Fundamentals of Operations Research

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Lecture No. # 14

Transportation Problem - Optimal Solutions

We continue our discussion on the transportation problem.

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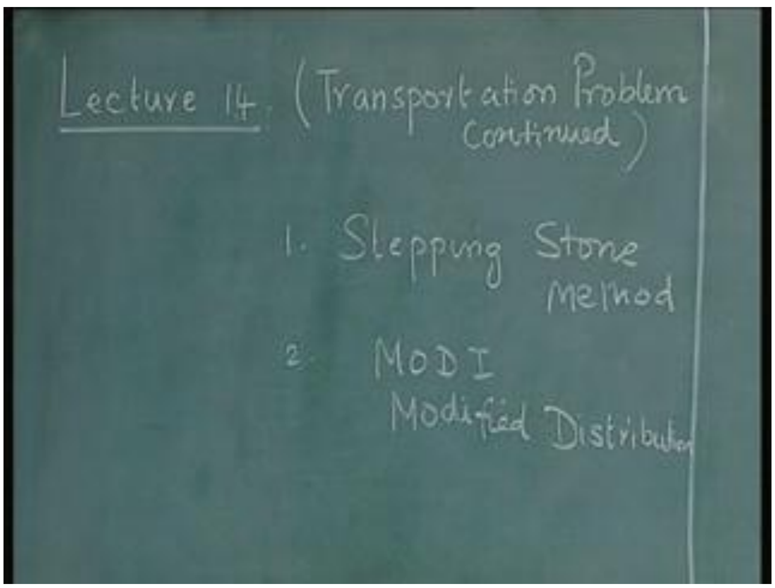

Today we look at 2 methods to get the optimal solution from a starting basic feasible solution. These 2 methods are called the stepping stone method and the modified distribution method, also called MODI. Let us look at the stepping stone method. We go back to the familiar example.

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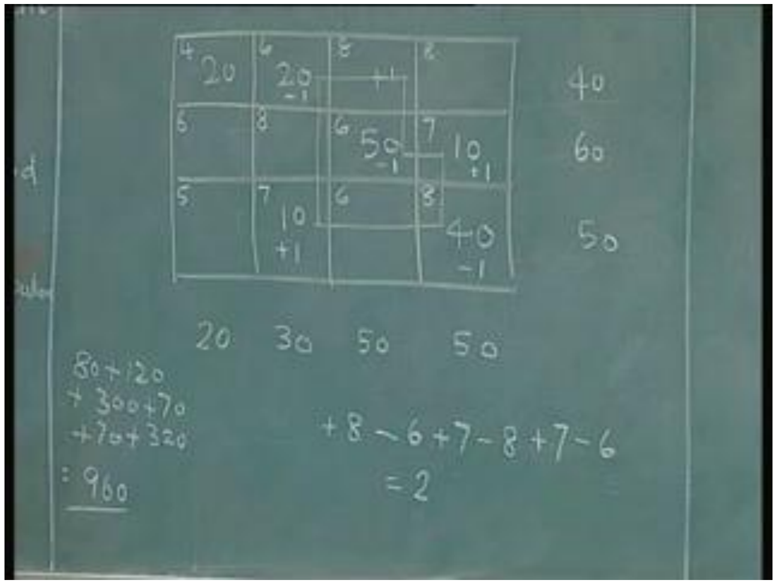

The 3 supplies are 40, 60 and 50 the requirements are 20, 30, 50 and 50 the cost are 4, 6, 8, 8, 6, 8, 6, 7, 5, 7, 6, and 8. We start with a basic feasible solution; the solution that we start with is given here, 20, 20, 50, 10, 10 and 40. You would remember that this is one of the solutions that we obtained using the Vogel's approximation method or penalty cost method. What we need is a feasible basic solution to begin with. It could be obtained through either North West corner rule or the Minimum cost method or the Vogel's approximation method but what we need is a basic feasible solution. Now this solution is basic solution because it satisfies all the supply and demand constraints. The row totals are 40, 60, and 50. Column totals are 20, 30, 50, and 50 respectively. It does not have a loop. It satisfies a non negativity restriction. It has exactly m + n – 1 allocation. It is not degenerated so we have non degenerate basic feasible solution to begin with. The cost corresponding to this is 80 + (6 into 20) = 120 + 300 + 70 + 70 + 320, 200, 500, 570, = 640 + 320 is 960. We have a basic feasible solution with cost 960.

Now from this basic feasible solution we would like to go towards an optimal solution, so first thing we need to do is to check whether this is optimum. If this is not optimal then it is obvious that one of the unallocated positions will have an allocation so what we try do is we try look at every one of the unallocated positions or non basic positions. Those that are allocated are the basic positions or the basic variables. The non basic variables in this case are X_{13}, X_{14}, 4 X_{21}, X_{22}, X_{31} and X_{33}.

So we take each one of the non basic variables that have an unallocated position and try to make an allocation there for example, if we try to make an allocation here which means we put a + 1 here now it means that already there are 6 allocations, now there is a 7 and any number of allocations greater than m + n – 1 in this, case 7 will have a loop so we need to identify a loop through this 1 so the loop that we have here is as follows. From this + 1 we go to this. From here we go to this 10, from here, we come to this then we come back to this and we come back to this and then we go back to this. + 1 50, 10, 40, 10, 20. Remember this is also a loop. A loop by

definition is, starting from any allocation you move horizontally or vertically and un-vertically alternately and then you are able to come back to the starting allocation. This is another loop.
Now to balance, if we put a, + 1 here this will become – 1. This will become + 1. This will become – 1. This becomes + 1. This becomes – 1 and it is now balanced.
So the net increase or decrease in the objective function as a case is + 8 because of this – 6. We are taking away 1 from here, so the cost comes down by 6 so, – 6 + 7 we add 1. So + 7 – 8 + 7 and – 6, so this is 8 – 6 = 2; 2 + 7 = 9 – 8 = 1; 1 + 7 = 8 – 6 is = 2. So the net increase is 2

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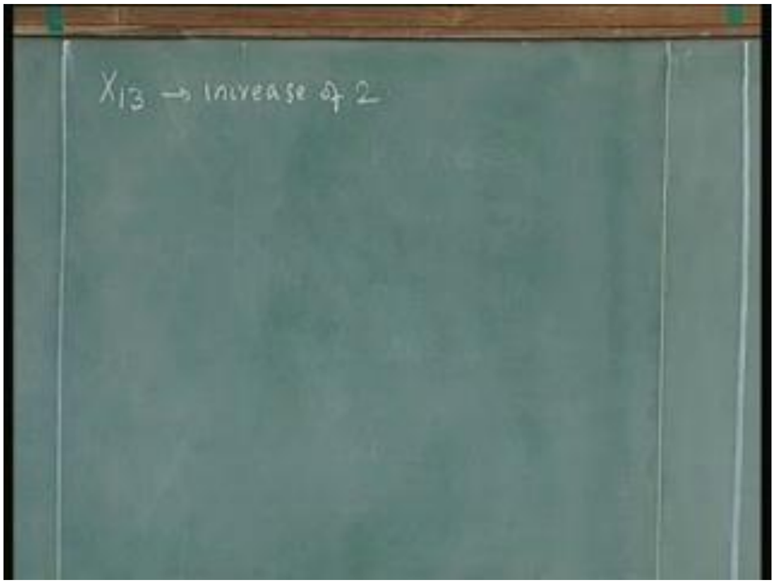

It means that if we consider allocating variable $X_{1\,3}$ putting a 1 to this position then the net increase is 2 now let us look at it is not $X_{1\,3} = 2$ so, when we enter $X_{1\,3}$, there is an increase of 2 per unit allocated here. Now we try to put it in the other position.

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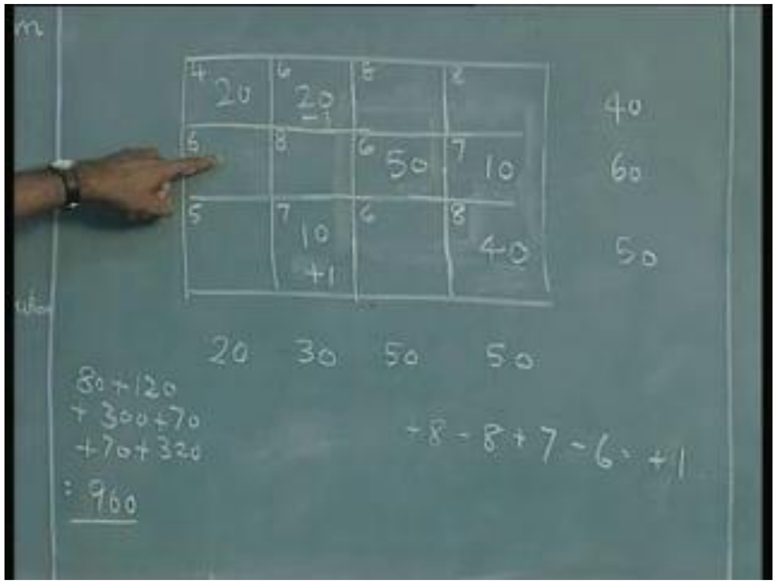

Now we try to put a + 1 in this position. Now we need to identify a corresponding loop and the loop is like this, from this to this, here 20 and back now. This is a loop. Please remember that in a loop from this + 1, you do not have to necessarily move into this. You could jump but the only thing is you move vertical and come to another position where there is an allocation. So net effect is, + 1 would make a – 1. Here to balance this, – 1 would make a + 1 here and this + 1 would make a – 1 here and the supply and demands are balanced. So the effective increase or decrease as the case may be this would give a + 8. This would give a – 8, this would give a + 7 this would give a – 6 so net is + 1.

