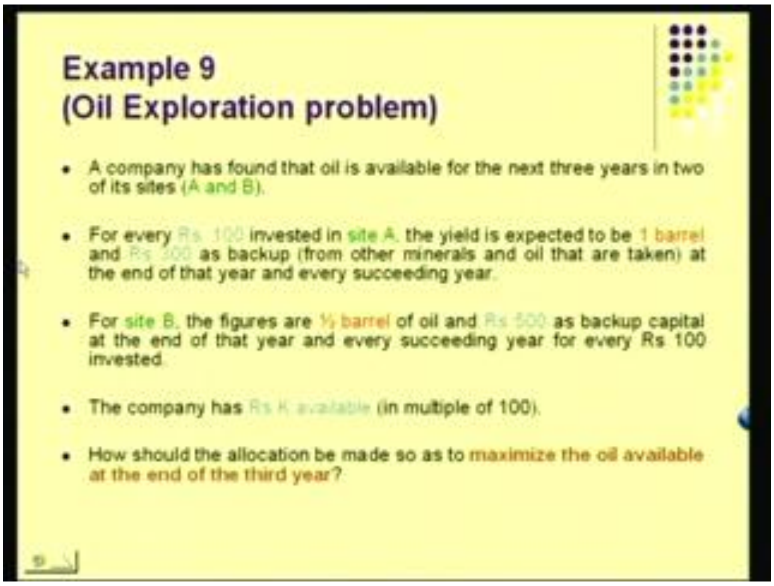

Now let us go to another example, the ninth example in our dynamic programming study which is called the oil exploration problem. This problem is as follows. Here a company is found where the oil is available for the next 3 years in 2 of the sides A and B. Now for every rupees 100 invested in site A, the yield of oil is expected to be 1 barrel in site A and rupees 300 as backup. This is obtained by selling other minerals and materials and other types of oil that can come along with the crude oil at the end of the year and every succeeding year. For example if rupees 100 is invested in year 1, it would give 1 barrel and 300 at the end of the first year, second year as well as the third year. This problem has a 3 year planning period. For site B, the figure is 1/2 a barrel of oil and rupees 500 as the backup capital. This 500 again is similar to 300 which come out by selling other materials which come out along with the oil. Now this happens not only for that year but for every succeeding year, for rupees 100 invested. Now the company has rupees K available at the beginning of the first year. You can also assume that K is a multiple of 100. How should the allocation be made so as to maximize the oil available at the end of the third year?

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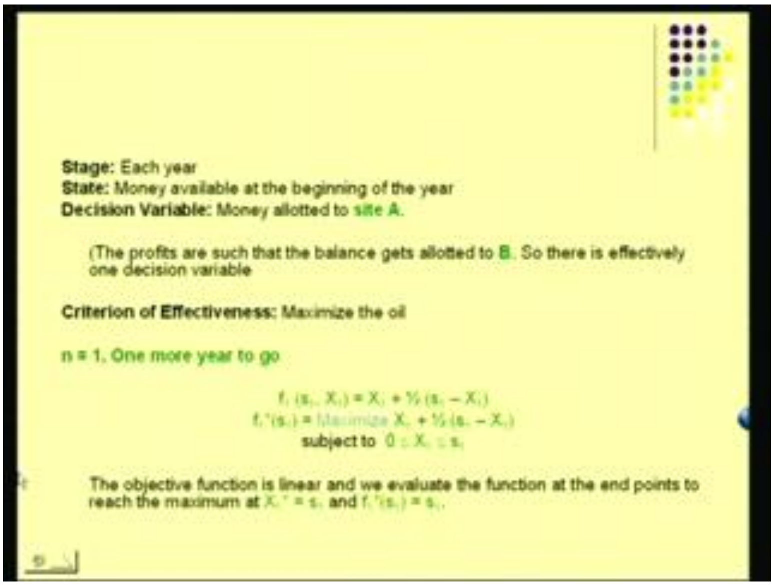

Now in this problem, stage is each year because we are going to make decisions at the beginning of every year as the amount of money that is allocated. Since decisions are made year wise, stage is each year. This problem state variable is the money available at the beginning of the year. We have already seen that the state variable always corresponds to the resource that is available and in this problem the resource available is money that is invested in the oil wells. State variable is the money available at the beginning of the year. Decision variable is the amount allotted to site A. We would normally have thought that there will be 2 decision variables. One would be the amount allotted to site A and the other would be the amount allotted to site B. Now this problem is such that for every 100 invested, you get 300 at the end of that year and every year. For every 100 invested you get 500 at the end of that year or every year. So in this situation we will not keep any money idle. So the only decision is the money is allotted to site A automatically.

The rest of the money would go to site B. So it is enough to define 1 decision variable from which the other decision variable gets defined. Profits are such that the balance gets allotted to B so there is effectivelY_1 decision variable, 1 independent decision variable in this problem at every stage. Criterion of effectiveness is to maximize the oil. Now n = 1; 1 more year to go. f_1 of S_1 X_1, S_1 rupees is available. Now S_1 is assumed to be multiple of 100 or S_1 multiple of 100 is available. X_1 is given or X_1 multiples of 100 is allotted to site A. For every X_1 I get 1 barrel. So the amount of oil I get at the end of the year is again at the third year is X_1 because of the investment in A and $S_1 – X_1$ will go to B. 1/2 a barrel, 1/2 into $S_1 – X_1$. So f_1 star of S_1 is the best value of X_1 that could maximize the total oil which would maximize X_1 + 1/2 of $S_1 – X_1$ subject to the condition that X_1 should be less than or equal to S_1. The amount allocated to A should be less than or equal to the amount that is available for allocation. Now in this case the objective is the linear function, so we evaluate the function at the end points which is 0 and S_1 and we observe that maximum is at X_1 star = S_1 and f_1 star of $S_1 = S_{1.}$ So here the decision is, whatever is available, give it to A, so that you maximize the amount of oil which is = $S_{1.}$

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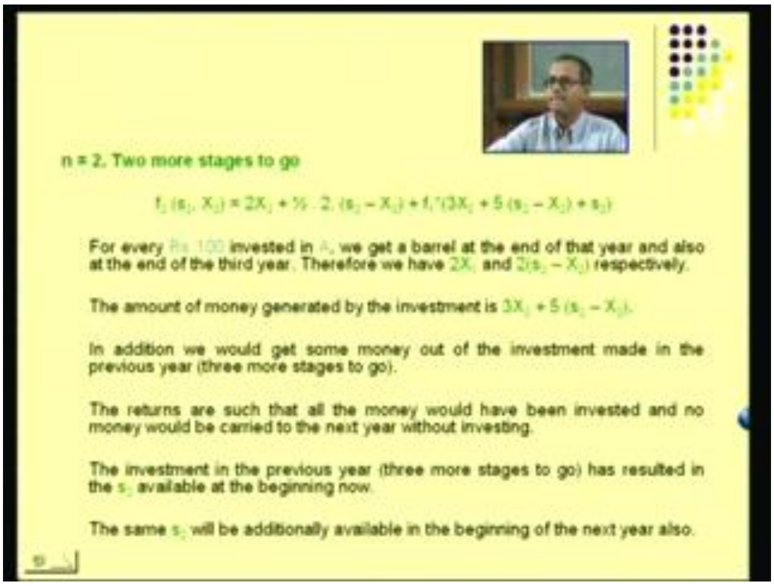

Now N = 2; 2 more stages to go. S_2 is available at the beginning of the second year. X_2 is allotted to A at the beginning of the second year. Again we assume that S_2 and X_2 are in multiples of 100. Now for X_2 allotted to A in the beginning of the second year, this would give 1 barrel of oil for the second year and 1 barrel of oil for the third year.

