Engineering Fracture Mechanics
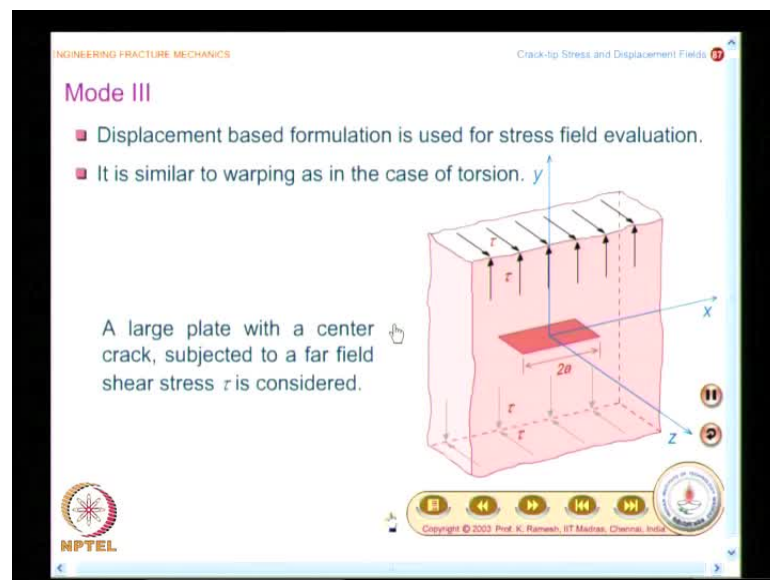
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Module No. # 04

Lecture No. # 20

Generalized Westergaard Approach

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In the last class, we have started doing the stress and displacement field. In the case of mode 3, we were half way in the solution development, and I asked the students to complete it and come, and I know the students psychology, you will wait for me to explain everything in the class and you get down to writing it. So, let us look at the mode 3 situation. In the case of mode 3, we have looked at, we have taken a through the thickness crack and it is subjected to anti-plane shear.

You know one of the students expressed a difficulty in visualizing this kind of a loading situation. In fact, if you look at your assignment two, I have given a variety of problems that are commonly encountered in engineering practice and I asked you to identify what is the mode of loading. And if you look at that exercise, you can easily verify for the problem of a shaft subjected to torsion, if you have a circumferential crack developed,

where it could develop? When you have to locate bearings or gears, you need to have some kind of a mechanism to locate and one of the usual approaches is to provide a circlip.

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So, you provide a groove on the circumference of the shaft; over a period of time, you could have a crack developed from this groove and you will have a circumferential crack and when it is subjected to torsional load, it is a candidate for a typical mode 3 loading situation. And so, you have to visualize from that point of view and this is shown as anti-plane shear in the diagram. And I said, for this case, we would go for a displacement formulation. So, you first look at the displacements and we have already said that the u is at displacement is actually a function of (x, y), very similar to your problem of a rectangular shaft in torsion, you have the warping function similar to that, you have a situation in this case also.

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From the displacements, you can find out the strains; these are all very standard quantities; we have looked that them in the last class, just for continuity I am showing these expressions. Once you get the strains, you can go for finding out the stress field.

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Now, the question is I need to have this w expressed and I want to have a nontrivial solution, so I should satisfy this equilibrium equation. When I substitute the expression for tau xz and tau yz in this, I get a Laplace equation; any harmonic function would satisfy this and we have defined this function as 1 by G imaginary part of Z 3.

So, in this case, the displacement is known; once you decide z which satisfies the boundary conditions, displacement is determined without any difficulty. You have to satisfy the Laplace equation and find out whether your w satisfies this and here you look at what Westergaard has proposed. You have X 3, the Westergaard stress function what we are going to use is for Z 3 prime that is given as tau z divided by root of z squared minus a squared and if you substitute it in the Laplace equation, this completely satisfies, if you substitute the what is w. And from your understanding of the crack problem, you know this stress function is very similar to what we have seen in the case of mode 1 loading as well as mode 2 loading.

There we have coined them as stress function; here you are using a similar form for defining the displacement and you have to take that this shear stresses out of plane shear and once you have this, the procedure is very similar to what we have looked at.

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So, what you will have to do is like we have done in the earlier cases, you have to shift the origin to the crack-tip. So, what you do is z becomes z naught plus a and you make a simplification; we are going to confine our attention to regions very close to the crack-tip, that means, z naught is very very small. So, when you do that, you have a simplification and Z 3 prime reduces to tau root of a divided by root of 2z naught.

And you can express this in terms of r n theta and write out what is Z 3 prime. So that turns out to be K 3 by 2 pi r root of 2 pi r multiplied by cos theta by 2 minus i sin theta by 2. This is the very simple and straight forward step; we have always been doing similar exercise in the case of mode 1 and mode 2, so you can comfortably write this.