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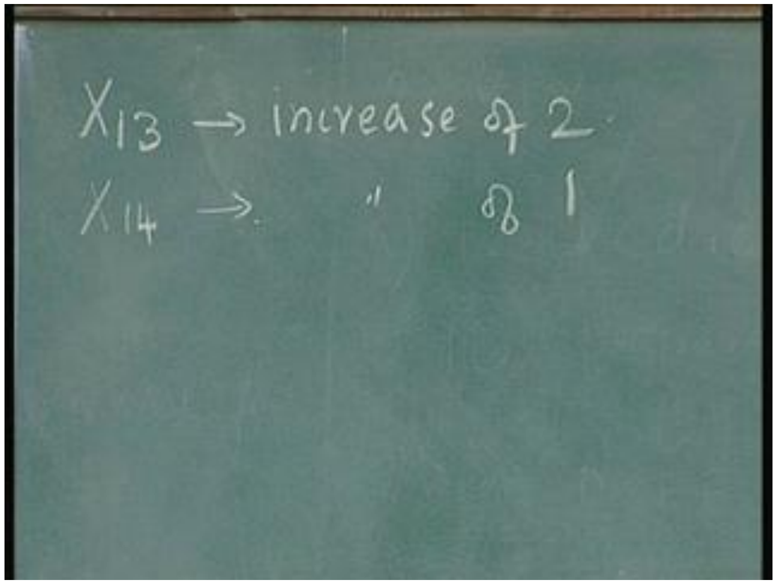

So if we consider $X_{1\,4}$ to enter the basis, we have an increase of + 1 for a unit increase in this position.

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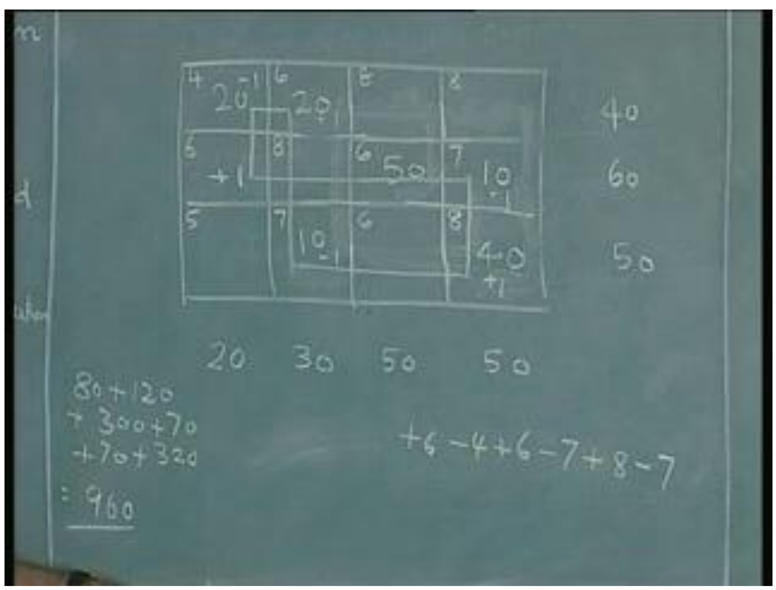

Now we look at this position for a possible increase, so we put a + 1 here. We need to identify a loop for this and let us see how the loop works out. From here we move to this 20. We come back to this 20. Then we go to this 10 and we go to this 40, back to this 10 and then back to this. It does not pass through the 50.

Now for the balance this + 1 would make a – 1 here, + 1 here, – 1 here, + 1 here, – 1 here and balanced. The net increase or decrease would be + 6 – 4 + 6 – 7 + 8 – 7 so we get 6 – 4 is = 2; 2 + 6 = 8 – 7 is = 1; 1 + 8 = 9 – 7 = 2.

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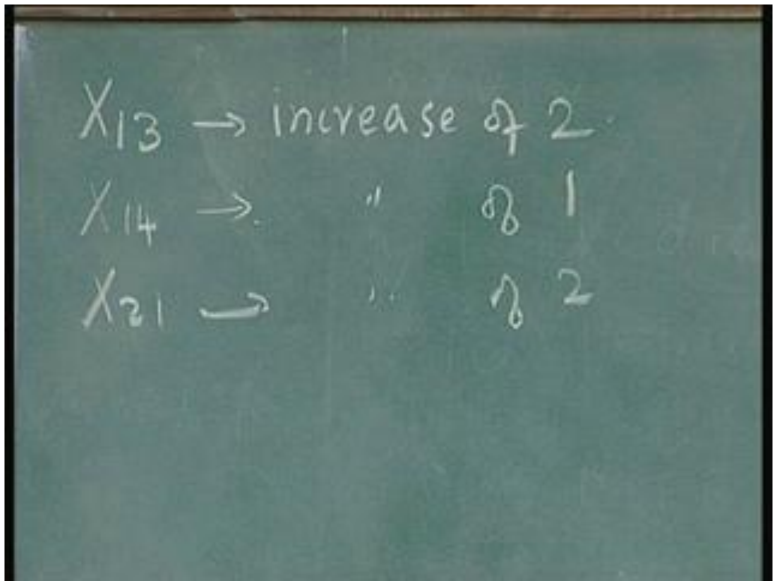

So if we consider 2 1 there is an increase of 2.

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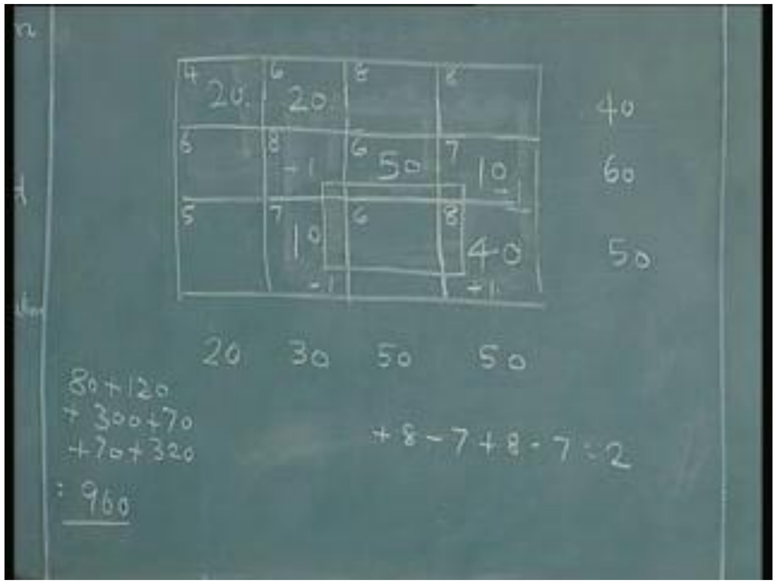

We consider another position which is this. We consider this position now and we put on + 1 here. Now this would mean we would complete a loop starting from this, we go to this position come down to this position, come back to 10 and go back again. The net effect will be this + 1 would become a – 1 here + 1 here and a – 1 here is still balanced. So the net effect is + 8 – 7 and from here + 8 from here – 7 here, so there is a net increase of 2.

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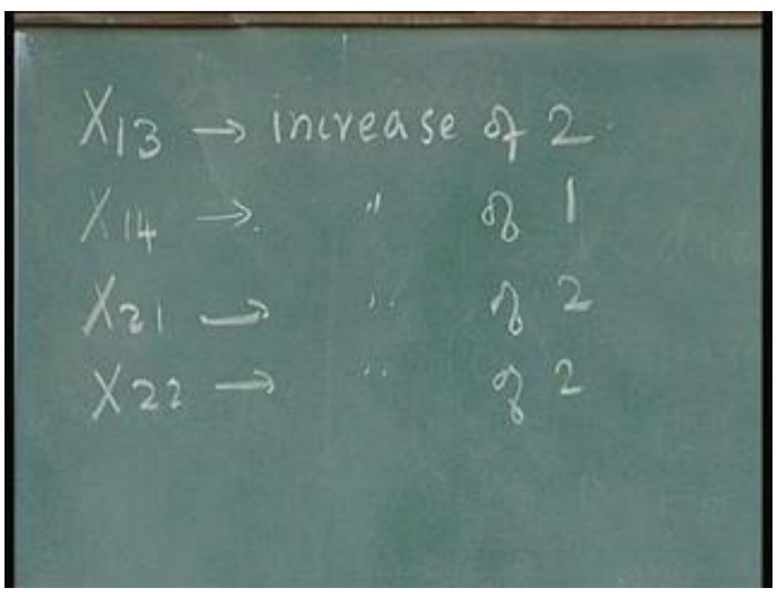

If we make the position $X_{2\,2}$ basic, there is an increase of 2. Now there are 2 more positions which need to be tried so we look at the remaining 2 positions here.

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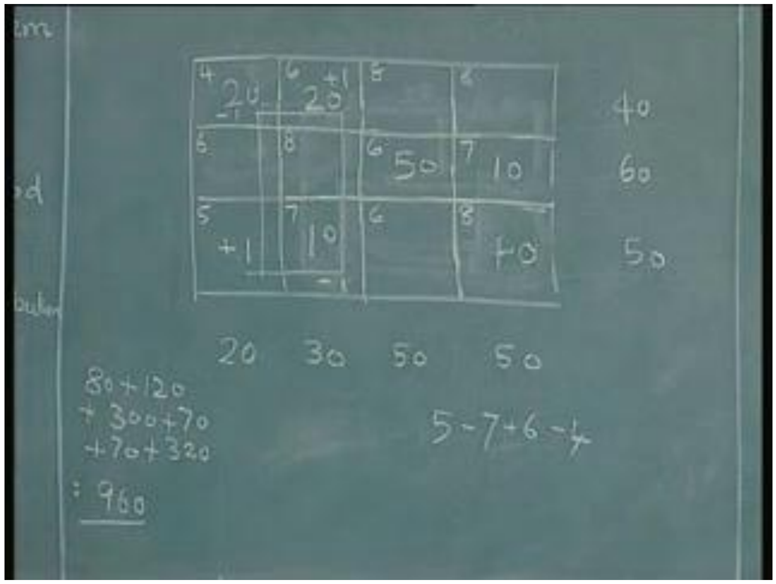

Now we tried to put a + 1 in this position. We need to make a loop so from here we move to this, we go back to this. We come back to this position and then we get back. So + 1 would become a – 1 here. For the balance now this – 1 will become a + 1 here and this + 1 will become a – 1 here so that supply and demand is balanced. The net would be + 5 because of this – 7 because of this + 6 because of this, – 4 because of this. So effectively the net increase or decrease is 0.