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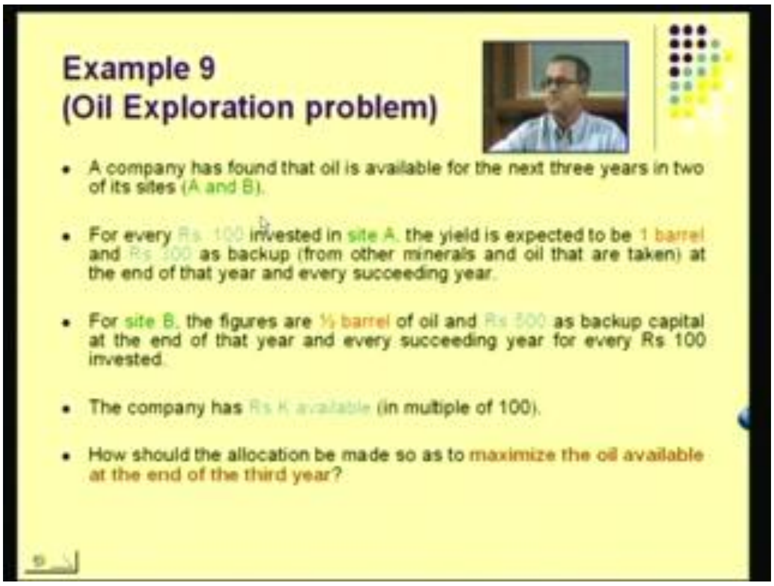

So this X_2 would give $2X_2$ whereas the problem says that year as well as every succeeding year. So this X_2 would give us 1 barrel or X_2 barrel in the second year and X_2 barrel in the third year. So we get $2X_2$ similarly we get 1/2 into 2 $S_2 – X_2$. Now with X_2 given to site A, $S_2 – X_2$ will go to site B and that will give us 1/2 a barrel, 2 years. So 1/2 into 2 into $S_2 – X_2$ is the oil that we get. Now what is the amount that we get? Now this X_2 would give 3 times S X_2 because for every 100 we get 300 as a backup capital at the end of the year so we get $3X_2$. Now the $S_2 – X_2$ that is allocated to site B would give us 5 times $S_2 – X_2$ because it says rupees 500 as the backup capital at the end of that year and the every succeeding year. Now

plus another S_2 comes in. Now let us explain how we get this S_2, so let us go back for every 100 invested in A. We get 1 barrel at the end of that year and also the end of third year therefore we have $2X_2$ and 2 into $S_2 – X_2$ respectively which are shown here. $2X_2$ and 2 into $S_2 – X_2$. Amount of money generated by the investment is $3X_2$ + 5 into $S_2 – X_2$. This is because of the 300 and 500 as back up capital. In addition we would get some money out of the investment made in the previous year because investments made in the previous i.e., investment made at the beginning of the first year would have given us some money and the same money we get at the end of the second year also. Now the amount that was available at the beginning of the first year would have been inspected and the return from that is the S_2 that we have with us right now. So similarly the same amount of S_2 would be generated as a result of the earlier investment at the end of the year 2. Also so the cash on hand at the end of year 2 gets another S_2 added to it. Returns are such that all the money would have been invested and no money would be carried to the next year without investment.
Investment in the previous year that is in the beginning of the first year has resulted in the S_2 available now because no money was left uninvested. The same S_2 will additionally be available in the beginning of the next year. Also we have another S_2 that comes here. So this is an important thing in this problem.

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Now f_2 star of S_2 is to maximize $2X_2$ + 1/2 into 2 into $S_2 – X_2$ + f_1 star of this quantity. Now we have seen f_1 star of S_1 is = S_1. So f_1 star of $3X_2$ + 5 into $S_2 – X_2 + S_2$ is $3X_2$ + 5 into $S_2 – X_2 + S_2$ which is here subject to the condition 0 less than or equal toX_2 less than or equal to S_2. Now this on simplification would give us maximize $7S_2 – X_2$ subject to 0 less than or equal to X_2 less than or equal toS_2. Once again the function we have is a linear function. Now X_2 is the variable with the negative sign. So the function will have a maximum at X_2 star = 0 and f_2 star of $S_2 + 7S_2$. We have 3 more stages to go, f_3 of K, X_3. We are at the beginning of the first year. We have K available which we have already seen. K is in the multiples of 100. Now X_3 is given to site A. X_3 is also in multiples of 100. So f_3 of KX_3 will be $3X_3$ because every X_3 would give X_3 barrels at the end of the first year, X_3 at the end of the second year and also at the end of the third year, $3X_{3.}$ Now $S_3 – X_3$ or $K – X_3$ is what is given to site B. So we get 1/2 a barrel. 1/2 barrel into 3 into $S_3 – X_3$ or $K – X_{3,}$ this is the oil that is obtained because of the investment. Now the money that would come in, available at the end of the

first year or at the beginning of the second year is 3 times X_3 because, 300 we get as backup capital and 5 times K – X_3 because of the 500 backup capitals. So f_3 star of K is the best value of X_3 that maximizes $3X_3$ + 1/2 of 3 into $S_3 – X_3 + f_2$. f_2 star of $3X_3$ + 5 into K – X_3
We know that f_2 star of S_2 is $7S_2$. So f_2 star of $3X_3$ + 5 into K – X_3 is 7 time S_3 X_3 + 5 – K into X_3. This when simplified would give us 73/2 K – 19/2 X_3. Once again we are maximizing. X_3 has a negative term. So the best value will be X_3 star is = 0 and Z = 73/2 K. In this problem the decision is allot 0 X_3 star, allot 0 to A and allot everything to B in the first stage. Similarly allot 0 to A and allot everything to B in the second stage and in the third stage in the last year allot everything to A. So the decision would be all the K that goes to side B gets X, multiplied. Once again in the second year all the K that goes to side B gets multiplied. And in the third year whatever money that is available is entirely into A. This again is an expected result because the amount of money that you get in B being higher. First 2 years we invest everything in B. We multiply the money, get maximum money and in the third year invest everything in A so that you get more oil. So this is how we solve this problem. What is new and special that we have learnt from this example? The first thing is that this is a new linear function. Therefore we do not differentiate. We simply evaluate the function at the range at the end points and then optimize which is a change from the previous example. Secondly the problem is such that the returns not only come that year but come at the end of the every succeeding year, so that has to be modeled carefully and has resulted in this + S_2 coming in. As part of the state variable when we have N = 2, 2 more stages to go.

