Fundamentals of Operations Research

Prof. G. Srinivasan

Department of Management Studies

Indian Institute of Technology, Madras

Lecture No. # 19

Dynamic Programming- Continuous Variables

In this lecture we continue our discussion on dynamic programming. We continue to solve examples where the decision variables take continuous values. In the previous example we had a linear objective functions with nonlinear constrains and variables were continuous. In this example we will have a nonlinear or a polynomial objective functions subject to a linear inequality as a constraint.

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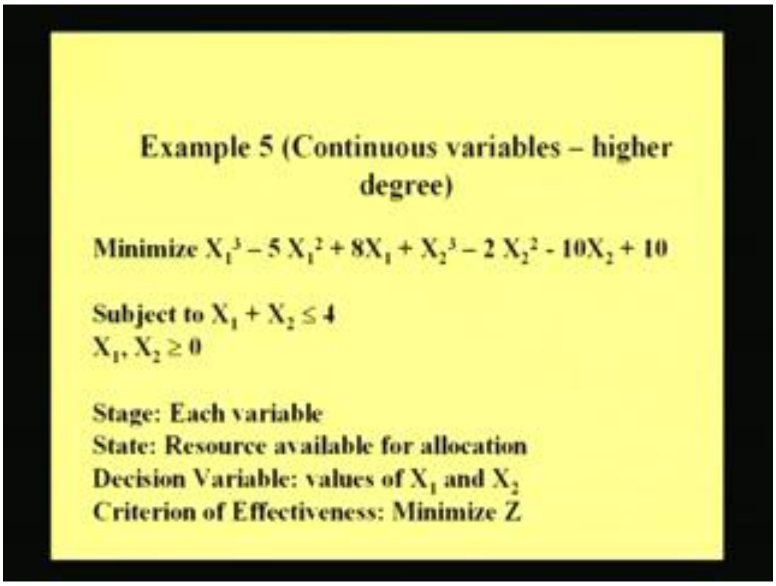

The objective function is to minimize Z = X_1 cube – $5X_1$ square + 8 X_1 + X_2 cube – $2X_2$ square – $10X_2$ + 10 subject to X_1 + X_2 less than or equal to 4; X_1 and X_2 continuous and greater than or equal to 0. As usual we define the stage, state, decision variable and criteria of effectiveness. We have 2 variables here so we solve in 2 stages and each variable represents a stage the state is the resource available for allocation the constrained X_1 + X_2 less than or equal to 4 does not tell us what the resource is therefore we generally define it as resource available for allocation. Decision variables of course, are the values of the variables X_1 and X_2 and the criteria of effectiveness minimize the objective function Z given by X_1 cube – $5X_1$ square + $8X_1$ + X_2 cube – $2X_2$ square – 10 X_2 + 10.

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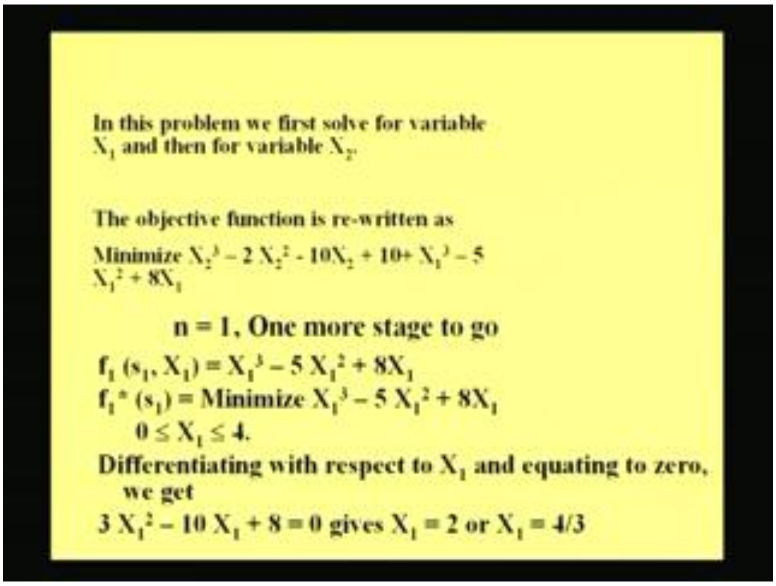

We have 2 variables in this problem or in our approach. We are going to first solve for variable X_1 and then for variable X_2 we may call it a kind of a forward recursion. Alternately we can also re write the objective function such that the objective function is X_2 cube – $2X_2$ square – $10X_2$ + 10 + X_1 cube – $5X_1$ square + $8X_1$. We will later see why we decided to rewrite the objective function. Since we have rewritten the objective function this way, it makes it easier for us to solve for variable X_1, first and then for variable X_2. Then constant now is added along with the X_2 terms and not along with the X_1 terms so as usual n = 1; 1 more stage to go F_1 or S_1, X_1 is = X_1 cube – $5X_1$ square + $8X_{1.}$ F_1 star of X_1 is the best value of X_1 that optimizes the function F_1 X_1 minimized, X_1 cube – $5X_1$ square + $8X_1$ subject to the condition. 0 less than or equal to X_1 less than or equal to X_1.,Differentiating with reference to X_1 and setting it to 0, we get $3X_1$ square – 10 X_1 + 8 = 0 which would give us X_1 = 2 or X_1 = 4/3. Now we have 2 values of X_1 which is $X_1 = 2$ and $X_1 = 4/3$. We also observe that when we solved for $3X_1$ square – $10X_1$ + 8 = 0, we can easily solve for X_1 by factorizing it. – 10 X_1 can be written as – $6X_1$ – $4X_1$ from which we would get $X_1 = 2$ or $X_1 = 4/3$.

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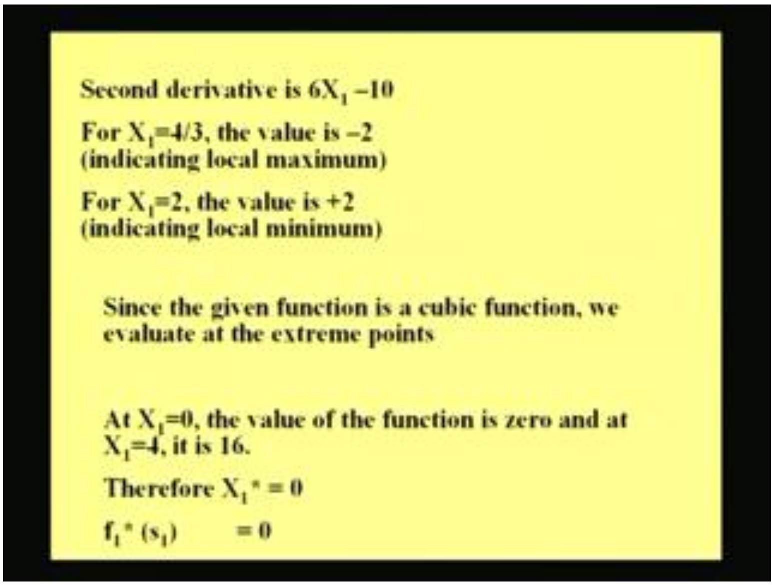

Now in order to find out which one is the minimum, we now have to find out the second derivative. Second derivative would be $6X_1 - 10$, now we substitute the 2 values that we have got. That is $X_1 = 4/3$ and $X_1 = 2$. when we substitute $X_1 = 4/3$ in $6X_1 - 10$, we get – 2 which indicates that 4/3 is a local maximum when $X_1 = 2$. The value is + 2 indicating 2 is a minimum. Now if we go back to the function X_1 cube – $5X_1$ square + $8X_1$ it is cubic in X_1. So we also have to find out in the range $0X_1$, 2, 4 the value of the function at the end. Unlike a quadratic in a cubic we need to find out the value at the end. So when we find the values at the extreme. At $X_1 = 0$, the value of the function is 0 added $X_1 = 4$. It is 16. Now if we compare the value that $X_1 = 0$ and $X_1 = 2$ we observe that the minimum actually happens at X_1 star = 0. So the minimum happens at X_1 star equals to 0 and F_1 star of X_1 is = 0.