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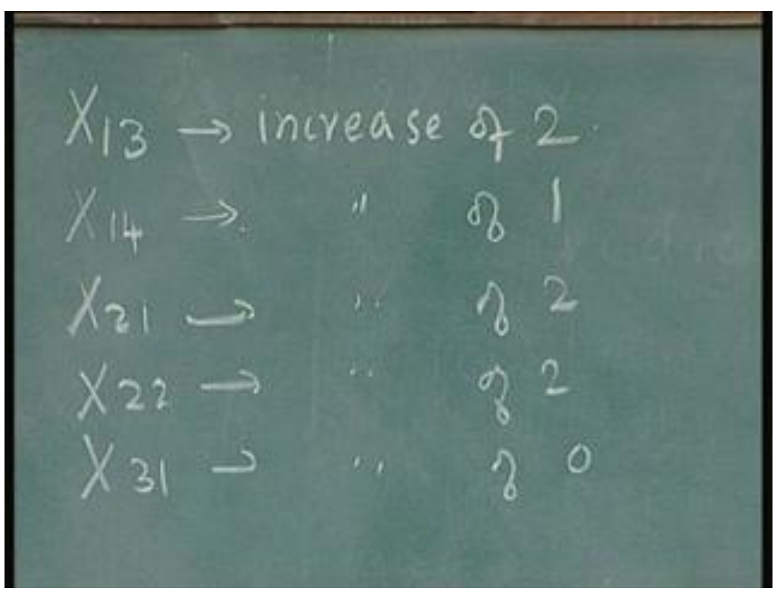

So putting $X_{3\,1}$ would give us an increase of 0 or decrease as the case may be. So we have one more position which is the last position here.

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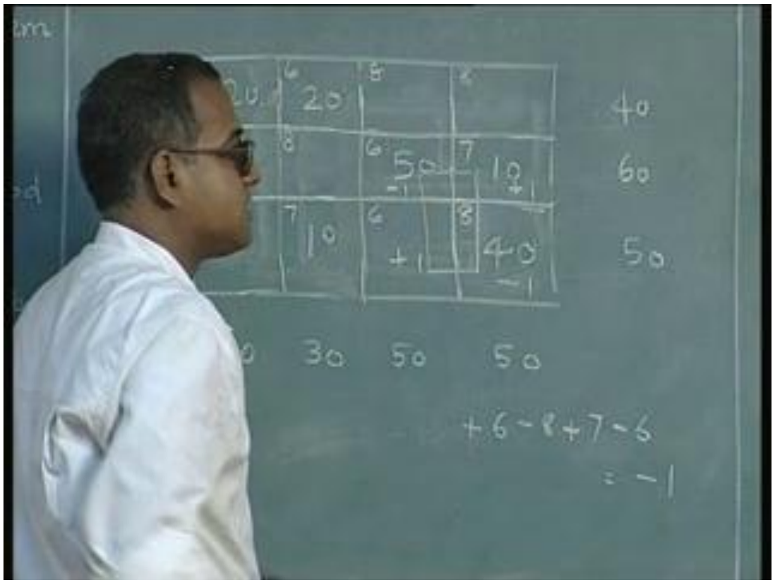

This is a last position so we put a + 1 here and the loop is like this. From here to this, from this to this, from this to this, from this to this, please remember that in a loop we have to move horizontally and vertically alternately and switch to position where there are allocations, so the net effect here would be + 6 – 1 here, + 1 here – 1 here so + 6 – 8 + 7 – 6. This is 6 – 8 is – 2 + 7 is = 5. 5 – 6 is = – 1.

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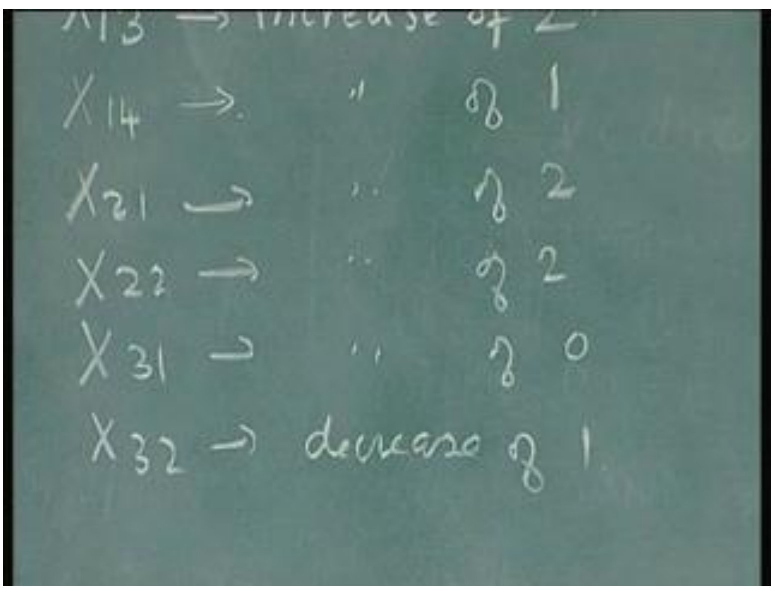

So X_{32} would give us a decrease of 1. We have evaluated it, trying to enter each one of the non basic positions. The 6 non basic positions we have looked at and then we realize that out of these 6 non basic positions 1, 2, 3, 4, 5 and 6.

We realize that putting a + 1 here can decrease the objective function and this is the only position where there is a decrease. There is also another position with 0 which right now we are not considering. So we look at trying to put 1 here. Now when we try to put 1, here this is the corresponding loop so we would per unit allocation here we know that there is a decrease of 1 possible. So we would like to put as much as possible in this so that we gain as much as we can.

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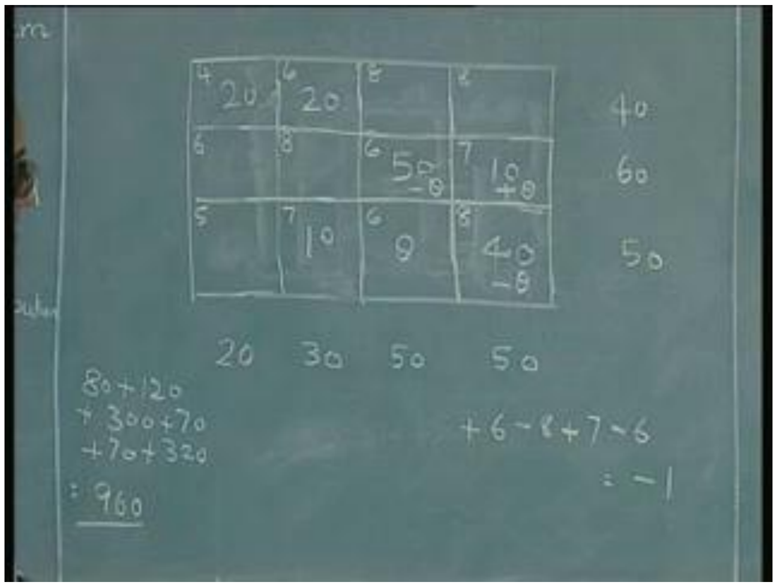

So we try to enter a theta here instead of 1 and then we complete the loop so theta would make this 40 – theta, 10 + theta and 50 – theta. Now we want our theta to be as large as possible so that we put maximum in this position. As we keep on increasing theta from 1, 2 we realize that these two are going to decrease. This is going to increase so when theta is 40 this will become 0 and increasing theta beyond 40 would make this negative which we do not want to happen.

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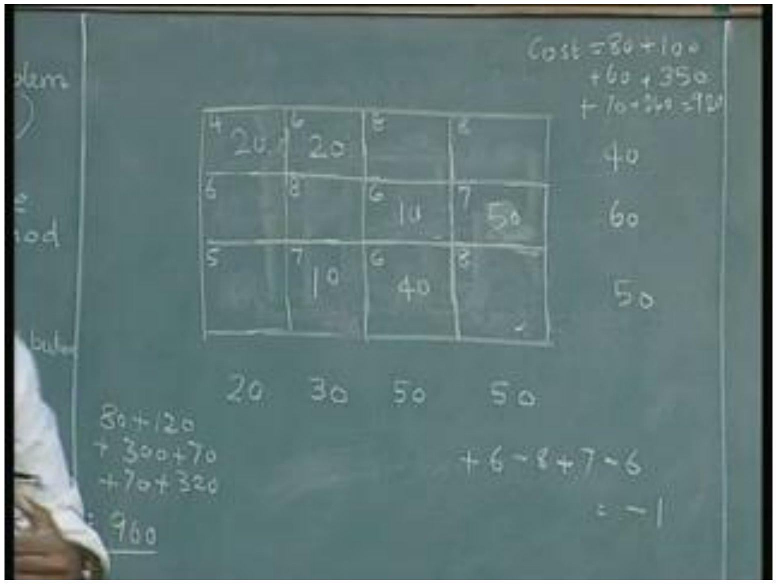

The maximum value that theta take is 40 which would make this 40. This 10, this 50 and this is 0 This is the next basic feasible solution which we have and for this solution, the cost associated with this solution will be 80 + 100+ 60 + 350 + 70 + 2 0 which is 200, 260, 610, 680 + 240 is = 920. So there is a reduction of 40 and that 40 is nothing but the product of theta into $C_j – Z_j$ as we have seen in linear programming. In this case, theta was 40, $C_j – Z_j$ was a decrease of 1, so the total decrease is 40, so 960 become 920. So now we want to check whether we go through one more iteration and we want to check whether this solution is optimal. To do that we there are now 6 non basic positions 1, 2, 3, 4, 5 and 6 and we try to put a + 1 in each of these positions and evaluate the effect of that.

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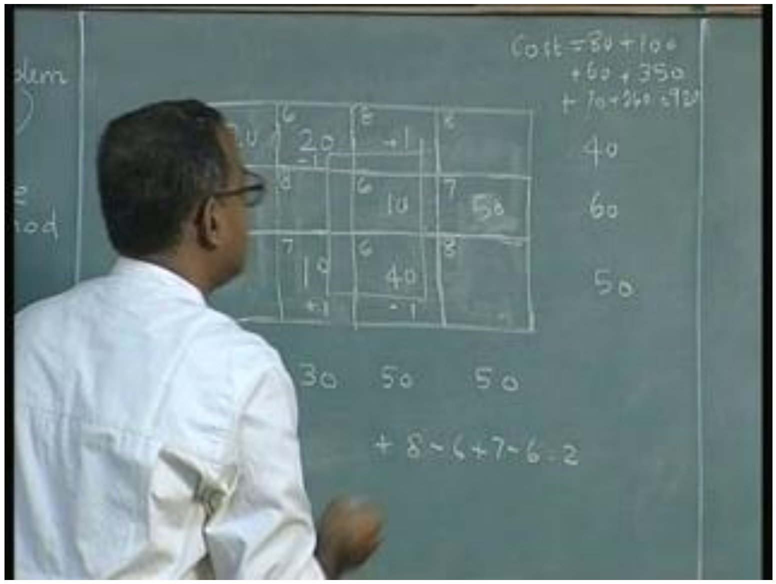

That is given by, if we put a + 1 here, now very quickly completing the loop and proceeding, the loop would be from here to this. Here this and back, so the net effect of the loop will be a, + 8 + 1 – 1 + 1 – 1 would give us **+ 8 – 6 + 7 – 6 which is 2.**

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So if we put in $X_{1\,3}$; there is an increase of 2.