Now dynamic programming also shows us a way to model situations such as this where the return is not only for the end of that planning period but also for succeeding planning period. We have seen in this example that it is also possible to model things like this where the return is not only for the end of that year but also at the end of every succeeding year. We go on to explain another example using dynamic programming.

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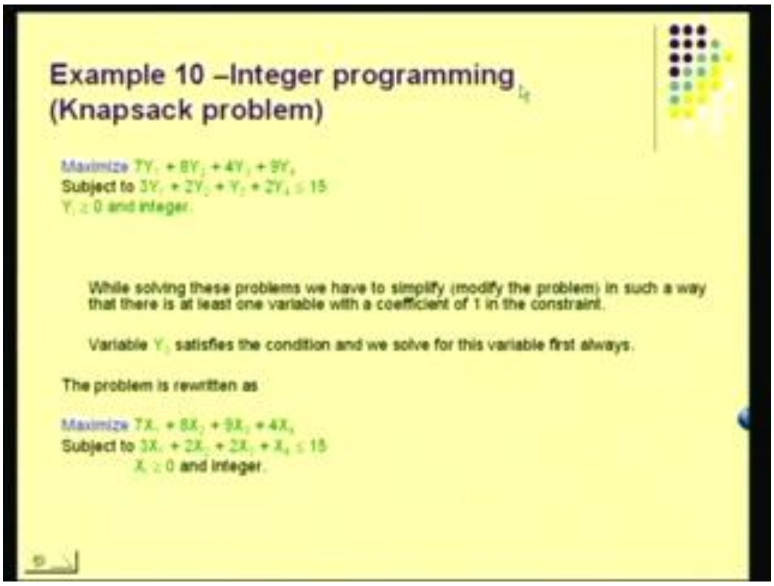

We take an integer programming problem or a Knapsack problem and try to solve. The Knapsack problem that we consider is maximize $7Y_1 + 8Y_2 + 4Y_3 + 9Y_4$ subject to $3Y_1 + 2Y_2 + Y_3 + 2Y_4$ less than or equal to 15. Now YJ is greater than or equal to 0 and integer. The integer is the key thing. So far in the last 4 examples we have seen problems that involve continuous variables. Now we go back to the integer and we describe examples. You remember that in the first 3 examples that we saw, all had discrete variables. Now we look at an integer programming problem. Single objective functional maximization is subject to a single constraint and an integer restriction on the variables. Now the problem is called as a Knapsack problem because the problem is about filling things in a knapsack. We are looking at 4 different types of items that are there and for example we want to pack or fill as much as we can into a sack. The weight that the sack can take is 15 and the weight of the individual item could be 3, 2, 1 and 2 respectively and if we decide to put YJ and integer value for example if we put 2 of the first item and then loose up 6 kgs of weight and so on. So we now want to find out how many quantity of each item we can put into the sack so that the weight restriction is not violated. Each item has a certain utility. So we assume that if Y_1 quantity of item 1 goes into the sack $7Y_1$ will be the total utility which we would like to maximize. Constraints can also be taken as a volume on restrictions instead of a weight restriction. Usually in all these problems, the objective function is like maximizing the utility and the constraint would represent either a weight restriction or a volume restriction. Now let us solve this problem. While solving these problems we have to modify the problem in such a way that there is at least 1 variable which is coefficient of + 1 in the constraint. Now this example has that variable Y_3 has a constraint coefficient of + 1. Now this would help us to solve the problem better, so we now bring this one as the last variable.

The Y_3 will now become the last variable. So the problem is rewritten as $7X_1 + 8X_2 + 9X_3 + 4X_4$. This Y_4 becomes X_3 and Y_3 becomes X_4. The variables have been changed, subject to $3X_1 + 2X_2 + 2X_3 + X_4$, the Y_3 becomes X_4. $2Y_4$ becomes $2X_3$ less than or equal to 15. XJ greater than or equal to 0 and integer. So we will now solve this problem because this problem is rewritten in such a way that the variable which has a + 1 coefficient in the constraint now appears as the last variable or the first variable that we will be solving. Now the stage is each variable because we solve 1 variable at a time. State is the amount of resource available. We already know that.

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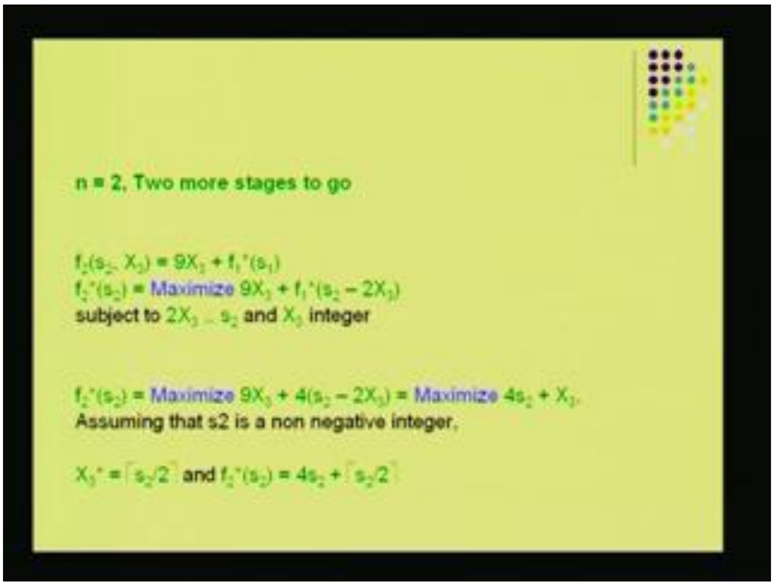

We do not know what exactly this would represent. This could represent a weight. This represents a volume. So we just say it is a resource and we say amount of resource available is the state variable. Decision variables are the actual values of X_1 to X_4 and the criterion of effectiveness is the objective function which maximizes Z. Z is $7X_1 + 8X_2 + 9X_3 + 9X_4$. Now n = 1; 1 more stage to go which means we are trying to solve this problem. Maximize $4X_4$ subject to X_4 less than or equal to S_1. X_4 greater than or equal to 0 and the integers. So f_1 or S_1 X_4, I have S_1 resources available. I want to give X_4 to it. So X_4. Now f_1 star of X_1 is the best value of X_4 that maximizes $4X_4$ subject to X_4 less than or equal to S_1 and X_4 is an integer. Assuming that S_1 is the non negative integer which is a fair assumption, the right hand side value is non negative. All the coefficients are positive or non negative and these X_1, X_2, X_3, and X_4 are also non negative integers. All the state variables will also be non negative integers. So assuming that S_1 is non negative integer, the best value X_4 star is = S_1 and f_1 star of S_1 is 4 times S_1. This is a very clear result. All the resource that is available goes to variable X_4. Now N = 2; 2 more stages to go, f_2 of S_2 X_3. Now if we go back to the problem, we are trying to solve $9X_3 + 4X_4$ subject to $2X_3 + X_4$ less than or equal to S_2. X_3, X_4 greater than or equal to 0 and integer.