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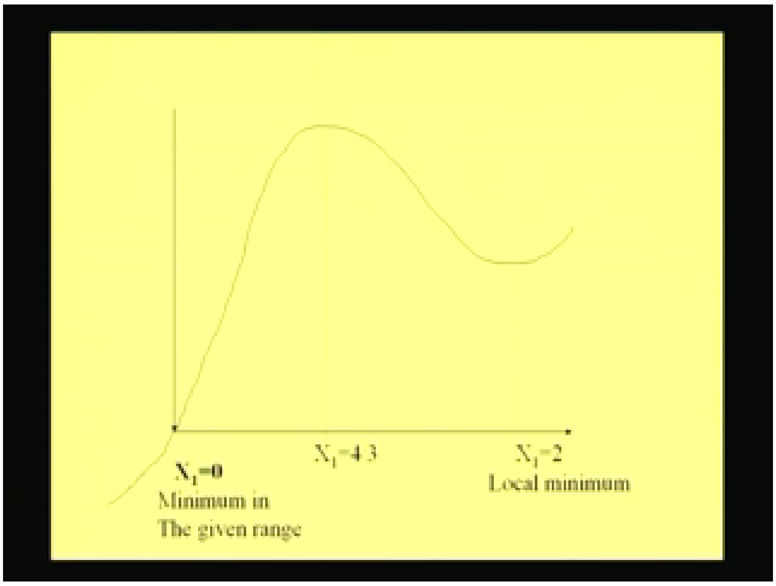

We also explain this using this figure. This is a kind of free hand diagram of the polynomial that we have and we observe that the 2 values that we obtained our $X_1 = 2$ here and $X_1 = 4/3$.

X_1 = 4/3 represents a local maximum. X_1 = 2 represents local minimum. X_1 = 0 is here and X_1 = 4 is farther. We also realize that for values less than 4/3, it is decreasing and it is at 0 and X_1 = 0. For values greater than 2, it will be increasing. So the minimum happens to be at X_1 = 0 with F_1 star of X_1 = 0.

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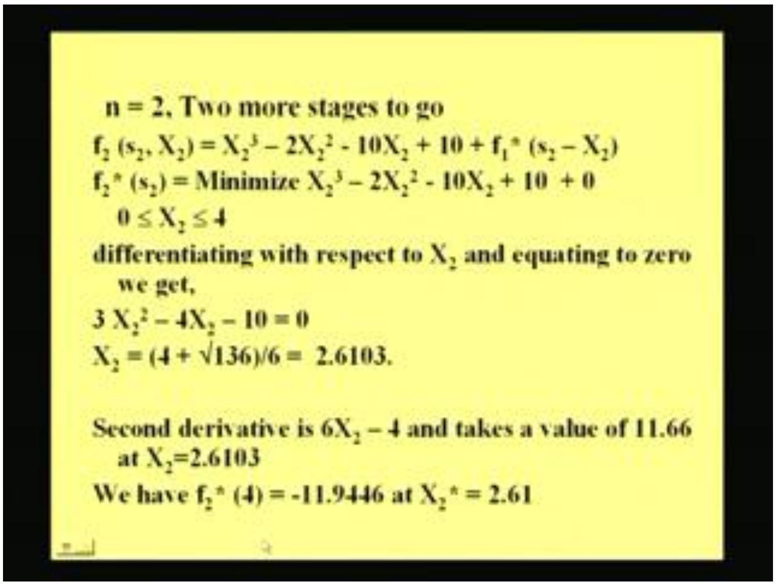

Continuing with N = 2 and 2 more stages to go F_2 or X_2, S_2 is = X_2 cube – $2X_2$ square – 10 X_2 + 10 + F_1 star of $S_2 - X_2$. Now F_1 star of $S_2 - X_2$ comes because we assume that there is an S_2 resource available, out of which an X_2 value of X_2 is allocated to variable X_2 and the balance $S_2 - X_2$ has S_1 for allocation to variable for X_1. Now F_2 star of S_2 are values of X_2, minimizes X_2 cube – $2X_2$ square – $10X_2$ + 10 + 0. The 0 comes because of F_1 star of $S_2 - X_2$. We have all ready seen here that F_1 star of S_1 is 0 therefore F_1 star of $S_2 - X_2$ is also 0. Now in the range 0 less than or = X_2 is less than or = 4. The 4 comes from the inequality here.

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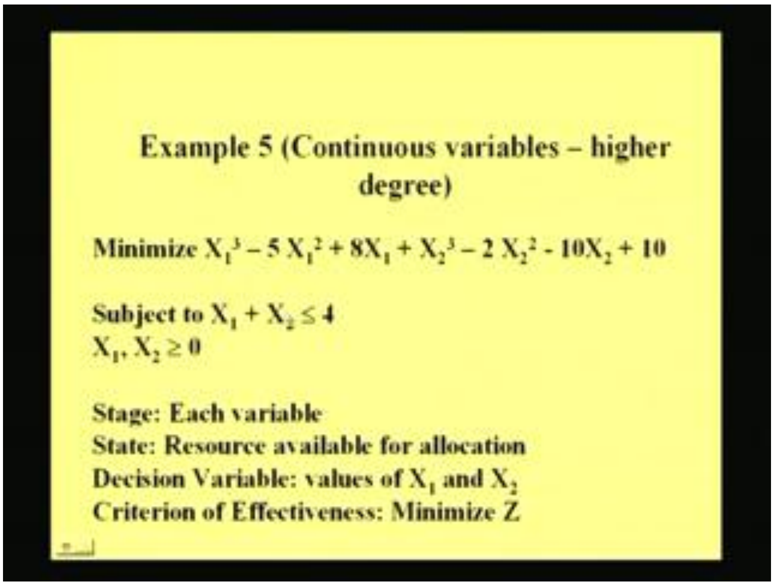

4 is the amount of resource available at the beginning. So S_2 takes value = 4. So X_2 is between 0 and 4 again. We have to differentiate with respect to variable X_2 and we equate it to 0. We get $3X_2$ square – $4X_2$ – 10 = 0 that comes from the first 3 terms. The terms X_2 cube – $2X_2$ square – $10X_2$ on differentiation will give $3X_2$ square – $4X_2$ – 10 = 0. Now this is a quadratic equation we have to solve for X_2, so we do not factorize it. So we use the formula – B + or – root of B square – 4 ac/2a to get X_2 is = 4 from + or – root of 136/6. So we get 2 values. One is 4 + root 136/6 and the other is 4 – root 136/6. Now we do not consider the negative value because X_2 should be greater than or equal to 0. So we consider only the positive value or the positive root and we have X_2 is = 2.6103. Now we have to find out whether this is a minimum and to do that, we further differentiate or we find out the second derivative. Second derivative, on differentiating this would give us $6X_2$ – 4. $6X_2$ – 4 at 2.6103, there is a positive value of 11.66 and X_2 is = 2.03. Therefore the optimum S_2 star is = 2.61 and F_2 star of 4 or the objective function value is – 11.9446. This we obtain by substituting 2.6103 in this expression X_2 cube – $2X_2$ square – $10X_2$ + 10.

When we substitute X_2 = 2.6103, we get the value of the function as – 11.9446. X_1 star is already 0, therefore the best values are X_1 star is 0, X_2 star is 2.6103 and F_2 star of 4 or the minimum value is – 11.9446.

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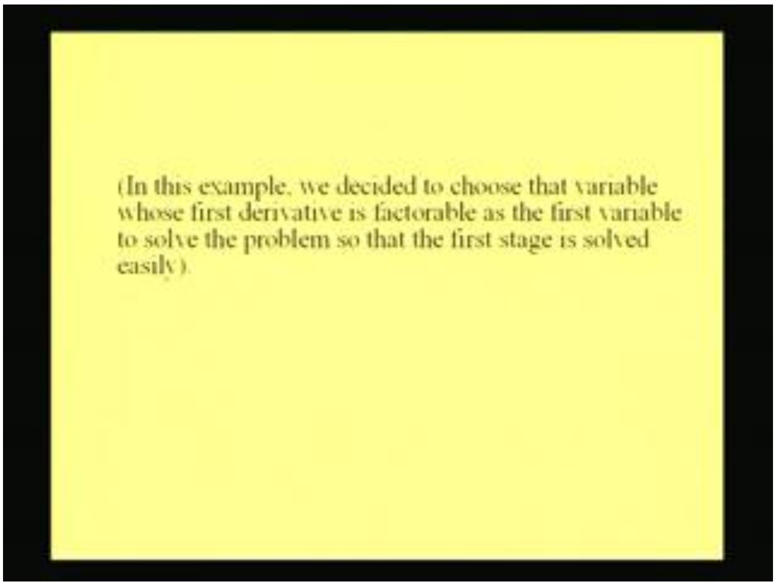

Now there are a couple of things we did in this example. In this example we decided to choose that variable X_1 first and then we chose variable X_2. If we look at the function carefully, normally in a backward recursive mode we would have solved for variable X_2 first and then at N = 2 we would have solved for X_1 but then here we did something. We rewrote the objective function or rearranged the terms. The X_2 terms were grouped together first and then the X_1 terms were written so that we could optimize X_1 first. The reason was very simple. When we had the first derivative here we could factorize it to get 2 or 4/3. If we had used X_2 first we would be forced to use the quadratic or the roots of the quadratic equation and we would have ended up getting decimals. So wherever we observe that it is possible to attract a round values and good values we can use that variable as a first variable to optimize, which is what we did here.