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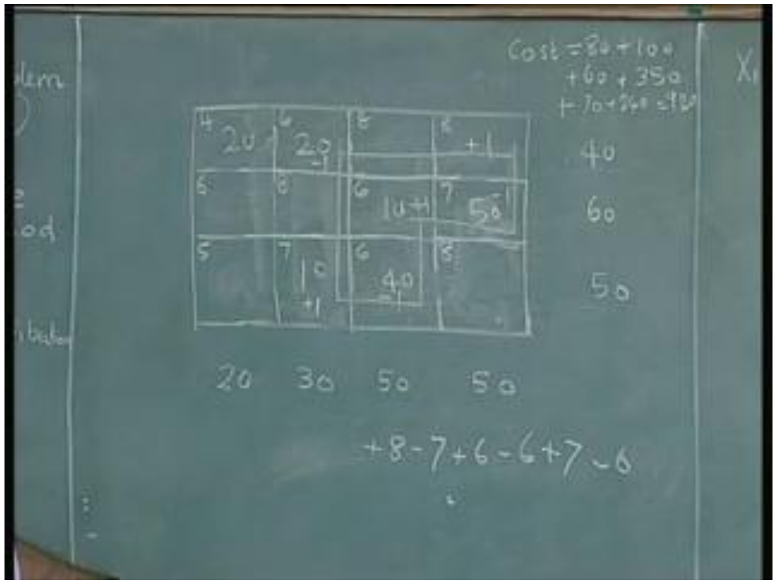

This is the next position. So we put a, + 1 here and your loop now comes from here to this, to this, back to this, from this and over. So + 1 – 1 + 1 – 1 + 1 – 1 the effect will be + 8 from here – 7 from here + 6 from this – 6 from this + 7 from this – 6 from this so 8 – 7 = 1; 1 + 6 = 7 – 6 is = 1; 1 + 7 = 8 – 6 is = 2.

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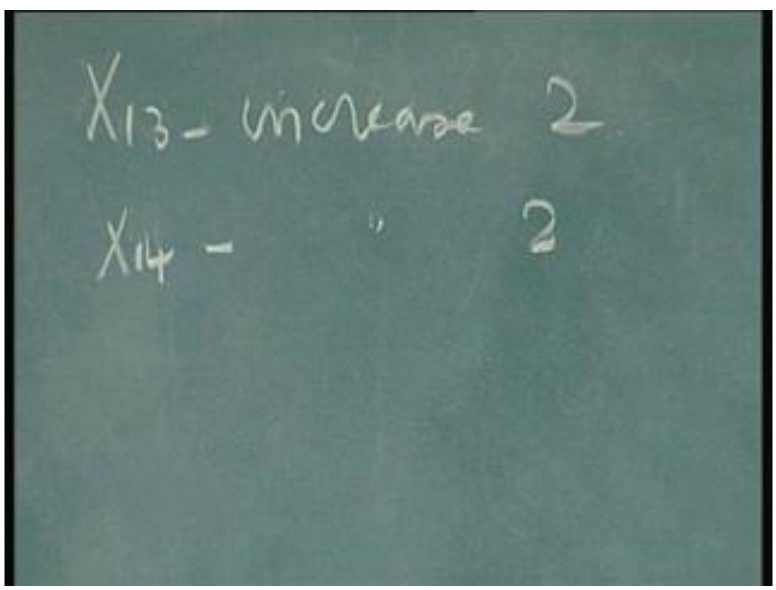

So when I put in $X_{1\,4}$. I have an increase of 2.

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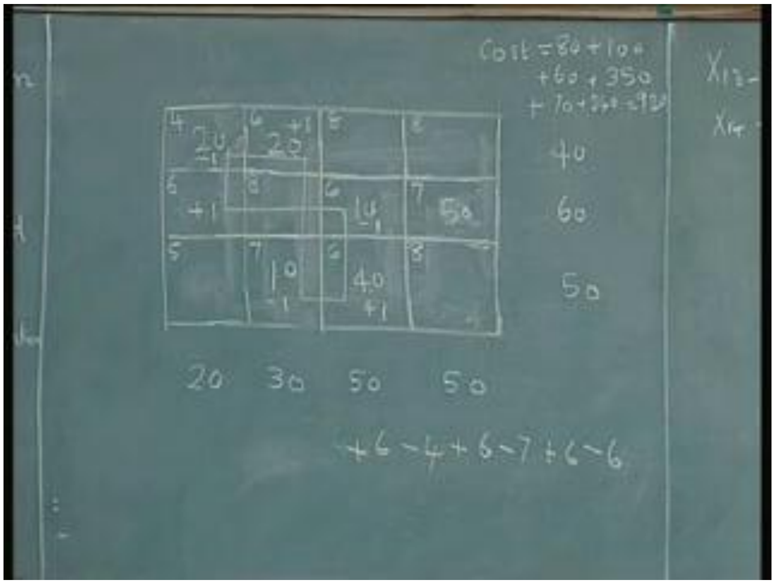

Look at third position here. This is my third position so I try to put a + 1 here and this is how the loop is. From this to this, this to back, to this, back to this goes here and comes back here. So this + 1 will become – 1 + 1 – 1 + 1 – 1. Considering the balance, the effect will be + 6 from here – 4 from here + 6 from here – 7 from here + 6 from here – 6. So it will be 6 – 4 = 2 + 6 = 8 – 7 = 1; 1 + 6 = 7 – 6 = 1.

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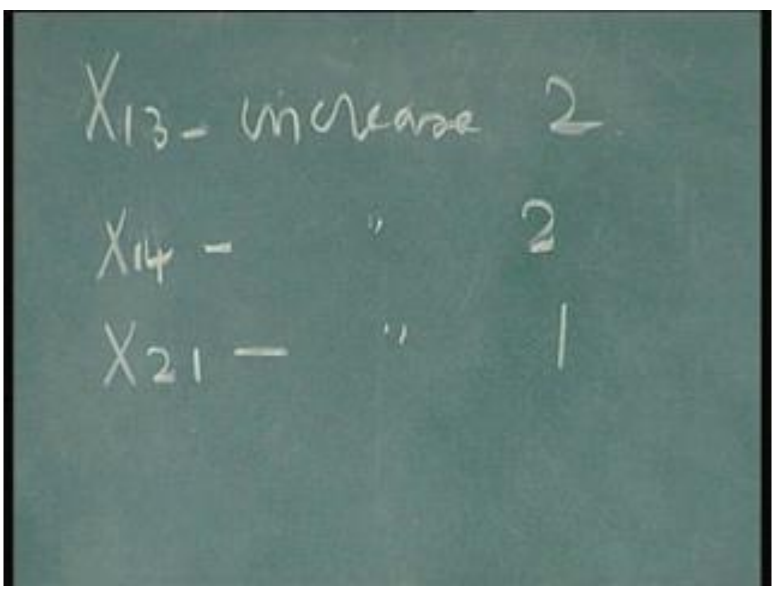

So putting it in X_2, one would give us an increase of 1. Let us go back and try the remaining 3 positions quickly.

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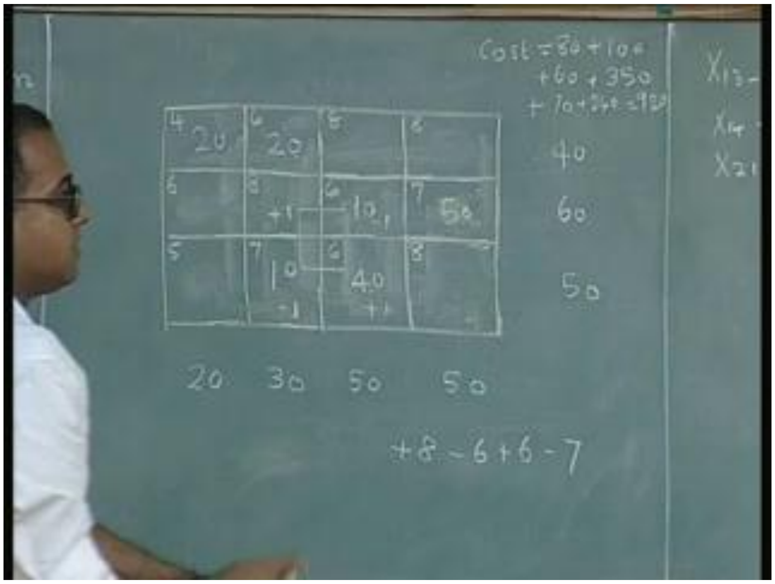

Now let us look at this position and put a + 1 here so the loop is like this, – 1 + 1 – 1 so the effect of this will be + 8 from here – 6 from here + 6 from here and – 7 from here so this is + 1.