So $9X_3$ + f_1 star of X_1. $9X_3$ comes from here; $9X_3$ + f_1 star of S_1, S_1 is the resource that is available after something is allocated to X_3. So f_1 star of $S_2 – 2X_3$, now $S_2 – 2X_3$ comes as follows. We are looking at $2X_3 + X_4$ less than or equal to S_2. So if X_3 quantity goes to variable X_3 then $2X_3$ of the resource is consumed. S_2 is assumed to be available so $S_2 – 2X_3$ is the amount of resource available for the next item. So you get $S_2 – 2X_3$ which is here $S_2 – 2X_3$ subject to $2X_3$ less than or equal toX_2. In this case we need this. $2X_3$ should be less than or equal to S_2 which is shown here and X_3 is an integer. So f_2 star of S_2 is to maximize $9X_3$ + 4 times $S_2 – 2X_3$. Now this comes because f_1 star of S_1 is $4S_1$ so f_1 stars of $S_2 – 2X_3$ is 4 into $S_2 –2X_3$. So we end up maximizing $4S_2 + X_3$. Now once again assuming S_2 is a non negative integer, X_3 star will take $S_2/2$ lower integer value of $S_2/2$. For example if S_2 is 3 units then X_3 star can be only be 1 unit. It cannot be 1.5 because X_3 is an integer. So we would get a lower integer value of $S_2/2$ and f_2 star of S_2 will be $4S_2$ + lower integer value of $S_2/2$. $4S_2 + X_3$ give $4S_2$ pus lower integer value of $S_2/2$.

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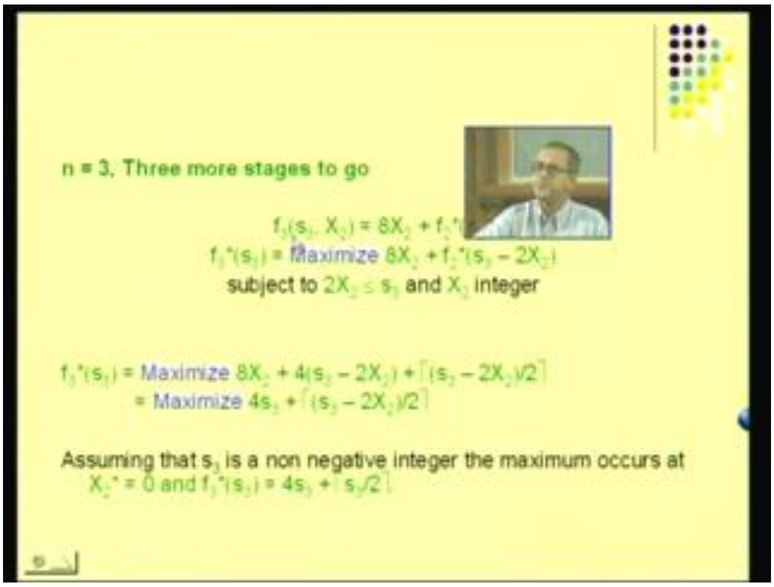

Now when we have 3 more stages to go, f_3 of S_3 X_2 = $8X_2$ + f_2 star of S_2. Now this comes because we are looking at this variable X_2. So we get $8X_2$, $2X_2 + 2X_3 + X_4$ less than or equal to S_3. So we have S_3 resource available. X_2 is given to variable X_2. So $2S_2$ is resource consumption so $S_3 – 2X_2$ is what is available as S_2. So we have 3 more stages to go. We have $8X_2$ + f_2 star of X_2. Now this S_2 that is available at the beginning of the next stage is resource S_3 available – resource consumed which is $2X_2$. So we have f_3 star of X_3 is the best value of X_2 that maximizes $8X_2$ + f_2 star of $S_3 – 2X_2$ subject to the condition $2X_2$ less than or equal to S_3 and X_2 is an integer. Now this comes because we have this $2X_2$. We have S_3 available. So $2X_2$ should be less than or equal to the resource available. X_3 and X_2 should be an integer because all XJ's are integers. Now we already know that f_2 star of S_2 is $4S_2$ + lower integer value of $S_2/2$, so f_2 star of $S_3 – 2X_2$ is 4 times $S_3 – 2X_2$ + lower integer value of $S_3 – 2\ X_2/2$. This on simplification would give us $4X_3$ + lower integer value of $S_3 – 2X_2$ divided by 2. Now once again assuming that S_3 is a non negative integer, the maximum occurs at X_2 star = 0. This is because the $8X_2$ and $8X_2$ are canceled out in this example. The only place where X_2 appears has a negative sign. It has a linear function. So the maximum occurs when X_2 star = 0 and f_3 star of S_3 is $4S_3$ + lower integer value of $S_3/2$.

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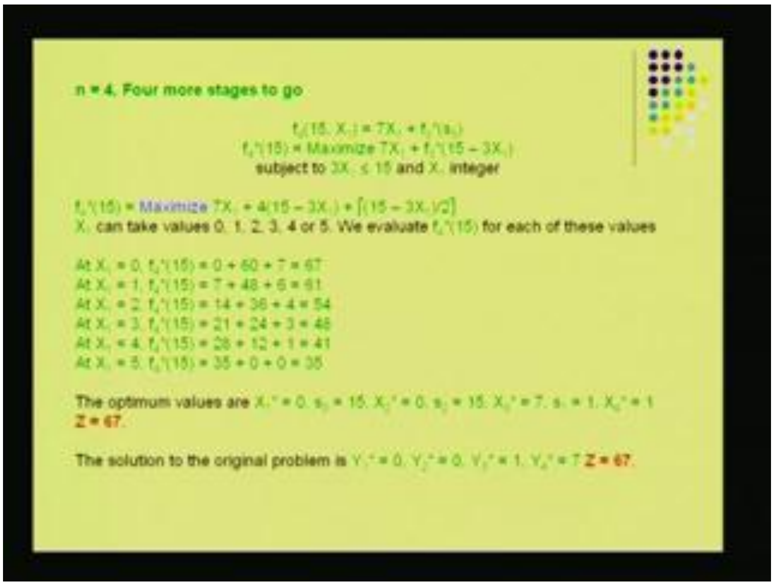