One should also understand that if we had worked out this problem in the normal way by choosing variable X_2 in the first stage or N = 1 and then choosing variable X_1 at N = 2, we would have got the same answer. The optimal solution in this problem is X_1 star is = 0. X_2 star is = 2.61 and the value – 11.9446. The other difference between the earlier problem and this problem is we need to look at that in this problem. Our constraint is an inequality. X_1 + X_2 is less than or equal to 0 whereas in the earlier problem, the constraint was an equation. So whenever the constraint is in equation at N = 1, one more stage to go. We did not optimize. We did not optimize or we did not differentiate in the earlier problem. We substituted the value S_1. Because of the equation all the resource S_1 will have to be utilized.

When we have an inequality, it is not necessary for us to use the entire research S_1 that is available. Here we have a research S_1 that is available. It is not absolutely necessary for us to use all the resource S_1 available. We have an inequality. We differentiate or we optimize at the first stage and then we got 2 values that happened in this example, where X_1 = 0 was the actual optimum.

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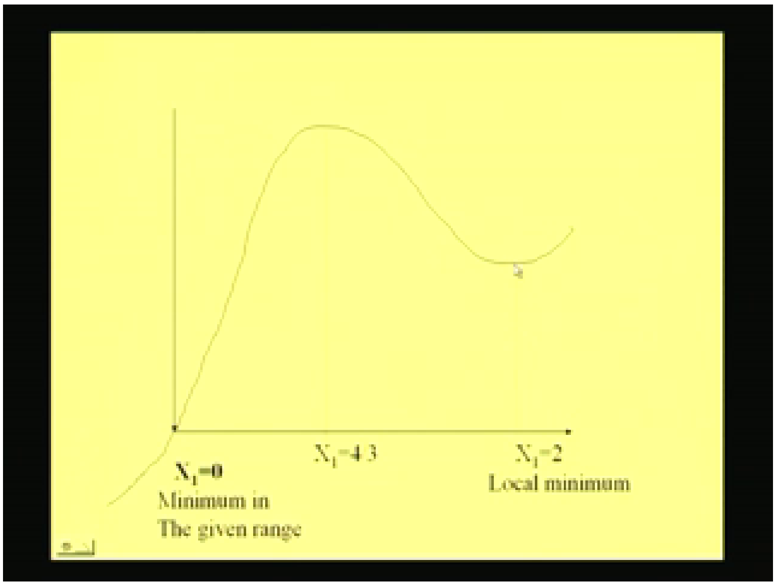

Another important thing from this problem is because this function is cubic, as explained in this figure it is not enough for us to restrict ourselves to minimum and maximum found by the differentiation. It can happen as in this example the extreme or the end is equal to 0 is actually the minimum, while the local minimum X_1 equals to 2 gains the higher value of the objective function. Now this is possible for higher order functions whereas if it were a quadratic, it would have been enough because the quadratic would have only one optimum moment. We have cubic and higher order functions, one need to look at the value the functions take at the end or at the extremes. So these are the things that we have learnt using this example. So in this example, we have tried to minimize the cubic functions subject to inequality. The most important thing is when we have an inequality; we do not substitute the value of the state variable. Instead we optimize at the first stage or we optimize at N = 1, this is the most important thing that we learned from this example.
Let us continue our discussion with 1 more example.

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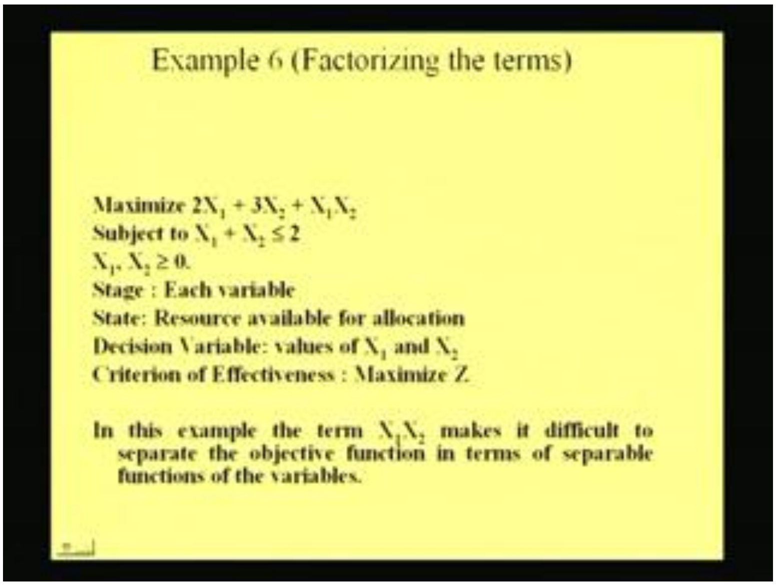

This is shown here. The problem is to maximize $2X_1 + 3X_2 + X_1 X_2$. Now, what is different in this objective function compared to the previous function, last two examples are that, we have a product form X_1, X_2 appearing in the objective function. Here in example 4, when we first introduced continuous variables, we did have a product form in the constraint but the difference here is, there is a separate X_1 term. There is a separate X_2 term, and then there is X_1 X_2. Now with this kind of an objective function or with this type of term, it may be difficult for us at the moment to identify the objective function and separate it with respect to each stage. Now we will see how we will handle this kind of peculiarity. As we go on with this example, we again have a constraint of the type $X_1 + X_2$ less than or equal to 2 and X_1, X_2 greater than or equal to 0.

We once again define state, stage, decision variable and criteria of effectiveness. Again we have 2 variables. So we solve it in 2 stages. So the 2 variables X_1 and X_2 give us the 2 stages. There are 2 stages and each stage corresponds to each variable. State once again is defined in a very general way as resources available for allocation. We do not know what this S_2 represents. So we call it a resource and the amount or resource available for allocation is state variable. The designation variables are the values of X_1 and X_2 which we are going to find out the criteria of effectiveness or the objective function is to maximize Z. Z is given by $2X_1 + 3X_2 + X_1 X_2$. Once again in this example the product term X_1, X_2 is present in the objective function making it difficult to separate the objective function in terms of 2 separate functions of the variables.

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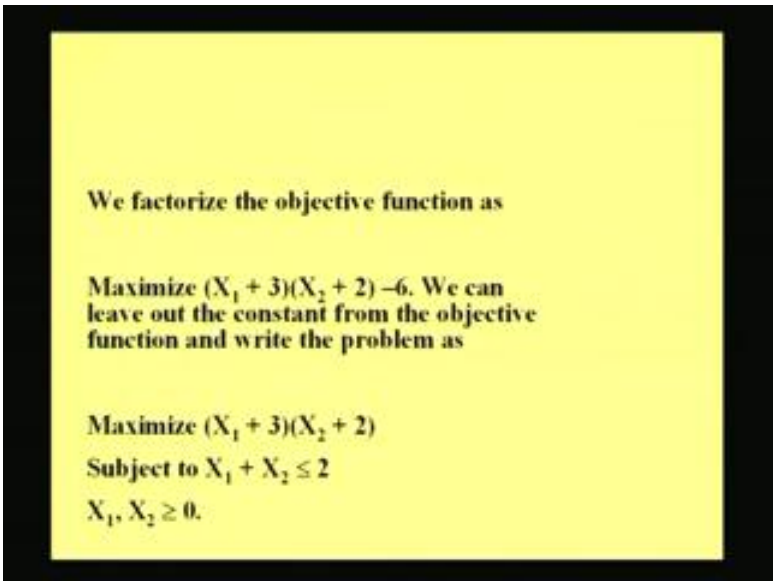