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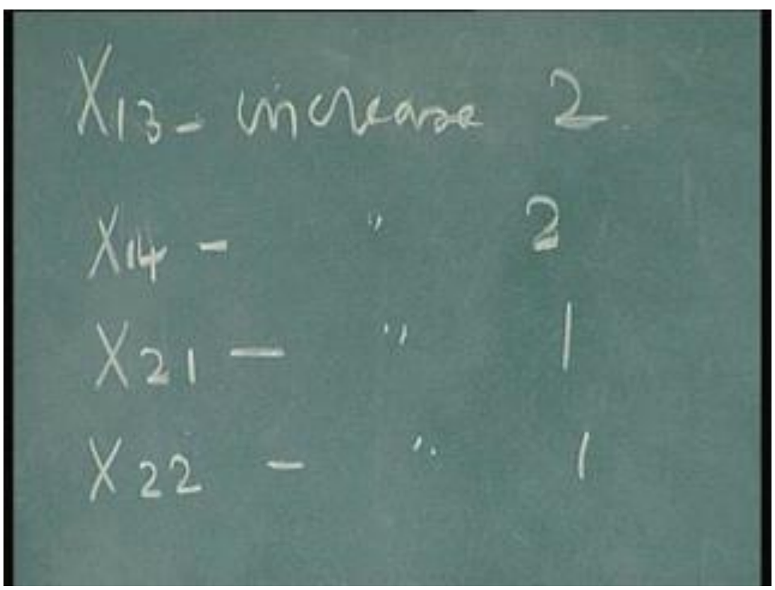

So trying to put it in $X_{2\,2}$ would give an increase of 1. There are still 2 more positions so we look at both these positions.

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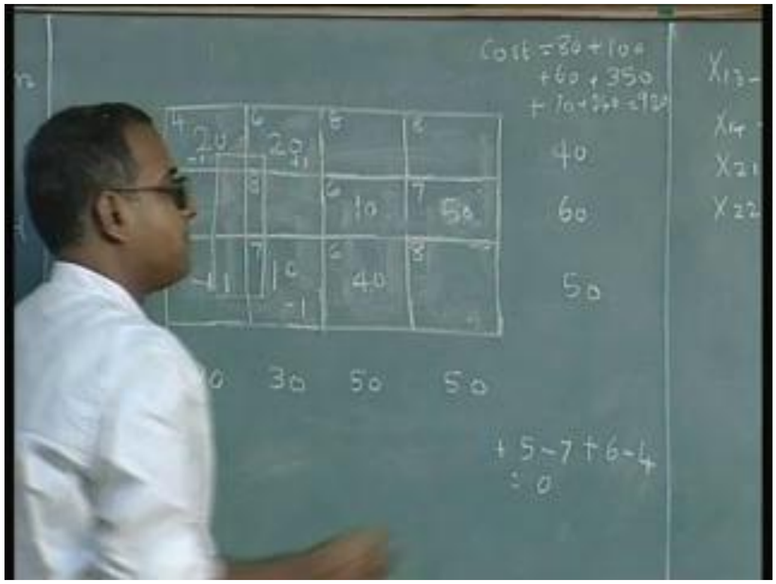

We look at this position so we put a + 1 here and the loop is like this, – 1 + 1 – 1 so the effect of it will be + 5 – 7 + 6 – 4 this will become 0.

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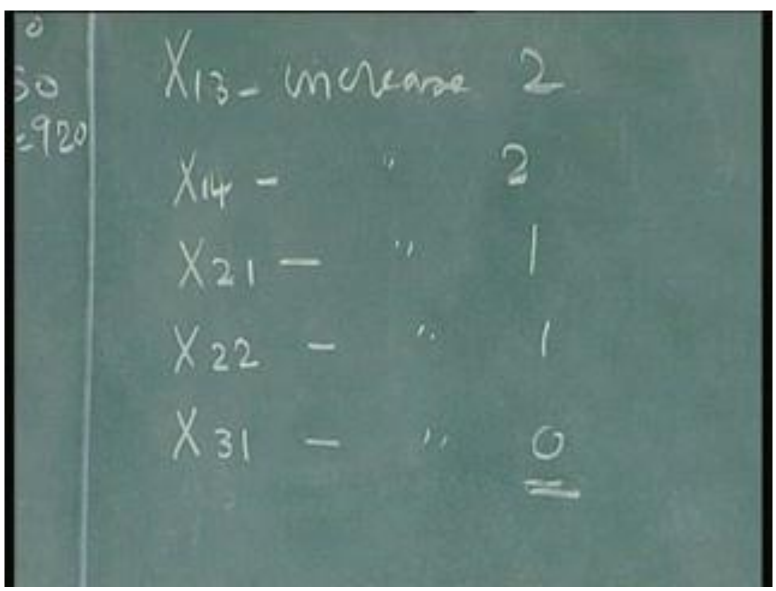

So $X_{3\,1}$ would give increase but 0 and this is a last position that we will look at.

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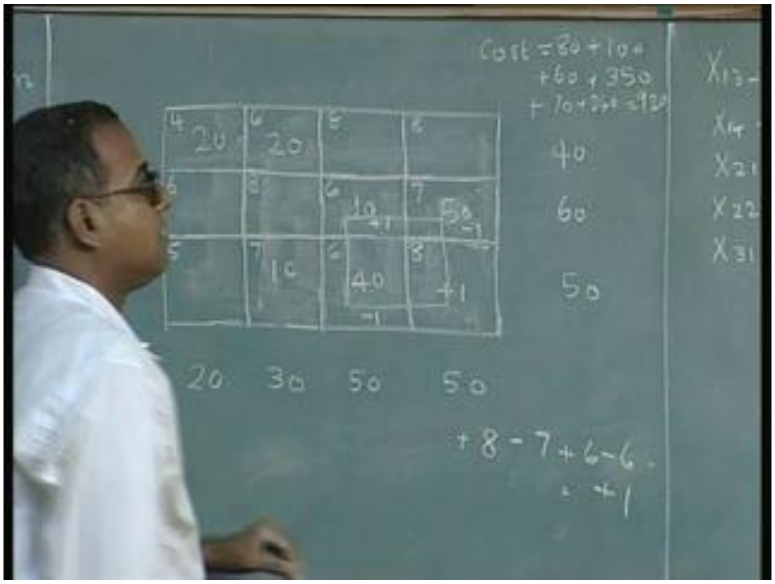

This is the last position so we put a + 1 and this is how your loop is – 1 + 1 – 1 so the effect of this will be + 8 – 7 + 6 – 6 which is + 1.

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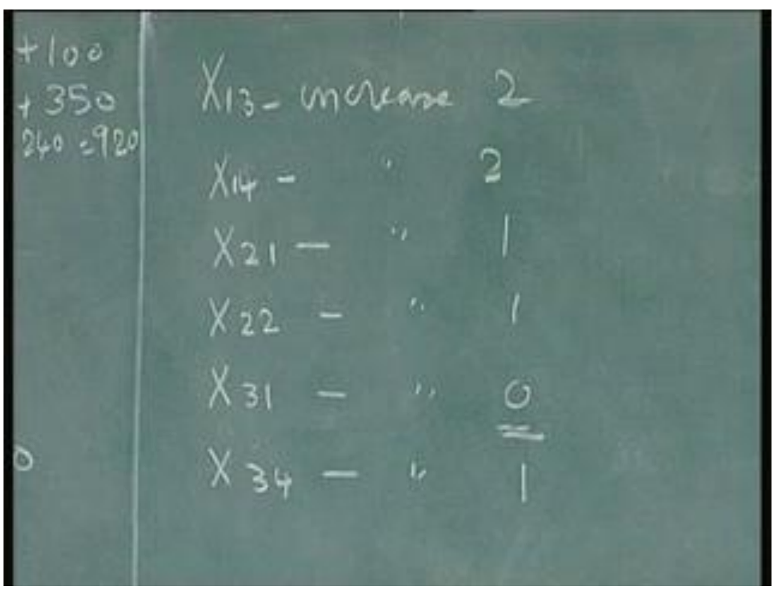

So $X_{3\,4}$ would give us increase of 1. Now we realize that there are 6 unallocated or non basic positions and putting + 1 in each would give us an increase of 2, 2, 1, 1, 0, and 1 respectively. None of them actually are capable of decreasing the objective function further from 920 to something lower than that so we realize that the optimum is reached. So putting any one of these does not decrease the objective functions. We have reached the optimum so the optimum solution is found and then the present solution is optimum because we are not able to identify a non basic variable which can bring down the objective function further. However we realize that at the optimum we have found one of the non basic variables that can contribute a zero increase or decrease. So this indicates alternate optimum. As we have seen in the simplex algorithm, the presence of a non basic variable with the 0 value of $C_j - Z_j$ indicates alternate optimum. Now this is a minimization problem so whatever we have found out here actually is $C_j - Z_j$ which represents the rate at which the objective function will go up or in this case go down because it is a minimization problem for a unit increase of the non basic variable. Because we have 1 non basic variable which can give us a zero increase or decrease at the optimum, this indicates alternate optimum and we try to find out the other alternate optimum solution.

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To do that we put a theta here, this is the position that gave us the alternate optimum. This will become 10 – theta, 20 + theta, 20 – theta. Now as we increase this, we realize that this and this are going to reduce and the limiting value for theta is 10 beyond which this will become negative.

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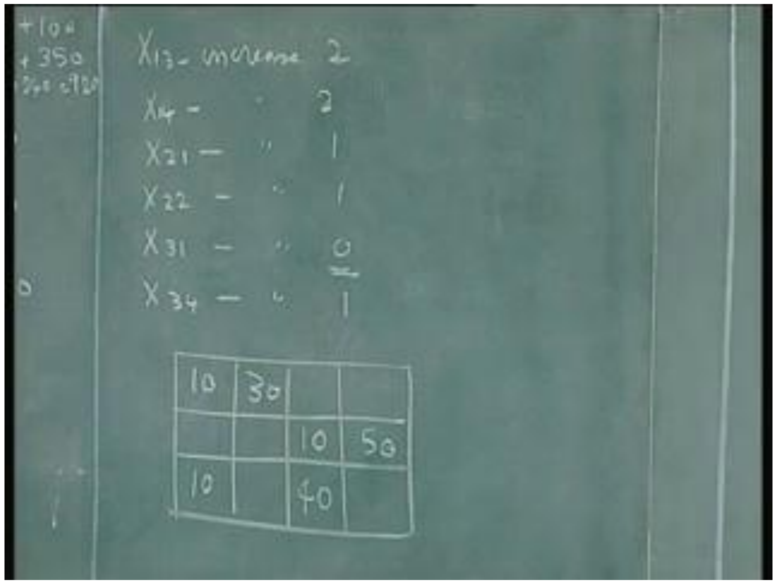

So theta takes value 10 and the alternate optimum would look like this (Refer Slide Time: 28:33) and the alternate optimum would give us a 10 here would give us a 40 here 50 here 10 here. 20 + theta will become 30 here and 20 – theta will become 10 here so this is one solution which is alternate optimum and this is the other alternate optimum. This is how the stepping stone method works. So in summary for a give basic feasible solution stepping stone method tries to enter every one of the unallocated position under the assumption that if the present solution is not optimal then at least one of the unallocated positions should have an allocation, so stepping stone method tries to do that. It exhaustively look at all the unallocated positions and if there is a decrease (because of putting a + 1 in the unallocated positions and completing the loop), it enters that and tries to move by reducing the objective function further.