Now n = 4; 4 more stages to go, we started with 15 here. Item 1 requires 3 units of the resource. So we have 15, X_1 is $7X_1$, $7X_1$ comes from here which is the utility associated with variable X_1. So $7X_1$ + f_3 star of S_2, now 15 is available. $3X_1$ is the resource consumption so $15 – 3\ X_1$ is the resource left over which becomes S_3 so f_4 star of 15 is the best value of X_1 that maximizes $7X_1$ + f_3 star of $15 – 3X_1$ subject to $3X_1$ less than equal to 15 and X_1 integer.
Now going back here, $3X_1$ is less than or equal to 15 and X_1 is an integer. Now f_4 star of 15 is to maximize $7X_1$ + 4 times $15 – 3X_1$ + lower value integer of $15 – 3X_1 – 2$. This comes because f_3 star of S_3 is $4S_3$ + lower integer value of $S_3/2$. So f_4 star of 15 is to maximize $7X_1$+ 4 times, $15 – 3X_1$ + lower integer values of $15 – 3\ X_1/2$. Now X_1, we can take only integer values such that $3X_1$ is less than or equal to 15, so X_1 can take values 0, 1, 2, 3, 4 or 5.

This being in the last stage, we evaluate the function at values 0, 1, 2, 3, 4 and 5 to get $X_1 = 0$. We would get 0 + 60 + 7. 0 comes from the first term, 60 come from the second term, 7 comes from the third. When $X_1 = 0$, 4 into 15 is = 60, now 15/2 lower integer value is 7, so we get 67. $X_1 = 1$ we get 7 + 48 + 6. 7 from $7X_1$; 48 from 15 – 3 into 1 = 12, 12 into 4 = 48. Now when $X_1 = 1$; 15 – 3 is 12, 12 divided by 6, we get 6. So this way we calculate the objective function and all integer values of X_1 and realize that at $X_1 = 5$, f_4 star of 15 is 35. Now the best value happens at $X_1 = 0$ and we get a value of 67. So the optimum value happens at X_1 star = 0. 15 is carried over to the next stage. Now X_2 star is 0, so we have X_2 star is 0. 15 is carried over to the other stage. At this stage the best value is lower integer value of $S_2/2$, so lower integer value of 15/2 is 7. So X_3 star is 7. So X_3 consumes 2 resources or 14 resources are consumed. S_1 becomes 1 and X_4 star is = S_1 so X_4 star is = 1. So the solution is X_1 star = 0; X_2 star = 0; X_3 star = 7; X_4 star = 1, Z = 67 but this is a solution for the modified problem. So the solution to the first problem should be rewritten once again and we know that X_4 becomes Y_3, X_3 becomes Y_4, so we get Z = 67 solution to the original problem. Now what are the new things that we have seen in this example 1? We have solved a problem where the variables take integer values and secondly we have a single constraint problem as always but we were able to solve the 4 variables in this case. Most of the integer programming problem is of this type. When we solve using dynamic programming we will be able to solve only for 3 stages. In this case we were able to solve for a 4th stage simply because here we had a situation particularly here, we had a situation where the $8X_2$ and 4 into $-2X_2$ cancelled out and therefore we were able to get a X_2 star = 0.

Normally we will be able to solve for 3. As a special case in this example we were able to solve for 4. It is also important in these examples that there is at least 1 variable which has a constraint coefficient of + 1, so that that variable is always pushed as the last variable or first variable in a backward recursive approach and we solve for it so that we always start with X_4 star = S_1 and f_1 star of S_1 is = some constant into S_1. Now this makes the solution easy. Now we could have this as a problem where all the YJ’s or if we take the modified problem where all the XJ’s are integer values. Now we could have easily solved the problem by the tabular approach. The tabular approach that we had seen earlier in the dynamic programming could have been used and we could have solved this but we did not do that simply because the resource being large particularly in the middle stages, the tables become extremely large.

Now as far as this problem is concerned, whether we had 15 or whether we had 115 as resource, the solution methodology is the same. Whereas if we had used the tabular method to solve this problem, if this right hand side value of the inequality becomes large then the tables become very large and so we do not use the tabular method to solve even though we know that we can solve this by the tabular method. Now single constraint integer programming problem can be solved comfortably up to 3 variables using BP when one of the variables has a + 1 coefficient in the constraint. Special cases, we can solve up to 4 variables as we have shown in this example. Here being a linear objective function in the linear constraint we do not use different ion and find out maximum or minimum. Maximum or the minimum happens at the extreme points in the first stage.

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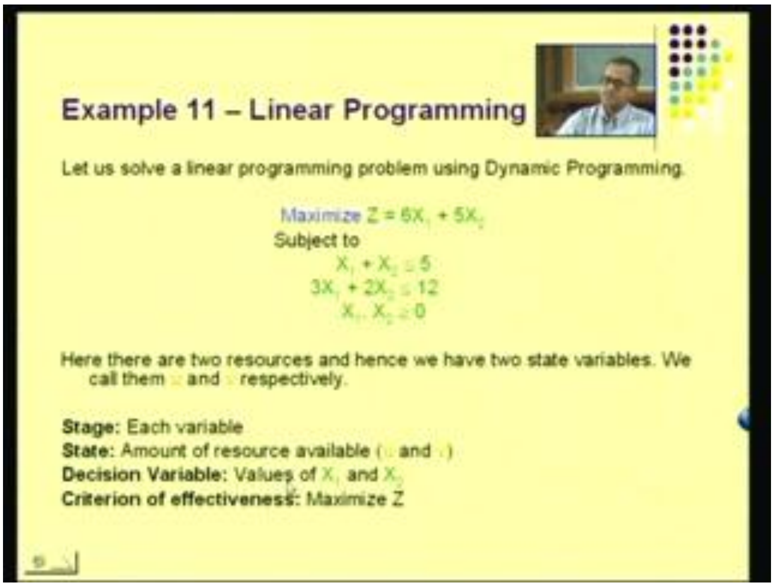

The last example that we will be seeing in the dynamic programming is to show how we can solve a linear programming problem using dynamic program. Now to do that we go back to the familiar example we have seen in this lecture series. Maximize Z = 6 X_1 + $5X_2$;
$X_1 + X_2$ less than or equal to 5;

$3X_1 + 2X_2$ less than or equal to 12;

X_1 X_2 greater than or equal to 0.