Now to overcome this, what we do is we try to factorize the objective function in this case. Now $2X_1 + 3X_2 + X_1 X_2$, what we can do is we can add 6 and subtract a 6 from this. So if we have $2X_1 + 3X_2 + X_1 X_2 + 6 – 6$ and we take only these 3 terms and + 6 and we realize that we can factorize it. After factorization we will get $X_1 + 3$ into $X_2 + 2 – 6$ or if we expand this, $X_1 + 3$ into $X_2 + 2 – 6$. We would get $X_1 X_2$ $2X_1 + 3X_2 + 6 – 6$ which is $S_2X_1 + 3X_2 + X_1 X_2$ which is what our original objective function is. So we factorize this and we get objective function which is in this form $X_1 + 3$ into $X_2 + 2 – 6$. We can always leave out the consonant from any objective function. So we write the problem so as to maximize $X_1 + 3$ into $X_2 + 2$ object of $X_1 + X_2$ less than or equal to $2X_1$ X_2 greater than or equal to 0. We have left out this – 6 therefore in the end, we have to add a + 6 to the objective function. So at the moment we leave it out. Now this is a form with which we are quite comfortable because we have 2 variables X_1 X_2. The constraint is of the form $X_1 + X_2$ less than R = 2. The objective function is clearly product of F_2 terms S_1 involving only X_1 and constraint and the other involving only X_2 and constraint. So now we can nicely do it the way we are comfortable with and we continue to solve this problem.

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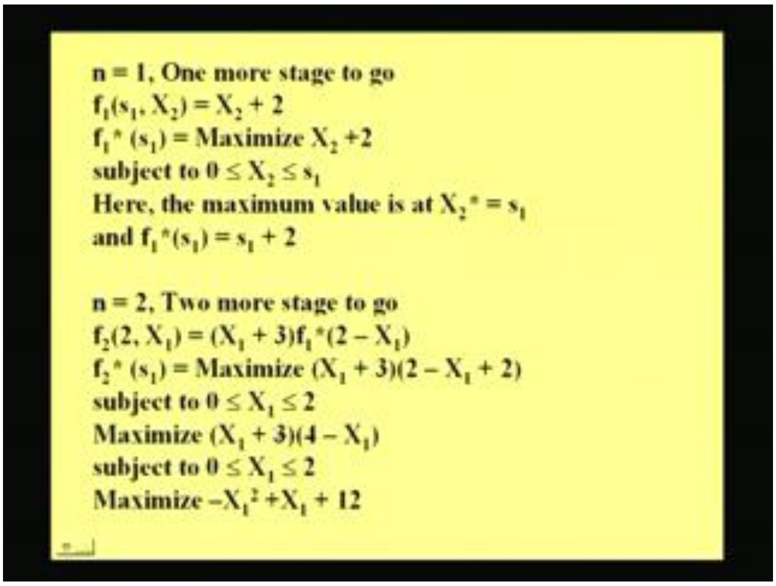

So when N = 1, one more stage to go, we try to optimize on the variable X_2 first so we have F_1 or S_1 is = X_2 + 2. We take only this term. We take X_2 + 2 terms. F_1 star of S_1 value of this function is to maximize X_2 + 2, subject to the condition less than or equal to X_2 less than or equal to S_1. We assume that a resource S_1 available here for variable X_2 which means something is allocated to S_1. The balance is available as S_1, subject to 0 less than or equal to X_2 less than or equal to S_1. Now here again we optimize it and it turns out that the optimum value is at X_2 star has an upper limit of S_1. So obviously, the best value of S_2 is going to be S_1. Due to the presence of the inequality here, we optimized it. We optimized this function subject to 0 less than or equal to S_2 less than or equal to S_1. The maximum value is got at S_2 star equal here and F_1 star of S_1 is = S_1 + 2. Now we go to the second stage N = 2, 2 more stages to go. F_2 of 2, S_1, 2 comes from this so, $2X_1$ is X_1 + 3 into F_1 star of F_2 – X_1. 2 – X_1 again comes because, out of this available 2, there is an X_1 greater than or equal to which is allocated to variable X_1, so 2 – X_1 greater than or equal to 0 is now available as S_1 or allocation for variable X_2.

Here again we are aware that S_1 is greater than or equal to. Now this S_1 is greater than or equal to because if we go back here both X_1 and X_2 are = 0. So if we can give an allocation, X_1 greater than or equal to or less than or equal to 2, the balance S_1 will have to be greater than or = 2 is applicable here. Similarly 0 less than or equal to S_1, less than or equal to 2 is applicable here. This would mean F_2 star of S_1 is to maximize X_1 + 3 into F_1 star of X_2 – S_1. We also know that F_1 star of a given S_1 is S_1 + 2, so S_1 star of S_1 will be 2 – S_1 + 2 subject to 0 less than or equal to 2. So to maximize S_1 + 3 into 4 – X_1 subject to 0, less than or equal to S_1, less than or equal to 2, this on multiplication would give us a – X_1 square + X_1 + 12. First derivative = 0.

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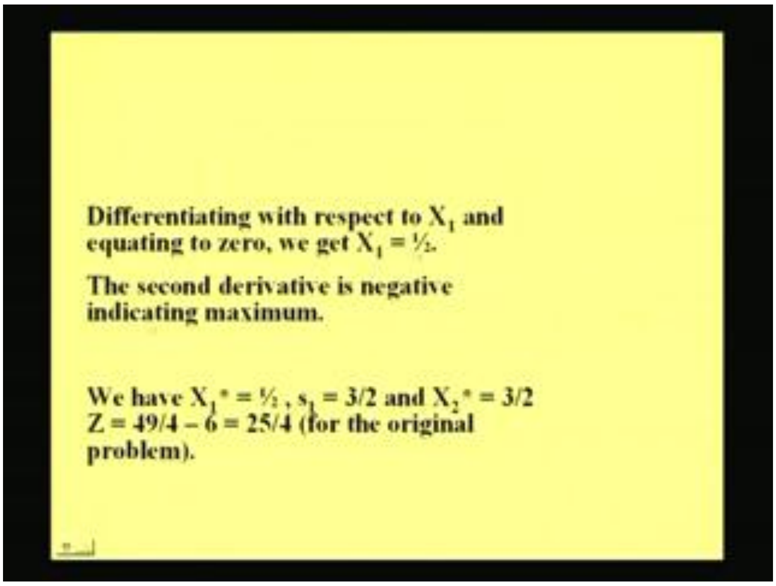

We get X_1 = 1/2. Now that comes because we have – $2X_1$ + 1 = 0 gives us S_1 = 1/2 and the second derivative will be – 2 indicating it is negative and also indicates the maximum. So we are into maximizing, therefore X_1 star = 1/2. Now if X_1 star is = 1/2; S_1 = 3/2 that comes because X_1 star is = 1/2. So we have used up 1/2 here. 3/2 is now available as S_1 for allocation of variable X_2. We also know that S_2 star is = S_1, therefore 3/2, which is available for allocation goes to variable X_2, so X_1 star is 1/2, X_2 star is 3/2. The value of objective function at present is 49/4, i.e., the value of the function – X_1 square + X_1 + 12 at X_1 = 1/2 is 49/4. But we also have to subtract a 6 because when we factorize, the original objective function turned out to be X_1 + 3 + X_2 + 2 – (we left out this) – (included only this). Therefore we have to include this – 6 here, so we subtract a 6 from the Z to get 25/4 for the original problem.