Now at the optimum when we are unable to find an entering unallocated position we say that the optimum is reached which is very similar to saying that I am unable to find an entering variable or we are finding all $C_j – Z_j$'s positive in this case for a minimization problem the positive $C_j – Z_j$ indicates optimum negative. $C_j – Z_j$ would enter in the previous iteration. We entered a negative value here which indicates a decrease. So we terminate and show the optimum when all the vales here are positive however the presence of one unallocated position or a non basic variable with $C_j – Z_j$ 0 indicates alternate optimum and we can enter that and try to enter the alternate optimum solution.

Now we look at a second method which is called modified distribution method or the uv method and we explain the second example using the same example that we have here. So we reconstruct the problem once again.

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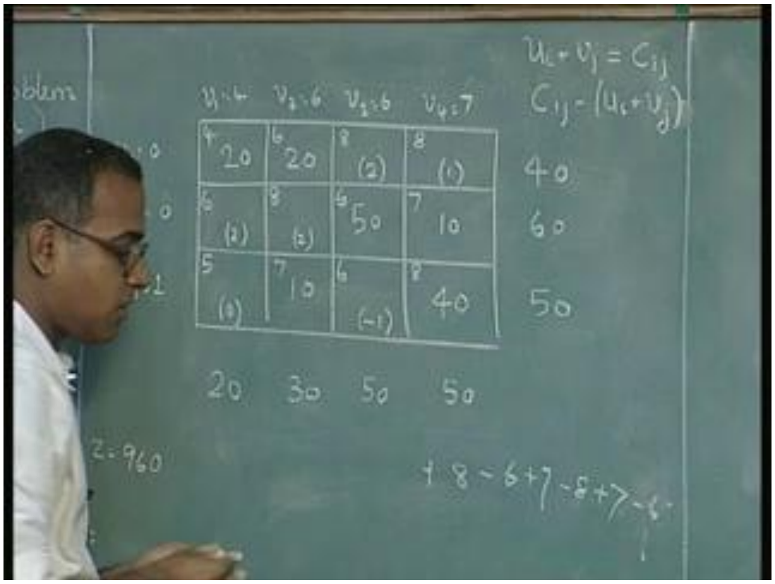

So the transportation matrix is like this supplies are 40, 60 and 50, demands are 20, 30, 50 and 50. Costs are 4, 6, 8, 8, 6, 8, 6, 7, 5, 7, 6 and 8 and we begin with a basic feasible solution. The same solution with which we started 20, 20, 50, 10, 10 and 40 so this has Z = 960 with which we started. Now what we are going to do in this uv method or modified distribution method is we try to compute some values of u and u. We associate u_1 = something, u_2 = something, u_3 = something, with the 3 supplies and we also associate v_1 = something, v_2 = something, v_3 = something, v_4 = something associated with these 3 or 4 requirements.

Now for consistency we always put $u_1 = 0$ we start with $u_1 = 0$ and now we look at the first row. Now there is an allocation here with associated cost being 4. Wherever there is an allocation, now find out either u or v whichever is to be found out such that $u_i + v_i$ is = C_{ij} in this case C_{ij} is 4; u_i is 0 so v_1 is 0 + 4 = 4. Now there is an allocation. There is a 6 here, this is a 0 here so 0 + 6 = 6; $u_1 + v_2 = C_{1\,2}$ and we have $X_{1\,2}$ has an allocation. Now we come back to this because for this allocation 10, C_{ij} is = 7. Now v_2 is = 6 therefore u_3 is = 1 so 1 + 6 = 7. Now having found $u_3 = 1$, we go back here we realize there is an allocation here. There is a 1 here so v_4 is 7 so 1 + 7 = 8.
Now that we have found out v_4 there is an allocation here. There is a 7 here so 7 + 0 = 7 and now we have an allocation here with a 6. There is a 0 here so v_3 is 6, 0 + 6 = 6.

Now we have 7 variables but we fixed one of them to 0. So we have 6 variables to find out. We have 6 allocations and because it doesn't have a loop, these 6 are independent allocations and therefore we will be able to find out the u_i's and the v_j's for any given basic feasible solution with independent allocations like what we have done. Now for all the unallocated positions, these 6 positions that we have, find out $C_{ij} – u_i + v_j$. Now for this one, $u_i + v_j$ is 0 + 6 = 6; 8 – 6 is = 2 so $C_{ij} – u_j + v_j$ is = 2; 0 + 7 $u_i + v_j$ is 7; 8 – 7 is 1; $u_i + v_j$ is 4; 6 – 4 = 2; $u_i + v_j$ is 6; 8 – 6 = 2; $u_i + v_j$ is 5; 5 – 5 is 0 $u_i + v_j$ $u_i + v_j$ is 7; 6 – 7 is – 1. So what we have found is, $C_{ij} = u_i\ C_j – Z_j$ corresponding to every non basic variable.

You will also observe if you go back to the stepping stone method that whatever we found out as the net increase is exactly the $C_{ij} – u_i + v_j$ you can go back and recollect that we recomputed the net increase or decrease corresponding each one of them and we got this.
For example just to illustrate one of this computations, if we had taken for example this if we had taken this, then your loop would have been from here to this, to this to this, to this to this, and back. So your net would be a + 8 from here – 6 because of this + 7 from here – 8 because of this + 7 from here and – 6 because of this 8 – 6 = 2; 2 + 7 = 9; 9 – 8 = 1 + 7 = 8 – 6 = 2.
So what we have found out as $C_{ij} – u_i + v_j$ or $C_j – Z_j$ is the net increase when we put a + 1 in this position. Now out of these, this is a_1 which has minus now. This increase, decrease and this variable can enter. Looking at it as $C_j – Z_j$, for a minimization problem a negative $C_j – Z_j$ will enter in simplex. We have seen for a maximization problem, positive $C_j – Z_j$ will enter for a minimization problem, negative $C_j – Z_j$ will enter, so this variable enters.

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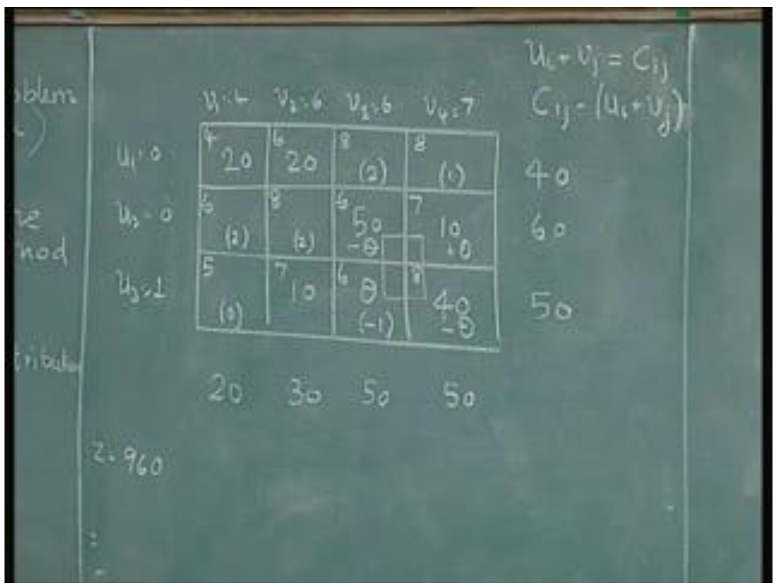

This variable enters we indicate a theta to find out the extent of entry the extent to which we can increase this variable to do that now we have to go back and compute a loop so this is the loop so this theta would mean 40 – theta, 10 + theta, 50 – theta. Now as theta increases these two decreases so, the limiting value of theta is 40 beyond which this will become negative. So we go back and re-do this table again.

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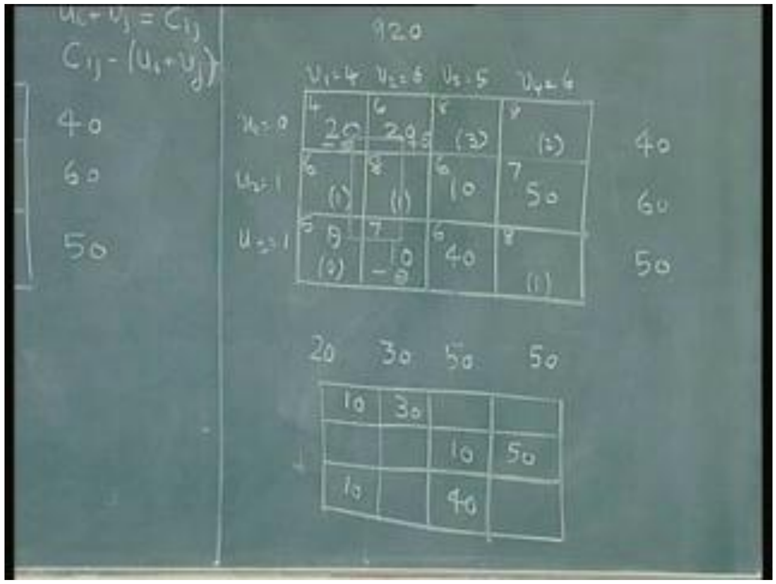

40, 60, 50, 20, 30, 50, 50, 4, 6, 8, 8, 6, 8, 6, 7, 5, 7, 6, and 8, so the new basic feasible solution would have a 20. Now theta being 40, 50 – 40 is = 10, 10 + theta will make it 50. This will become 0 and this will become 40 and this will remain as 10.
This is the new basic feasible solution and if we find out the cost associated with this, 80 + 120 is 200, 260, 260 + 350 is 610, 680, and 920. So the cost associated with this is 920 cost associated with earlier one was 960, there is a gain of 40 and in linear programming the gain is a product of $C_j - Z_j$ and theta $C_j - Z_j$ is 1 or – 1 in this case. Theta is 40, so the net decrease is 40 so 960 becomes 920.