Now this is a linear programming problem. This does not have integer restriction on the variables. Variables are continuous. Now there are very special things about this which is very different from the example that we have seen. We are looking at 2 constraints here and not 1 constraint. There are 2 constraints, 2 resources, 2 values at the right hand side. We have 2 state variables. So far in all the examples we have had only 1 state variable. In this case we have 2 state variables. Instead of using the notation‘s’ for the state variables we use notations U and V respectively for the state variables. So in this problem we define state stage decision variable and the criterion of effectiveness. Stage is each variable. There are 2 variables here. We will be solving for 1 variable at a time, so stage is each variable. State is the amount of resources available. There are 2 resources U and V namely first and the second. So the 2 resources are state variables. There will be 2 state variables for this problem. Decision variables are the values of X_1 and X_2 and the criterion of effectiveness is to maximize the objective function which is Z which is given by $6X_1 + 5X_2$

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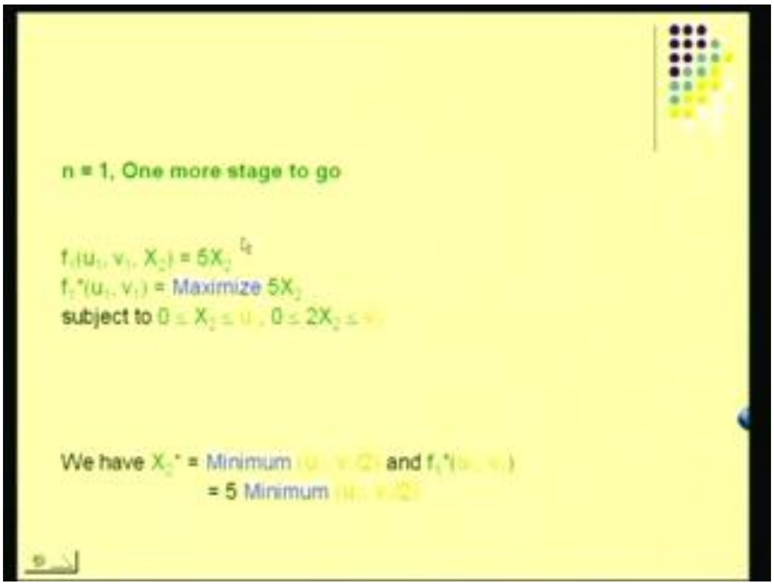

Now N = 1; 1 more stage to go. f_1 of U_1 V_1 X_2 = $5X_2$; we are trying to solve a problem, maximize $5X_2$ subject to X_2 less than or equal toU_1, $2X_2$ less than or equal toV_1. X_2 greater than or equal to 0. So we want to maximize $5X_2$ subject to the condition X_2 less than or equal toU_1. $2X_2$ less than or equal toV_1; X_2 greater than or equal to 0. Now here what will happen is the maximum value assuming U_1 and V_1 are non negative values which is also not a very bad assumption because these values are non negative. The coefficients are all non negative and the variables are non negative. So the state variables will be non negative values. Now the maximum value that X_2 will take is actually the minimum of U_1 and $V_1/2$ because X_2 less than or equal toU_1; $2X_2$ less than or equal toV_1 would give us X_2. The maximum value X_2 can take is the minimum of U_1, $V_1/2$. So X_2, the star X is the minimum of U_1 and $V_1/2$ and f_1 star of U_1V_1 is maximize or 5 times minimum of U_1, $V_1/2$. The best value of X_2 is a minimum of U_1, $V_1/2$. So X_2 the star is being minimum of U_1 $V_1/2$. f_1 star of U_1,V_1 will be 5 times minimum of U_1, $V_1/2$. Now N = 2; 2 more stages to go and we come back to the problem where we are solving $6X_1 + 5X_2$ subject to $X_1 + X_2$ less than or equal to 5; $3X_1 + 2X_2$ less than or equal to 12. X_1 X_2 greater than or equal to 0. So we are solving this problem.

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Now we go back and say $6X_1$ because for X_1 the objective function is $6X_1$ and whatever resource is left over this X_1 is given here, $5 – X_1$ goes as U_1 and $12 – 3X_1$ goes as V_1. So we have f_1 star of $5 – X_1$ and $12 – 3X_1$ which go as U_1 and V_1. f_2 star of 5, 12 is the best value of X_1 that maximizes this $6X_1$ + 5 times minimum of $5 – X_1$; $12 – 3X_1/2$. We have already seen that X_2 star is minimum of U_1 $V_1/2$ and the value is 5 times minimum of U_1, $V_1/2$. So 5 times minimum of U_1 $5 – X_1$; $V_1/2$; $12 – 3X_1/2$ subject to the condition 0 less than or equal toX_1 less than or equal to 5. 0 less than or equal to 3; X_1 less than or equal to 12, which comes from here. X_1 less than or equal to 5, $3X_1$ less than or equal to 12. Now what we need to do is this. Now we look at both these functions $5 – X_1$ and $12 – 3 X_1/2$ or we need to find out the range at which one of them becomes minimum.

Now the point at which they are equal is X_1 = 2. At X_1 = 2, we have $5 – X_1$ which is 3.12 – 6/2 which is also 3, so at X_1 = 2, these 2 are equal. So f_2 star of 5, 12 is to maximize $6X_1$ + 5 into $5 – X_1$ the first function. In the range 0 less than or equal to X_1 less than or equal to 2 and maximize $6X_1$ + 5 into $12 – 3 X_1/2$, the second function is in the range 2 to 4. 4 comes in because this would give us X_1 less than or equal to 5. This would give us S_1 less than = 4, so 4 becomes the upper range. So we have 2 functions here. We want to maximize $6X_1$ + 5 into $5 – X_1$ in the range 0 to 2 and $6X_1$ + 5 into $12 – 3 X_1/2$ in the range 2 to 4. At X_1 = 0 we have Z = 25. We are in this range. So 0 + 25 is 25. At X_1 = 2 we have 12 coming from this and 15 coming from this giving us 27.

From the other expression, also we have 12 coming and 15 coming from the second term which is 27. At X_1 = 4 which is in the other expression, we have 6 into 4 = 24; $12 – 3 X_1$ is 0. So the best value is that X_1 star = 2; Z = 27. When X_1 star = 2, U_1 is $5 – X_1$ which is 3; V_1 is 12 – 3; $X_1/2$ which is V_1 is $12 – 3X_1$ which is 6, so from the previous table minimum of U_1, $V_1/2$ is X_2 star. Minimum of U_1, $V_1/2$ minimum, so minimum of 3 and 6/2 which is 3, so we get $X_1 = 3X_2$ or the X_1 = 2; X_2 = 3 and Z = 27 which would give us 12 + 15 which is 27. So this is how we solve linear programming problems using dynamic program. Just to illustrate this we have taken a 2/2 problem. We have also taken a maximization problem. We have taken a 2 variable, 2 constraint problems. We also have taken a very simple problem where all the coefficients are positive terms and non negative terms in this problem. Problems become a little more complicated when we have negative terms here. Problems become complicated when we have greater than or equal to constraints. In fact you would have seen in all our problems whether they were problems such as linear programming or integer programming or nonlinear objective function or constraints or problems that are descriptive in nature which had non linear or linear terms, the resource constraints were all less than or equal to constraints. We did not encounter a greater than or equal to constraint in our example. For a first course less than or equal to constraints are easier to handle and we have taken a variety of examples but all them are consistent about the fact that the constraints were of the less than or equal to 5. Now do we use DP or dynamic programming to solve large linear programming problems? The answer is no. The reason is we have as many state variables as the number of constraints. Each constraint represents the resource. So we have as many state variables as the number of constraints and therefore the problem now gets too many constraints whenever we solve problems of a larger size. Now this is called curse of dimensionality. Now we could solve comfortably a 2/2 problem but beyond that it becomes little bit more involved to solve linear programming problems. But before we wind up dynamic programming, let us also look at some additional comments.