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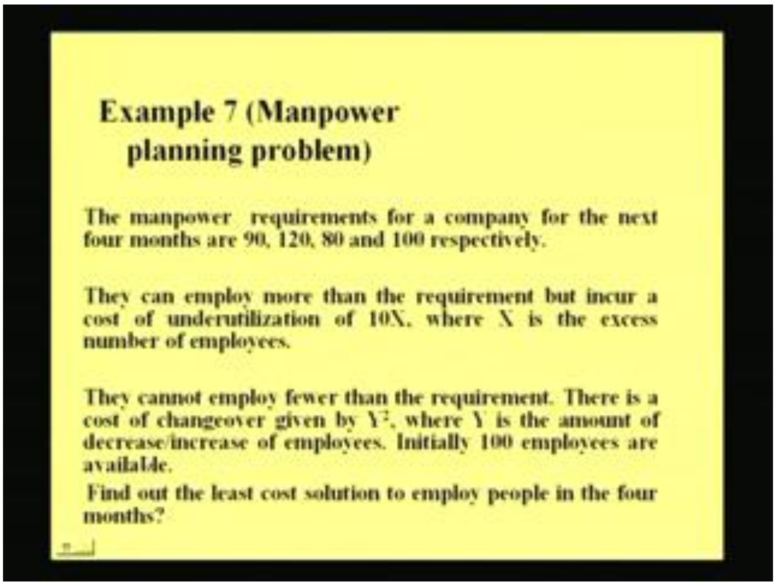

Now the most important thing in this is when we have an objective function of this type, where this is the first time we considered an objective function which has X_1 terms, X_2 terms and a product, the presence of the product along with the X_1 terms and the X_2 terms makes it difficult. Therefore factorization came to our rescue. If the function is such that we can factorize it, we can use it to our advantage and then convert the objective function into this form where we can separate the X_1 term and the X_2 terms which in this case turned out to be a product of S_1 and a function of S_2. If we had started with this, then it would have become very difficult trying to optimize 2 variables in 1 stage or we end up having to identify different types of state variables. So to avoid or overcome that difficulty by factorizing and by bringing it into or at form, we can separately take out the X_1 term as well as the X_2 term and then we can separately optimize for each one of them. So it is important when we have these kinds of objective functions. We should explore possibilities.

We can convert them into the form S_1 of X_1, S_2 of X_2, so in this case it was the product form. Also we need to understand that whenever we have an inequality coming in, we do not substitute. We need to optimize. In this case we did optimize X_2 + 2 subject to 0, less than or equal to 0. We found out that, the value of X_2 star was the available resource but we did optimize to find out that the maximum value R is = S_1. Couple of things that we can learn from this example is that when we have an objective function, when we could use factorization to convert into a form which is familiar and comfortable. We then have an inequality. We need to optimize N = 1 stage. We continue our discussion in dynamic programming through another example from which we would learn a few new things.

Now this problem is a typical manpower problem. So the problem is as follows. Man power requirements for a company for the next 4 months are 90, 120, 80 and 100 respectively. They can employ more than the requirement in any month, but if they do that, they incur a cost of under utilization. Now this cost of underutilization is 10 times X, where X is the additional number of employees over and above the minimum required and the minimum required in the first month is 90. If they end up employing 95 people then because of the additional 5, they would incur an under utilization cost of 10 into 5 which is 50. They cannot employ less than the minimum required. So they can only employ 90 or more for the first month. They cannot employ fewer than the requirement. Now there is also a cost of changeover given by Y square where Y is the amount of decrease or increase over the employees. For example if they choose to employ 95 employees for the first month instead of 90 and say 120, they employ in the second month then there is a difference of 15 from 95 to 120. Remember 90 is the requirement and if they employ 95 then from this 95 to 120, there is a change over cost and increase cost which will be 15 square which is = 225 . If they choose to employ 100 here then there is a decrease of 20 so, that would result again in a changeover cost so 20 square which is = 400. So whether there is an increase or decrease, there is a change over cost given by Y square where Y is the amount of decrease or the extent of decrease or increase of the employees. Initially 100 employees are available so 100 employees are available here at the beginning of the first month to find out the least cost solution to employees in the fair month where we will now we look at.

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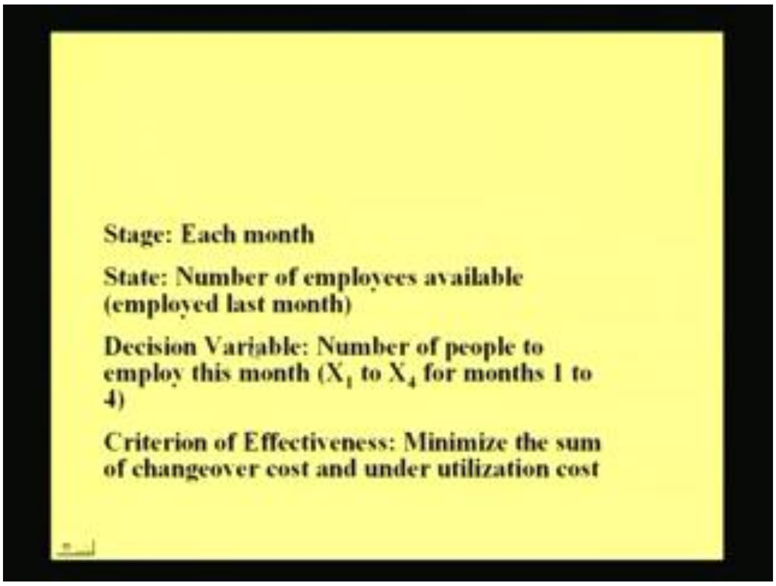

We had defined decision variable and criterion of effectiveness now. There are 4 months, so each month would act as a stage. We have 4 stages N = 1 to N = 4 which are represented by the 4 months. State in this example, is the number of employees available in the month or the number of employees who have been employed in the previous month. For example when we look at the beginning of the first month now, 100 employees are available in the beginning of the first month so that is the state variable. If we look at the beginning of the second month then the number of people who were employed in the first month would now be available at the beginning of the second month and therefore that would be the state variable. State variable here is the number of employees available at the beginning of the month which is the number of employees who were employed in the previous month. Decision variable is X_1 to X_4 is the number of people to be employed in the current month or in the last month. So we define X_1 to X_4 for 4 month, S_1 to 4. Criterion of effectiveness or the objective function as an example would be to minimize the sum of the change over cost and the underutilization cost. There are 2 costs. There is a change over cost. There is an underutilization cost. So we want to minimize the sum of these 2 costs as the minimum total cost. We use a backward recursive approach.

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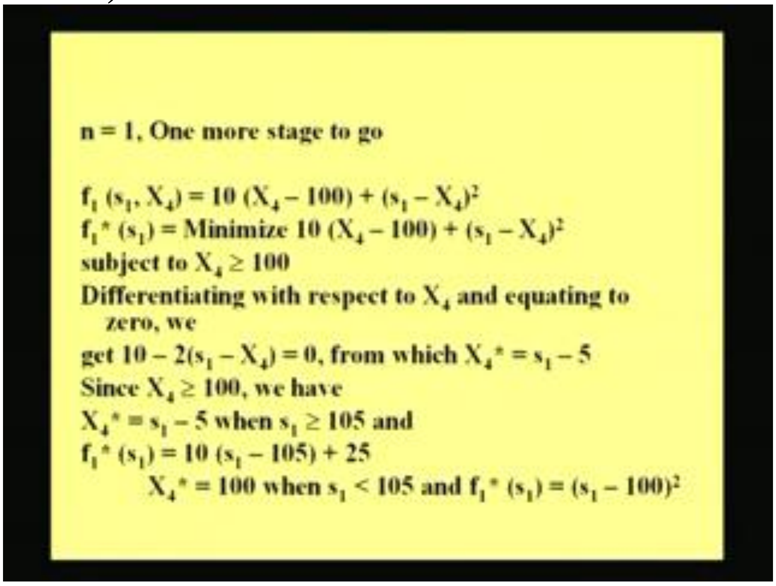

So when N is = 1 more stage to go or 1 more month to go, we are effectively looking at the 4th month, so because of the backward recursion we have 1 more month to go. We are here. So we are looking at the fourth month, so we assume that we are at the end of the 3rd month. F_1 or S_1X_4, S_1 is a state variable which is the amount of people available at the beginning of 4th month which is also the number of people employed in the 3rd month.

We will see that later. So the number of people available at the beginning of the 4th month is X_1. X_4 is the decision variable which is a number of people who are going to be employed in the 4th month. The requirement for the 4th month is 100, so number of people who are going to be employed in the 4th month will have to be more than 100. So if it is more than 100 then $X_4 – 100$ is the excess number of employees who contributed to the underutilization cost. So underutilization cost is 10 times X_4 which 100. Now there is a change over from X_3 to X_4 because X_3 is the number of people in the month 3. X_4 is the number of people in the month 4. $X_3 – X_4$ is the change over so X_3 is the same as S_1. A state variable is the same as the number of people employed in the 4th month. We have $S_1 – X_4$, power the whole square. This is the change over cost.