Now we want to check if this is optimal using the uv method. So we start with $u_1 = 0$ always. Now there is an allocation here, this would give us $v_1 = 4$; 0 + 4 is 4, there is an allocation here therefore v_2 is 6; 0 + 6 is = 6. Now once we have found out v_2, we go back vertically here. There is an allocation under 7, so u_3 is 1, 6 + 1 is = 7. Having found u_3 there is an allocation here. There is a 6 here so v_3 is 5, 5 + 1 is = 6. Having found this v_3, we come back to this. There is an allocation here. Total is 6, u_2 is 1 and once we have u_2 is = 1, we go back. Here is an allocation at 7. There is a $v_4 = 6$, so we have found out all over u_i's and v_j's. Find out $C_{ij} - u_i + v_j$ for all the non basic or unallocated positions. Now these are the basic or allocated positions for the non basic variable or unallocated position, find out $C_{ij} - u_i + v_j$; 0 + 5 = 5; 8 – 5 is = 3; 0 + 6 is = 6; 1 + 5 is = 6; 1 + 6 is = 7 ; 1 + 5 7 6 8, 0 + 6 = 6, so you get a 2 here. 4 + 1 = 5, you get a 1 here, 1 + 6 = 7, so you get a 1 here, 1 + 4 = 5, 5 – 5 is = 0; 1 + 6 = 7, 8 – 7. Now we find that all $C_{ij} - u_i + v_j$ are positive with 1 0 there is no negative $C_{ij} - u_i + v_j$ so there is no entering variable or every non basic variable taken individually is not capable of decreasing the objective function further. So we say that the optimum solution is reached because all the $C_{ij} - u_i + v_j$'s are greater than or equal to 0, for minimization problem $C_j - Z_j$ positive indicates optimality.

But once again there is a $C_{ij} - u_i + v_j$ with 0 at the optimum. So it indicates alternate optimum. So we try to find out that alternate optimum to put a theta here. So when we put a theta here now, this is the loop so this will become 10 – theta, 20 + theta and 20 – theta.

Now this will become a new solution which is this. So we go back here and we can easily create. So now this 40 will remain as 40. There will be a 50 here; there will be a 10 here. Now this theta becomes 10. 10 – Theta is = 0, 20 + theta will become 30, 20 – theta will become 10. So this is the other optimal solution, the same solution that we saw using the stepping stone method. So this is called the uv method or the modified distribution method. One could use both, either the uv method or the modified or the stepping stone method. One could use any one of them but consistently we use the uv method and we later try to show why the uv method is desirable as the problem size increases.

Let us look at one more example to show how we get the optimal solution here. Now we take a case which has a degenerate basic feasible solution. We have already seen that we could have a degenerate basic feasible solution. A degenerate basic feasible solution is one where we have fewer than m + n – 1 allocation.
Let us take an example with a degenerate basic feasible solution for the same problem that we have used.

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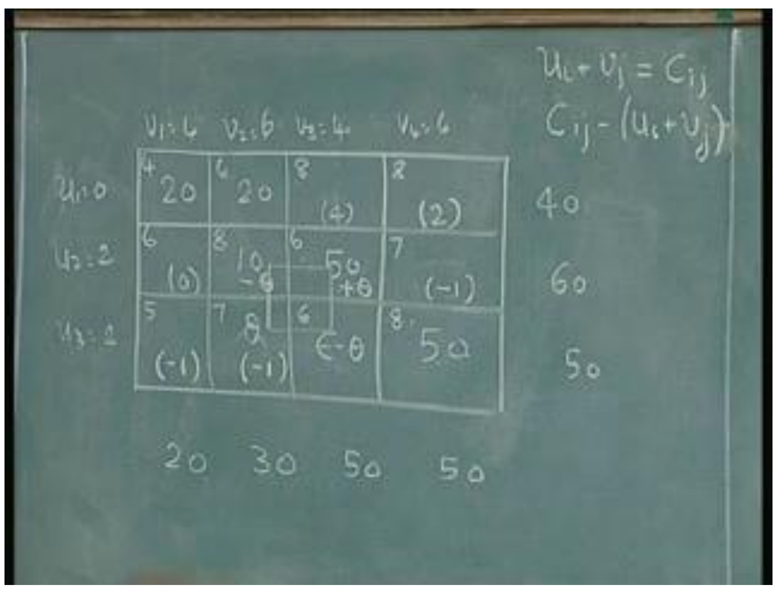

So let us look at the supplies, which are 40, 60 and 50. Requirements are 20, 30, 50 and 50 and these are 4, 6, 8, 8, 6, 8, 6, 7, 5, 7, 6 and 8 and we look at a different allocation for the degenerate basic feasible solution. Let us look at this solution which has 20, 20, 10, 50 and 50. Now let us look at this basic feasible solution to the transportation problem. This is a basic feasible row. Sums are 40, 60 and 50 respectively. Column sums are 20, 30, 50 and 50 respectively. Now this is degenerate because we have 5 allocations m + n – 1 is 6. Now the cost associated with this is 80 + 120 is = 200; 280 + 300 is = 580; 580 + 400 is = 980. Now let us try to solve this problem using the uv method. So we first put u_1 = 0, would give us v_1 = 4. There is an allocation here so this would give us v_2 = 6 so that 0 + 6 is = 6; now there is an allocation here so we get u_2 = 2; 6 + 2 = 8 there is an allocation here so we get v_3 = 4; 4 + 2 = 6. Now here, having identified this v_3 = 4; we do not have another allocation here which would give us this and to do that we need to put 1, presently all these 1, 2, 3, 4, 5 and 6 positions are unallocated there are 7, 1, 2, 3, 4, 5, 6 and 7 positions that are unallocated and we need to identify a place from which we will be able to compute either u_3 or v_4. So we try to put an epsilon. For example in this position, we tried to put an epsilon in this position so that this for now we have assumed that epsilon = 0 as an allocation. So u_3 now becomes 2, so 4 + 2 = 6 and now that we have an allocation here, 2 + 6, v_4 will be 6; 2 + 6 = 8

So whenever we have to degenerate basic feasible solution which means we have 1 allocation less than what we normally have, we need to put an epsilon in a particular place so that we able to get the u's and v's and we have to be very careful that we do not put epsilon in the wrong place. Epsilon should retain the fact that these 6 positions are independent. For example if we had put epsilon here then we realize that this would already have been a loop. An epsilon 50, 10 and 20 would create a loop and if there is a loop, there will be inconsistency in the u's and the v's. If you had put the epsilon here, then you realize that 0 + 4 is not = 8 and that is violated. So we have to make sure that the epsilons are put in such a way that they are independent and we are able to get the u's and v's and there is no inconsistency with respect to the u's and v's.
Now we go back and find out the $C_{ij} – u_i + v_j$ as corresponding to the 6 unallocated positions. So this is 0 + 4 = 4; 8 – 4 is = 4; 0 + 6 is = 6; 8 – 6 is = 2; 2 + 4 = 6. We have a 0 here, 2 + 6 = 8. So we have – 1 here, 2 + 4 = 6. We have a – 1 here, 2 + 6 = 8. We have a – 1 here.

There are 3 places which have a – 1, $C_{ij} – u_i + v_j$ is – 1, – 1 and so on. We could take any of these positions. There is a tie for 3 positions. We for example start with this position and we identify this as a position that can enter and we try to put a theta here. So when we try to put a theta here, we need to close the loop now. This would give us this loop. This would mean that theta epsilon – theta 50 + theta and 10 – theta. This would balance the supply and demand required.
Now what happens is as we put theta, these two starts decreasing. So the smallest value that theta can take is epsilon. Beyond which this will become negative.

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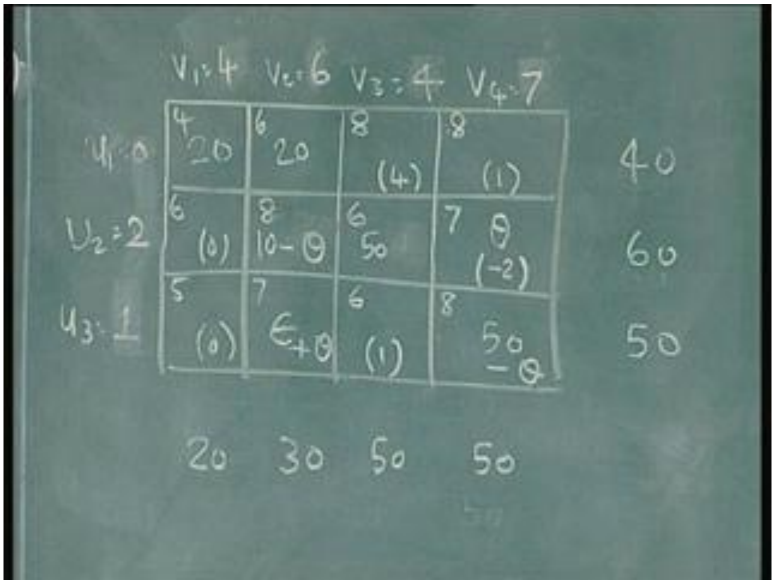

The value that theta takes is epsilon and your new solution now becomes 4, 6, 8, 8, 6, 8, 6, 7, 5, 7, 6, 8, 20, 20, 10. Theta is epsilon, so this becomes 10 – epsilon this is 50 + epsilon. This is epsilon and this epsilon – theta becomes 0. What is the value of epsilon? When we put epsilon here we meant to indicate that this position is basic but with value 0. The epsilon has to be used very judiciously now epsilon will take value 0 in this position but will retain itself as epsilon here indicating that this is a position where there is an allocation with 0. If we go back and evaluate the cost corresponding to this, you will get 200, 280, 280 + 300 is = 580; 580 + 400 is = 980. Cost did not change because $C_j – Z_j$ is 1 or – 1 but theta was epsilon which is 0 here, so the cost did not change. The only thing that effectively happened here was that the position of the epsilon got shifted; there was no reduction in the objective function. The exact reason is that degeneracy would not go through iterations where the objective function value does not reduce for a minimization. So the typical effect of degeneracy is just a shift of the epsilon.