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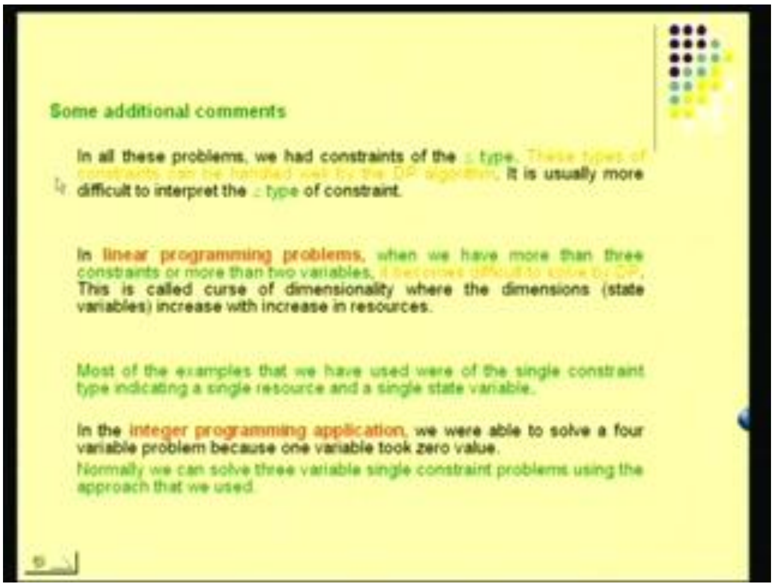

In almost all the examples we had constraints of less than or equal to type. These constraints can be handled very well by the DP algorithm. It is very difficult to interpret the greater than or equal to type of constraints even as a state variable. In linear programming problems when we have more than 3 constraints or more than 2 variables it becomes difficult to solve by DP. This is called curse of dimensionality where the problem dimensions, the state variable increases with increase in the number of resources. Most of the examples we have used were of the single constraint problems indicating a single resource and a single state variable. In the integer programming application we were able to solve a 4 variable problem, 1 variable definitely took a 0 value. Normally we solve 3 variables, single constraint problems using the approach that we used.

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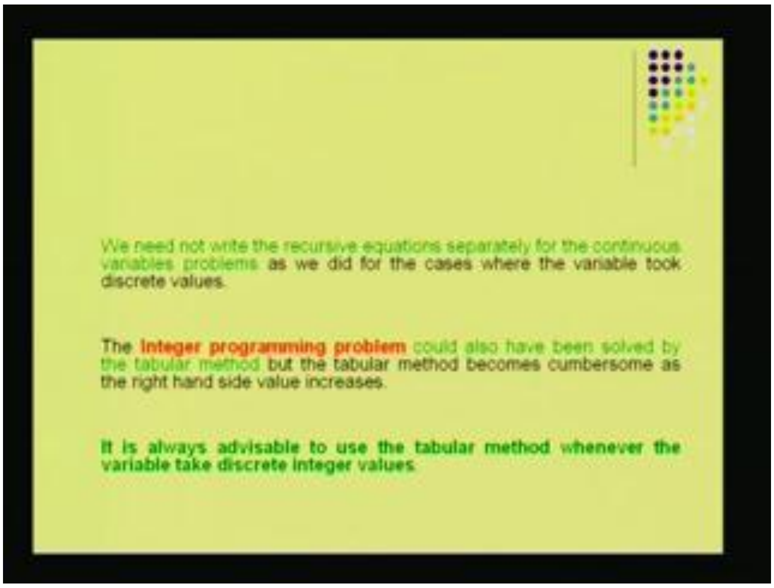

If whenever we solve problems with continuous variables, we need not write the recursive relations or separately as we did for the cases where we took discrete values. The integer programming problems could have been solved by the tabular method but the tabular method becomes cumbersome as the right hand side value increases. It is always advisable to use the tabular method whenever the variable takes integer values.

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Now at the end of the dynamic problem, we move to the last topic of the first course. In the fundamentals of operation research we addressed deterministic inventory models. We now look at the very basic of inventory control in this lecture, in the introductory part of it and in the subsequent lectures. Now the inventory control deals with ordering and stocking policies for items used in manufacture of industrial products. In every manufacturing environment, we realize about nearly 80% or more of the items are bought out from outside and the rest enter as raw material are manufactured and assembled into the final product. Now items bought from outside or bought from vendors have the following costs associated with the purchase. There are 4 normal costs that are important to us. The actual cost of the product or the item is shown here. There is an ordering cost that the organization incurs, the amount of money that is spent in placing an order for the items. All these items are special items that need to be ordered and the vendors make these have a supply. There is a carrying cost or holding cost for the items. Items are not bought on a daily basis or bought frequently. They are bought in certain quantities and are stocked within the organization. So there is a cost associated with carrying or holding these items and sometimes there are shortage costs or backorder cost when the items are not available and the production stops for want of these items.

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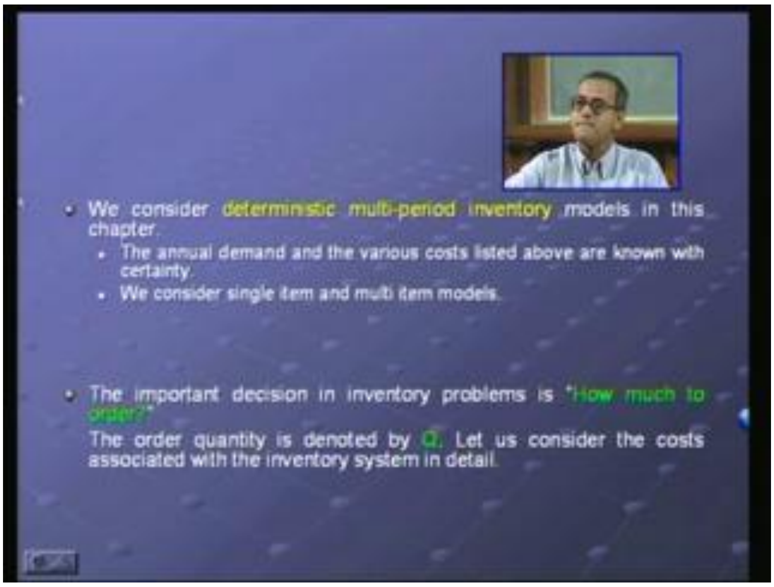

Now in this course in the first course, the introductory course on recent research, we introduce inventory models. We are going to consider deterministic multi-period inventory models in this chapter. We are going to look at inventory problems where inventory decisions are made more than once during the planning period not the static problems with the dynamic problems and we are also going to look at some deterministic problems where all the data are available at the beginning of the planning period. Now the assumptions annual demands for the items are known. The various costs associated with the inventory, the 4 costs that we looked at are cost of the product, ordering cost, carrying or holding cost and shortage cost. They are known with certainty and do not change during the planning period and we also consider single item as well as multiple item inventory models in the introductory portion.