This objective function is the sum of the underutilization cost and the change over cost. So F_1 star of S_1 is the best value of X_4 which is to minimize the sum of the underutilization cost and the changes over cost to minimize 10 times $X_4 – 100 + S_1 – X_4$ the whole square. So subject to the condition, X_4 is greater than or equal to 100. Company should employ at least the minimum number or more and cannot employ less than 100 in this case. So once again differentiating w.r.t 4 and equating to 0, we would get 10 – 2 times $S_1 – X_4$ will be = 0, that comes with this expression. This (Refer Slide Time: 35:58) would give us 10. This would give us – 2 times S_1 into $X_4 = 0$ from which X_4 star is $= S_1 – 5$. Now we also have to find out the second derivative and show that the second derivative is positive which would indicate a minimum, which we would see from here. – 2 into – X_4 would give us a + $2X_4$ which would give us a minimum. So X_4 star is $= S_1 – 5$ in this case.

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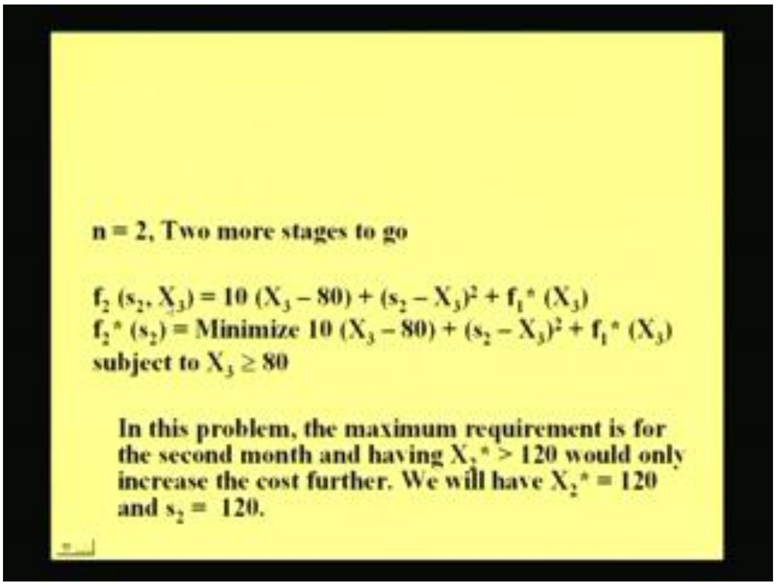

Now we also have 2 cases coming in here since X_4 is greater than or equal to 100, we need to have a couple of things. When S_1 is greater than or equal to 105 then X_4 star will become S_1 – 5. For example if S_1 were 110, then the best value of X_4 star would be 110 – 5. On the other hand if S_1 was 100, then S_1 – 5 would give us 95 which is going to violate X_4, greater than or equal to 100. So in such a case, X_4 star will take a value 100. X_4 star will take values depending on what the value of S_1 is, so if S_1 is greater than or equal to 105 then X_4 star will be S_1 – 5 and F_1 star of S_1 will be 10 times S_1 – 105 + 25. This comes because when we substitute X_4 = S_1 – 5, becomes 10 into S_1 – 5 – 100 which is 10 into S_1 – 105. X_4 is S_1 – 5, S_1 and S_1 will cancel out each other and we will get a 25. So when X_4 star is = S_1 – 5, F_1 star of S_1 is 10 times S_1 – 105 + 25.

If S_1 is less than 105, S_1 – 5 will become less than 100 but then it will violate the restrictions, so F_4 star will take a value 100. So F_4 star will take a value 100, if S_1 is less than 105. Substituting here, the under utilization cost becomes 0 and only the change over cost will be there. F_1 star of S_1 will become S_1 – 100 the whole square. Now N= 2; 2 more stages to go, F_2 of S_2 X_3 star. Now when we have 2 more stages to go, we assume that we are at the beginning of month 3, and we have certain number of resources available to us which is the number of people who worked in month 2. So, when we are here, when have N =2; 2 more stages to go, and we have to make a decision for variable $X_{3,}$ assuming that S_2 people are available.

F_2 of S_2, X_3= 10 times X_3 – 80, 80 come as the minimum requirement for month 3 + S_2 – X_3, the whole square represents the changeover. There is S_2 that was available at the beginning, X_3 is employed + F_1 star of $X_{3,}$ this comes in because the number that we employ this month, X_3 is now available at the beginning of the next month as X_1 so the function is 10 into X_3 – 80 + S_2 – X_3, whole square, + F_1 star of X_3. F_2 star of S_2 is to minimize 10 into 3 – 80 + S_2 – X_3, the whole square, + F_1 star of $X_{3,}$ subject to the condition, X_3 greater than or equal to 80. Now you go back to the problem. We observe that the 4 requirements are 90, 120, 80 and 100. One look at these numbers is going to tell us that for the second month it is not advisable for us to have more than 120 people at all because if we have more than 120 people then we are going to unnecessarily have underutilization cost and the change over cost is going to be more because the next month’s requirement is 80. One look at these numbers it would tell us that for the second one, we are going to employ only 120 people. We use this information in this problem to simplify this problem further. So we go back. In this problem, the maximum requirement is for the second month and having an X_2 star greater than 120 would only increase both the under utilization cost and the change over costs further. We will have X_2 star = 120 and S_2 = 120. So we can comfortably substitute S_2 = 120 in this. Now we also have 2 functions for F_1 star of X_1. F_1 star of S_1 took 10 into – 105 + 25 in a certain range and F_1 star of $S_{1,}$ S_1 – 100, the whole square in a different range where S_1 is less than 105. Now we have to substitute both these expressions here. So we have 2 expressions for F_2 star of S_2. Now we understood that S_2 is 120 because X_2 star is 120. So we are going to substitute S_2 = 120 and we are also going to substitute the 2 expressions for F_1 star of X_3. F_1 star of X_3 is the same as F_1 star of S_1. So we substitute both these expressions. So the first one will have one F_2 star of S_2, minimize 10 times X_3 – 80 + 120 – X_3 the whole square + 10 into X_3 – 105 + 25. We have used the first expression here, 10 into S_1 – 105 + 25 but that is within the range greater than or equal to S_1, greater than or equal to 105. This implies X_3 greater than or equal to 105.

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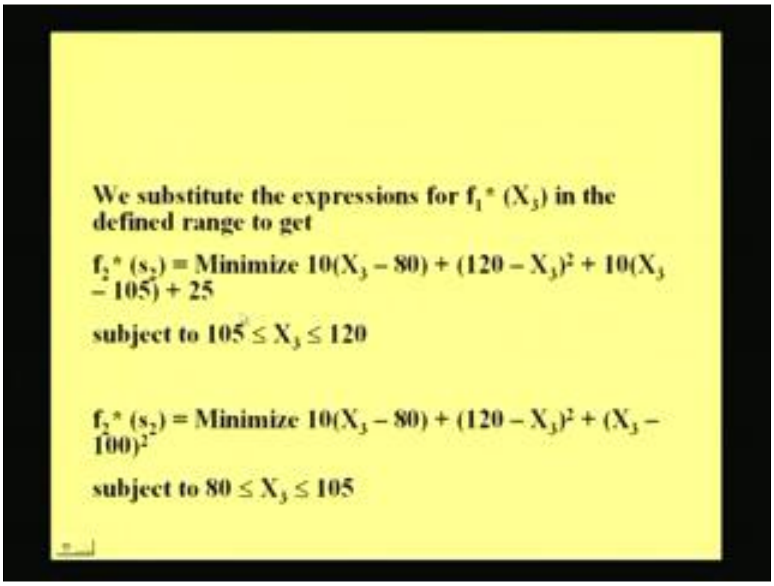

X_3 has to be less than or equal to 120 because that is the maximum. Therefore in the range X_3 between 105 and 120, we have this function which minimizes 10 times X_3 – 80 + 120 – X_3 the whole square + 10 times X_3 – 105 + 25. On the other hand in the range X_3 less than 105, let us go back. S_1 is less than 105, F_1 star of S_1 is S_1 – 100 the whole square. S_1 is the same as X_3 in the range, X_3 less than 105 or in the range of X_3 between 80 and 105, 80 comes from the fact that minimum requirement is 80. Now in the range 80 to 105 that we have here, now the function becomes minimized. 10 into X_3 – 80 + 120 – X_3, the whole square + X_3 – 100 whole square. X_3 – 100, the whole square comes from S_1 – 100 the whole square.