The objective function value remains the same. We now proceed with this and try to get to the optimal. Now $u_1 = 0$ would give us $v_1 = 4$. Based on this $v_2 = 2$, based on this. Coming back $u_2 = 6$; 2 + 6 = 8. There is an allocation here $v_3 = 0$; 0 + 6 = 6. From this, $u_3 = 5$; 5 + 2 = 7. Now from this $v_4 = 3$; 3 + 5 = 8. Find out $C_{ij} – u_i + v_j$. Now this is 6, 0 + 4 = 4. Let us compute the u's and the v's again. This is 0 always, $u_1 = 0$. This gives us $v_1 = 4$, there is a 6 here, $v_2 = 6$. So $u_2 = 2$ from here $v_3 = 4$. Now 6, so this is + 1 and this is 7, 1 + 7 = 8 so 0 + 4 is = 4; 0 + 6 is = 6; 2 + 6 is = 8; 2 + 4 is = 6; 1 + 6 = 7; 1 + 7 = 8.

Find out $C_{ij} – u_i + v_{j,}$ 0 + 4 = 4; 8 – 4 is = 4; 8 – 7 is = 1; 2 + 4 is = 6; 6 – 6 is 0; 4 + 1 = 5; 5 – 5 is = 0; 1 + 4 = 5; 6 – 5 is = 1; 2 + 7 = 9; 7 – 9 is – 2. Now this is the only position that shows a decrease. So we put a theta here and we need to complete the loop. So 50 – theta, epsilon + theta 10 – theta. Now as theta increases you realize that these 2 are going to decrease so eliminating the value of theta is 10, beyond which this will become negative so theta takes value 10 in this.

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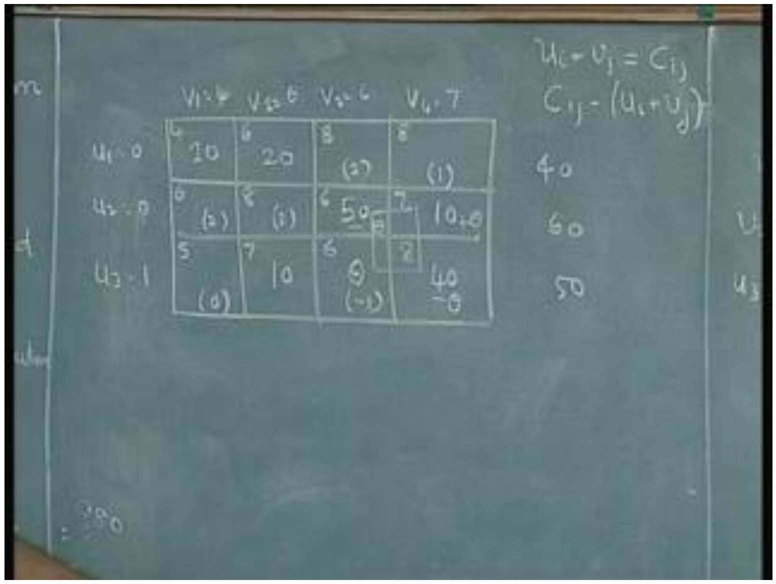

The new solution becomes 6, 8, 6, 7, 5, 7, 6 and 8. The 20 remains as they are now. Theta is 10. This becomes 10, 50 – theta will become 40, epsilon + theta epsilon was 0, so 0 + 10, 10 – theta will become 0 and we have another 50. We realize that here we had 5 allocations 1, 2, 3, 4 and 5 but here we have 6 allocations. We have come back now here. It has come out of degeneracy now. This is an example where it has come out of degeneracy from 5 allocations, we have got 6 allocations. We can proceed as we normally did towards the optimal solution. The gain would be now, in this case theta is 10; $C_{ij} – Z_j = 2$. So gain will be 20 so 980 would become 960 from now on, we proceed $u_1 = 0$; $v_1 = 4$; $v_2 = 6$. From here $u_3 = 1$; 6 + 1 = 7; $v_4 = 7$; 7 + 1= 8; $u_2 = 0$; 0 + 7 is = 7. Now $v_3 = 6$; 6 + 0 is = 6. Once again 4 + 0 is = 4; 6 + 0 is = 6; 0 + 6 = 6; 0 + 7 = 7; 1 + 6 = 7; 1 + 7 = 8. Find out the $C_{ij} – u_i + v_j$; 0 + 6 = 6, 8 – 6 is = 2; 8 – 7 is = 1; you will get a 2 here, you get a 0 here, 0 + 6 = 6, you get a 2 here, now 6 + 1 = 7 you will get a – 1 here. There is a – 1, there is 1 entering non basic variable which was a decrease in the objective function for a minimization problem. So put a theta here. The loop will be 40 – theta, 10 + theta, 50 – theta. Once again as we increase theta these two decreases, so the limiting value of theta is 40.

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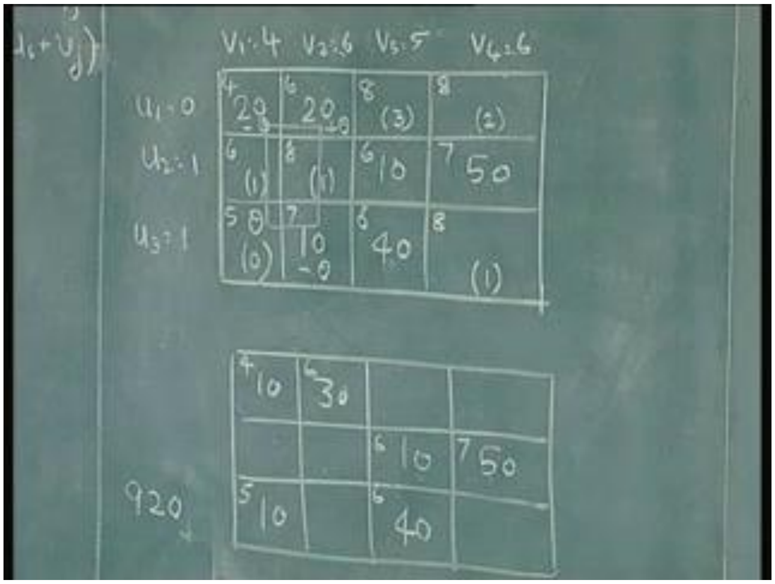

Now the new solution will becomes 4, 6, 8, 8, 6, 8, 6, 7, 5, 7, 6 and 8. We have 20 here. We have 20 here; we have a 10 here, now theta is 40. So this becomes 0, 10 + theta will become 50; 50 – 40 will become 10 and this theta is 40. Now the $C_j – Z_j$ is – 1, theta is 40. So there is a further reduction of 40 so 980 had already become 960; 9 60 – 40 is = 920. Now that is given by 80 + 120 = 200, 260, 610, 680 + 240 which is = 920. Once again finding u_i's and v_j's, we have 6 allocations. We do not worry about degeneracy. So u_1 = 0; v_1 = 4 from this v_2 = 6, from this u_3 = 1; 6 + 1 is = 7; from here v_3 = 5; 5 + 1 is = 6, from here u_2 = 1; 5 + 1 is = 6; v_4 = 6; 6 + 1 is = 7. Once again verifying 0 + 4 = 4, there is an allocation, 0 + 6 = 6, allocation 1 + 5 = 6, there is an allocation 1 + 6 = 7 allocation 1 + 6 = 7 and 1 + 5 is 6. Now once again finding $C_{ij} – u_i + v_j$ 0 + 5 = 5; 8 – 5 is = 3; 0 + 6 = 6; 8 – 6 is = 2; 1 + 4 = 5; 6 – 5 is 1; 1 + 6 = 7; 8 – 7 is = 1; 1 + 4 = 5; 5 – 5 is 0 and 1 + 6 = 7; 8 – 7 is 1.

All $C_{ij} – u_i + v_{j,}$ none of them are negative. All of them are positive or 0, now this indicates optimum. The same optimum solution that we had now indicates optimum because there is no non basic variable or an unallocated column or a potential entering variable that can reduce the objective function further. However we realized as we did earlier that there is one non basic variable or unallocated position which has a $C_{ij} – u_i + v_j$ or $C_j – Z_j$ value of 0 at the optimum so this once again indicates alternate optimum as we had seen, so we evaluate the alternate optimum by trying to put a theta here. Now we once again complete the loop. So this theta would mean this as a loop so 10 – theta 20 + theta and 20 – theta. The new solution will look like this. So this 40 will remain as 40, 10, 50 theta is 10 because once again as we increase theta, these two values reduce and the limiting value is 10 beyond which this will become negative so theta is 10, 20 + 10 is = 30 and 20 – theta is 10.

So we once again have a solution with 6 allocations. Once again with the same 920, 40, 6, 6, 7, 6 and 5 so this is 40 + 180 is = 220; 220 + 60 is = 280, 280 + 350 is = 630; 630 + 50 is 680; 680 + 240 is 920. Once again same value because alternate optimum $C_j – Z_j$ was 0 even though theta was 10, so the value is the same 920 from this. This is how we solve the problem if we have a degenerate basic feasible solution. Now we began this example with degenerate basic feasible solution and we saw a couple of things in 1 iteration that the epsilon nearly shifted its position. Every degenerate basic feasible solution would require an epsilon and epsilon is necessary to get the (m + n – 1)'th position or the missing position. The epsilon should be put in such a way that we are able to get the u's and the v's to retain the independence of the allocations making sure that there is no inconsistency in evaluating u's and the v's, so we need to put the epsilon in that manner. Because of degeneracy we could get into the situations where there will not be a decrease in the objective function in a subsequent iteration and the position of epsilon will change. That can happen when after we introduce a theta we get an epsilon – theta in 1 position. There we have to be very careful and to shift the position of epsilon from one to another. Rest of the allocations will remain. We could also have situation where the problem comes out of degeneracy by itself .As we saw we had 5 allocations and somewhere we got the 6th one so that will happen. Sometimes we have epsilon plus theta coming in as we put the theta and close the loop. If the problem has an alternate optimum then at the optimum that is when all the $C_{ij} – u_i + v_j$'s are strictly positive there will be 1 $C_{ij} – u_i + v_j$ with 0 which will enter and it will give us the alternate optimum.

So what we have seen today are the 2 methods. The stepping stone method and the modified distribution method also called the uv method and we also saw how we handle problems with degenerate basic feasible solutions using the uv method. We continue our discussion and compare these two methods in the next lecture.