There are 2 decisions in inventory problems. The first and the most important decision is called how much to order. The second decision is called when to order. Now orders have to be placed for these 2 items. So the 2 questions would be how much I order every time I place an order and when I decide to place an order. Now the answer to the question how much to order is given by something called the economic order quantity or the order quantity which is denoted by the letter Q. Now let us go back and look at the various costs that we decide. We introduced 4 types of costs. Cost of the product, ordering cost, carrying cost and shortage cost. Now obviously the ordering quantity or the economic order quantity depends on these 4 costs. So let us get into these 4 costs in detail and see.

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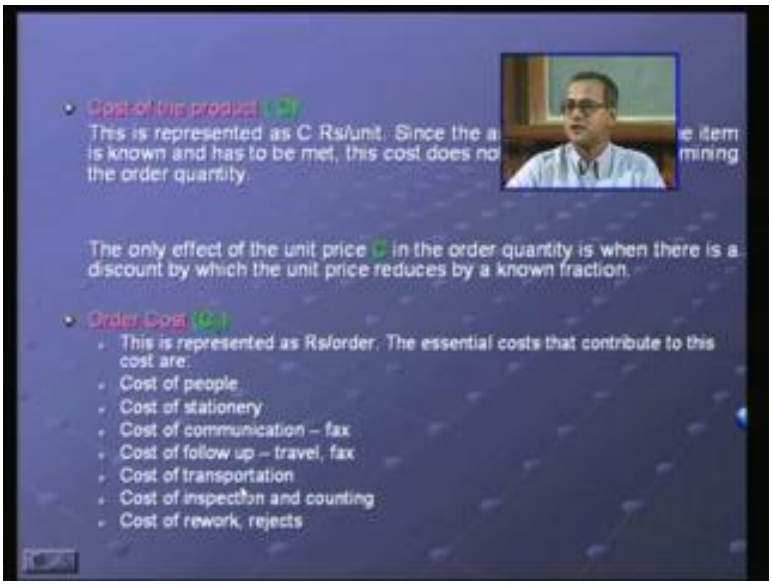

What constitutes these 4 costs? Now the cost of the product of the item is usually represented by the capital C and that will be the notation that we will be using. Now this is given as rupees C per unit or C rupees per item or the annual demand for the item is known and we have to meet the annual demand. This cost does not play a significant part in determining the ordering quantity. No matter what the order quantity is, cost is going to be the same or we later show that the order quantity does not depend on the actual cost of the product. However the only effect of the unit price C in the ordering quantity is when there is discount. Now when there is a discount the unit price reduces by a known fraction. Therefore it influences the ordering quantity. The only situation where the price will have a C in the determination of the order quantity is when we are looking at discount models. We will be looking at discount models subsequently in this lecture series and we will see the effect of the discount and the economic order quantity. In the next class we will look at order cost.

Order cost is the cost that is incurred whenever an order is placed for an item. Now this is represented by the notation C0 or C subscript 0 or O in this lecture series. C0 is the order cost and its unit is rupees per order. Every time there is an order placed, there is an amount of money spent. It is either money per order or rupees per order. Now there are many costs that constitute the order cost. Now these are the following order. Cost of people, there are normally people who work in an organization who are in charge of purchase and who place these orders. So cost of hiring these people and cost of their salary and pay role is included as part of the ordering cost. However small it is, cost of office and cost of stationery also becomes a part of the ordering cost. If there is a cost of communication now, the purchase orders are made and they have to be communicated to the vendors which would involve cost of fax, cost of sending the courier or cost of making long distance calls and so on. There are also costs of follow up.

Once the purchase order is made, the organization follows up with the vendors. So there is a cost of follow up associated with this and this cost of follow up would mean sometimes courier, fax, telephone calls as well as travelling. Sometimes people have to go to the vendor's place and then get the items. So it involves a certain travel. There is a cost of transportation because the items have to be transported from the vendor to the organization.

There is a cost of inspection and counting which we have. Whenever the items come in, they are inspected and counted, so there is a cost of the pay roll or the time that is spent in these activities. Sometime there could be rejects which are sent backs or some rework which has to be made which would contribute a little bit to the total cost. All these are the components of the order cost.

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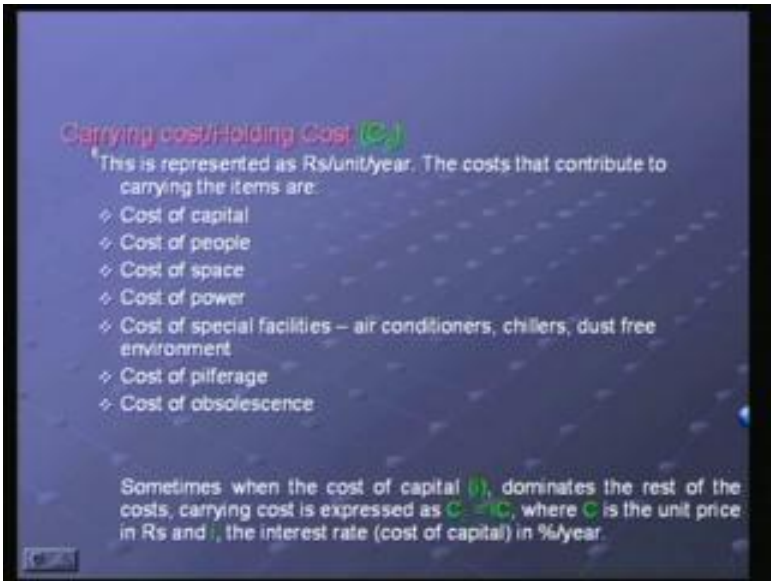

The other one is called the carrying cost or the holding cost which is often represented as C_c. C subscript c. Cost of carrying, this is represented as rupees per unit per year, money per unit per year, Cost that contribute to the carrying of the items are many cost. First and most important in the cost of capital when items are bought, a certain amount of money is spent and a certain amount of interest is paid on the money that is being borrowed. Cost of capital is the most dominant cost or the holding cost. Other cost would also include cost of space, Cost there would be a warehouse, Cost of people who manage the warehouse, cost of power and other electrical utilities. Sometimes we would need cost of special facilities such as air conditioner, chillers, and dust free environment and there could be pilferage obsolescence. Now all these constitute the cost of carrying or holding which is represented by $C_{c.}$

Now we look at the other cost such as the shortage cost as well as the inventory models in detail in the next lecture.