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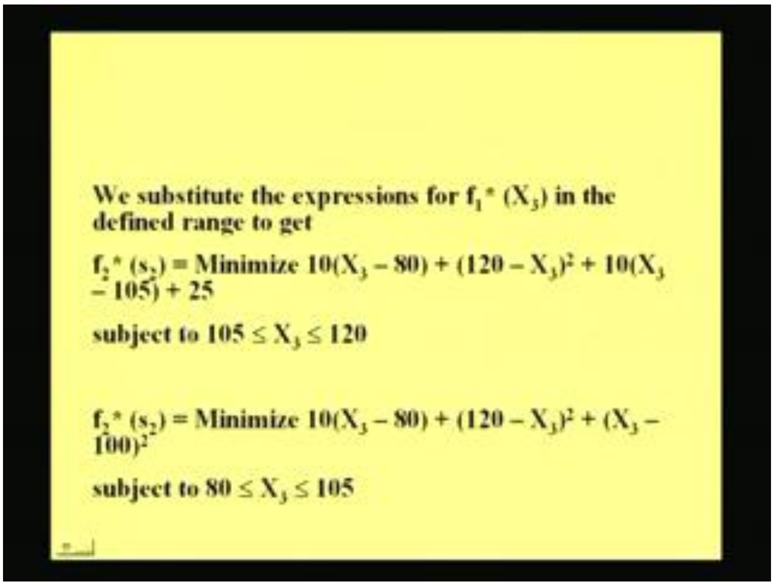

So we have written both these functions for F_2 star S_2 depending on the range of values of X_3. So if X_3 is in this range, we have one function. X_3 is in the other range. We have another function, so we have to separately optimize both these functions and find out what is the minimum value of X_3.

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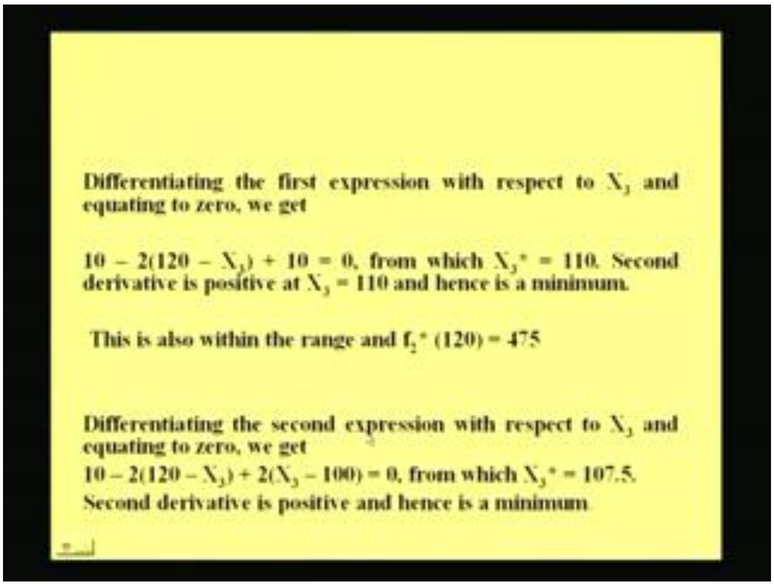

In order to do that we differentiate the first expression with respect to X_3 and we equate it to 0.

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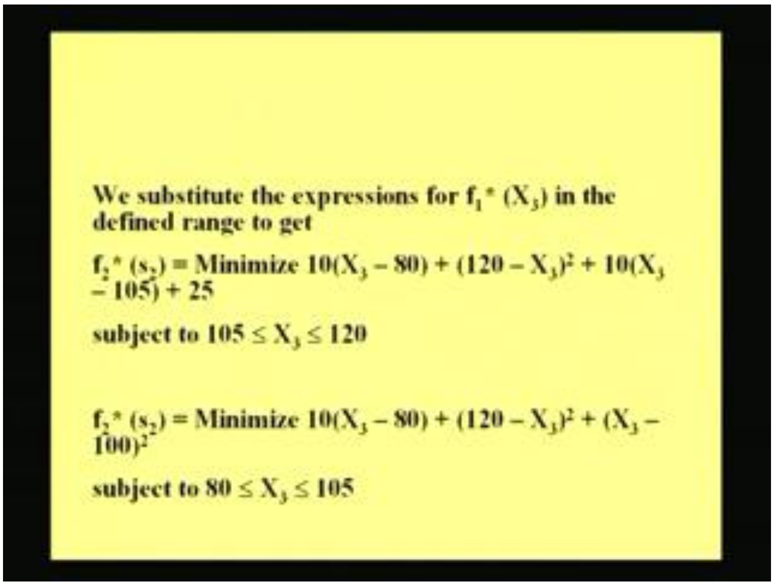

So this would give us a 10. This would give us 2 times 120 – X_3 into –1. This would give us another 10. So we get 10 – 2 times 120 – X_3 + 10 = 0 from which X_3 star is = 110. Second derivative is positive because we have a + $2X_3$ term that would give us a + 2 which is positive at X_3 = 110 and then the value of X_3 star is = 110 is indeed a minimum. We also need to check that, the X_3 for this function will be within the range 105 to 120 and it happens if 110 is within the range and so we do not have to worry.

If X_3 star had exceeded 120 or were less than or equal to 105, then we substitute as the extreme points to find out which one is the minimum. Now in this case we found out that the optimum value which is 110 is in the range and we comfortably use this 110 and substituting we get F_2 star of 120 = 475. We go back and substitute here we get 400, 0 50, another 25 so put together we get 475. So now we go to other expression we have to get the value of X_3 keeping in mind the range. This would give us a 10, this would give us – 2 times 120, – X_3 and this would give another 2 times X_3 – 100. So we get 10 – 2 times 120 – X_3 + 2 times X_3 – 100 = 0 from which X_3 star = 107.5. Second derivative is positive because there is a $2X_3$ here and a $2X_3$ here so $4X_3$ will be + 4 and so it is positive. We need to look at something else. The value is 107.5 but we know it is valued between 80 and 105.The optimum exceeds the range. So we need to go back and substitute whether it is minimum at 80 or whether it is minimum at 105. Because this function is quadratic and because at 107.5, it is a minimum, we observe that within this range the value at 105 will be smaller than the value at 80 so X_3 star is actually 105 in this case and not 107.5. So we will take 105, X_3 star will be 105. When we substitute F_2 star of 120 is 487.5.

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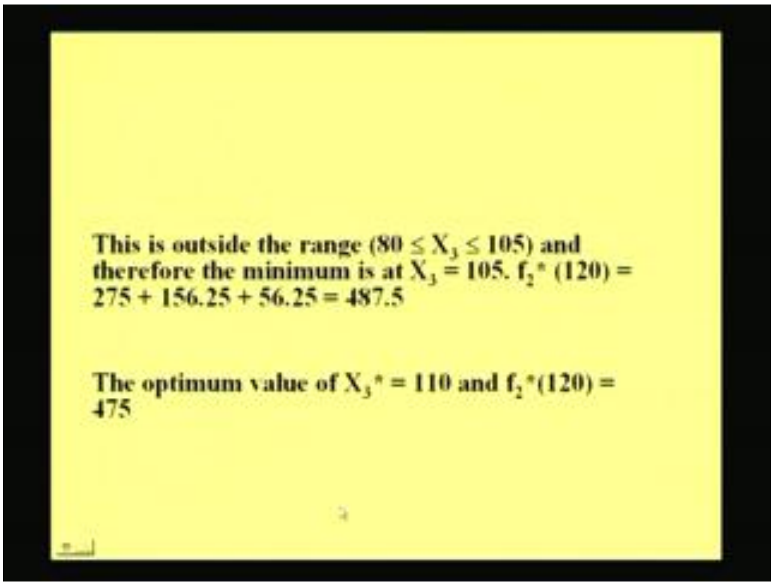

In this case the optimum value of X_3 star is 110 which is the best value corresponding to the lower of 475 and 487.5. So F_2 star of 120 is 475, X_3 star is = 110. Now we proceed again at N = 3, 3 more stages to go mean that we are somewhere here (Refer Slide Time: 38.05). We are here at the beginning of the second month and we have a state variable and a decision variable too to make. So at N = 3 more stages to go, assuming F_3 is a state variable and X_2 is the decision variable. 10 times X_2 – 120, 120 is the minimum requirement, X_2 is what we are going to employ for month 2 + S_3 – X_2 the whole square. This represents the change over cost + F_2 star of X_2. Now F_3 star of X_3 which is the best value of this function is given by an X_2 such that it minimizes 10 times X_2 – 120 + X_3 – S_2 the whole square + F_2 star of X_2. F_2 star of X_2 comes because whatever we have as X_2 in this month is available as S_2 in next month will be available X_2 in the beginning of next month, subject to the condition, the minimum X_2 greater than or equal to 120 is also satisfied.

Now we have already seen that because of these 4 values 90, 120, 80 and 100. We know it is not advisable to have more than 120 in the second month and therefore we have already seen that 2 star is = 120. So all we need to do here is to show or to take the value F_2 star is = 120 which we know in this case, X_2 star is = 120. So we are not going to optimize this and find out 120, we may do that.

Since we know that 120 is maximum and we have already used this in the earlier result, we have X_2 star is = 120 and F_3 star of X_3 will be S_3 – 120, the whole square + 475. When X_2 star is = 120, the under utilization cost is not present. There is no under utilization because we are not employing more people. Only the change over cost is there. So whatever is available as state variable is X_3, from that X_3, we are going to bring it to 120, S_3 – 120 the whole square + F_2 star of X_2 has been found out to be 475, so we add this 475 to get F_3 star of X_3 = S_3 – 120 the whole square + 475.

Now we go to the last one N = 4, 4 more stages to go, which means we are at the beginning of the problem, which means we are at the beginning of month 1. The requirement during this period is 90. People available at the beginning of the planning period are available at the beginning of month 1, is 100, so we go back to N = 4.

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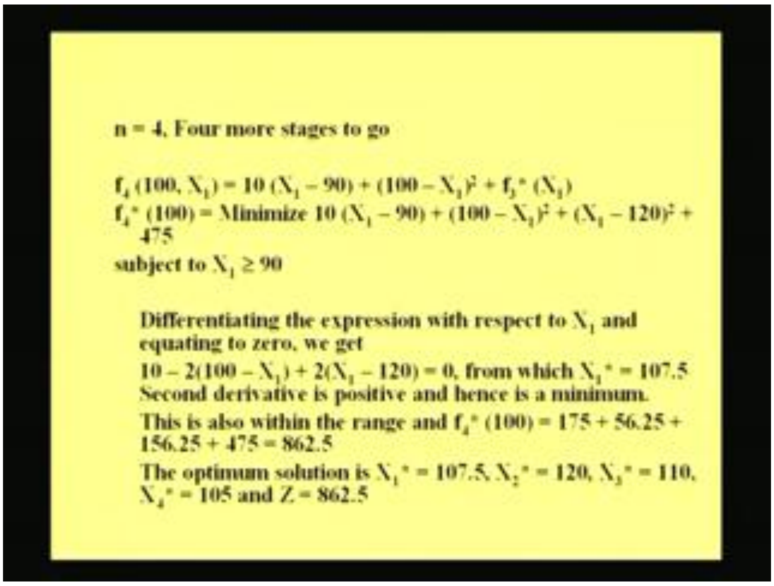

F_4 of $100X_1$, this 100 comes in because 100 is the amount of people available at the beginning of the first month and X_1 is the decision. X_1 is the number of people that we are going to employ in the first month. So F_4 of 100, X_1 is 10 times $X_1 – 90$. 90 is the minimum requirement for month 1, so if we end up employing more than X_1 then there will more than 90. There will be an $X_1 – 90$ which will add to the under utilization. So there is going to be a change over cost, $100 – X_1$, the whole square represents the change over cost + F_3 star of X_1 F_3 star of X_1 comes because this F_3 star of X_1 is available as X_2 at the beginning of the stage F = 3. X_1 is going to go as S_2.

Earlier we had defined F_3 star of S_3. Now this X_1 is going to go as F_3 so we write F_3 star of X_1. F_4 star of 100 is the best value that this function can take for a given state variable which has a value 100 and that would minimize this function, 10 times $X_1 – 90 + 100 – X_1$, the whole square + $X_1 – 120$ the whole square + 475. $X_1 – 120$ the whole square + 475 comes from the earlier statement, $X_3 – 120$ the whole square + 475. So the whole thing repeats again and most importantly subject to the condition, X_1 less than greater than or equal to 90. Now this 90 is the minimum requirement that we need to have for this. Now once again differentiating with reference to X_1 and equating it to 0, we get this expression. It would give us a 10. This would give us 2 times $100 – X_1$ into – 1 and this would give us 2 times $X_1 – 120$. Constant does not yield anything. So we get 10 – 2 times $100 – X_1$ + 2 times $X_1 – 120$ equals to 0 from which X_1 star is = 107.5. Second derivative would be positive because this is a $4X_1$ terms this is a $2X_1$ term. So together we have $4X_1$. So the second derivative will be 4 and it is positive and therefore 107.5 is the minimum.

Now it also satisfies the condition X_1 greater than or equal to 90 and therefore we are comfortable and it is also less than 120 which is one of the assumptions we had made. So it also satisfies that and therefore does not give any trouble or any difficulty. We comfortably assumed the value 107.5. This is within the range and F_4 star of 100 on substitution would give us 107.5 – 90 is 17.5 multiplied by 10 gives us 175. 100 – 107.5 is 7.5 square which is 56.25, 107.5 – 120 is 12.5 square is 156.25 + 475 gives us 862.5 as the cost. Now the optimum solution is X_1 star is = 107.5 that we found out. Now X_2 star is 120 which we assumed in this problem. X_3 Star is 110 that we found out. X_4 star was 105 which we also found out right at the beginning because when X_3 star is 110. We go back here at N = 1 X_3 star is 110. It implies that S_1 is 110, if S_1 is greater than 105, therefore X_4 star will be $S_1 - 5$. It will become 105 and the value of the objective function is 862.5

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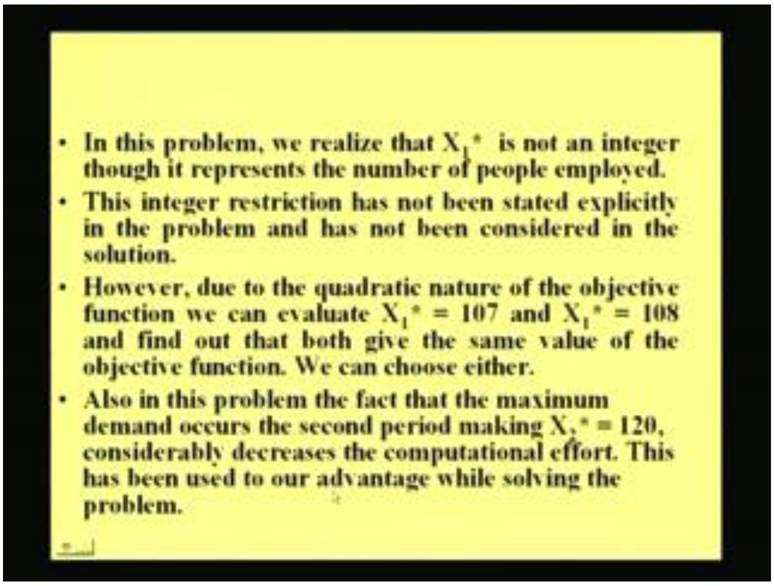

But there are a few other things. We realize that X_1 star is not an integer. It represents number of people employed. In the problem if we go back to the original problem, it does not explicitly state that the decision variables are integers therefore we ignored it and solved it as a continuous variable. Due to the quadratic nature of the objective function, we can actually evaluate X_1 star = 107 and X_1 star = 108 finds out actually. They both gave the same value of the objective functions. We can actually choose either of them. This problem was made much simpler by the fact that the maximum demand occurred at the second period making X_2 star = 120. This considerably decreases.

We have used this for our advantage while solving this problem. So a quick summary of this problem had a quadratic kind of an objective. It also gave us situation where as in this stage, the function F_1 takes the common 2 different values depending on the range of the variables. We have to carry it forward to the next stage and then we need to solve that. We continue our discussion in the next class with more examples.