And you have an expression for tau xz and tau yz since you know Z 3 prime, you could write them also, tau xz is nothing but imaginary part of Z 3 prime. So, when you look at the imaginary path here, this is minus of K 3 divided by root of 2 pi r sin theta by 2 and tau yz is the real part of Z 3 prime, which you can write it as K 3 by root of 2 pi r cos theta by 2.

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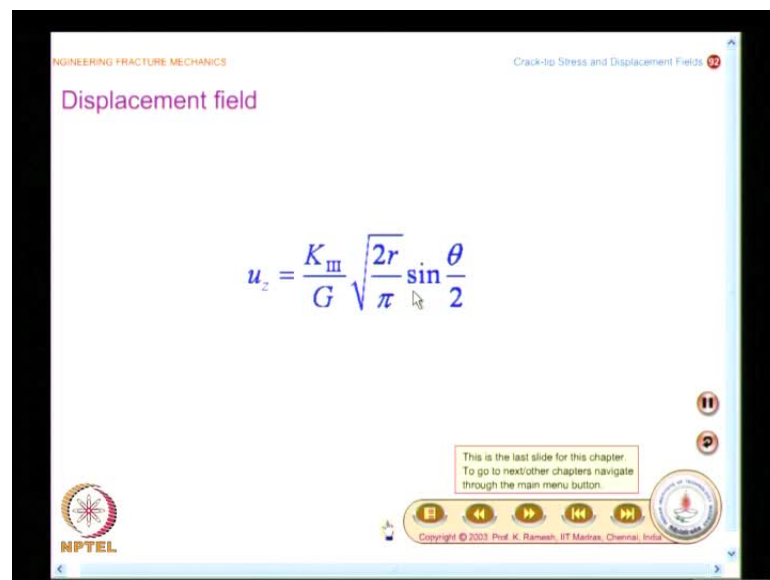
So, once you look at these expressions what do you notice? This also has root r singularity; you have essentially shear stresses and the strength of the stress field is dictated by the stress intensity factor K 3 and that distribution is decided by your function in terms of r n theta. And you have that theta function as sin theta by 2 and cos theta by 2 and the displacement is straight forward; we already know this, we write it in terms of r n theta. So, this turns out to be u z equal to K 3 by G root of 2r by pi into sin theta by 2. In fact, if you compare these expressions with what you had seen in the case of mode 1 and mode 2, the displacement field is bounded; when r tends to 0, displacement is a finite quantity in-fact, it goes to 0; whereas stresses become theoretically infinite, because it is a singular point.

So, the similar scenario you also come across in mode 3 situation; the stresses are very high near the crack-tip, the displacement field is bounded and because we have taken z naught very small in comparison to the crack length, the solution is valid only in the near vicinity of the crack-tip that also we have emphasized.

Now, what we will have to do is, we will have to go and reinvestigate the kind of questions I raised when we looked at the mode 1 loading situation. The question I raised was, whether the solution obtained by Westergaard was sufficient or not.

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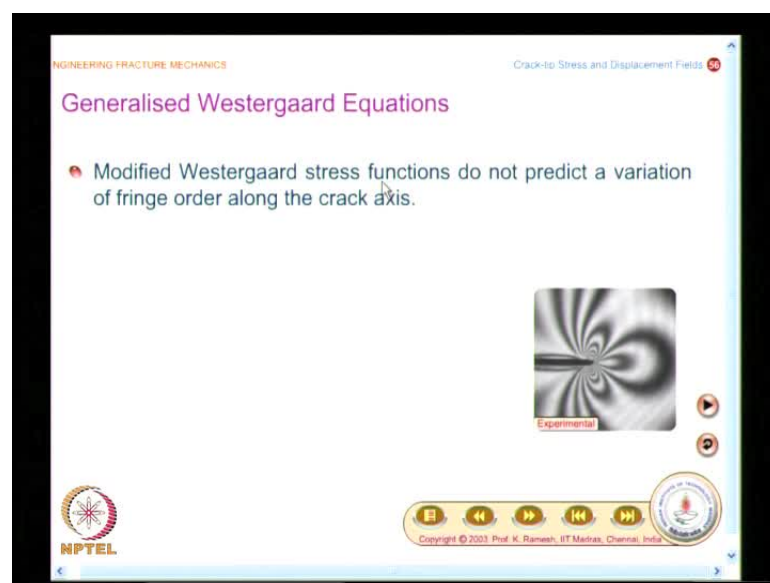
So, we will now take up our discussion on generalised Westergaard equations. This is necessary, because we have looked at Westergaard stress function; we have also looked at modified Westergaard stress functions, what are the modifications that we have looked at? We have looked at a modification introduced by Irwin; he added minus sigma naught x to the sigma x stress term.

So, he added one more term in the series, this was one of the modifications; the other modification was introduced by Tada, Paris and Irwin what he did was, instead of simply taking only the first term when you are simplifying z naught is very small compared to a, he had expressed the denominator as a binomial series and allowed as many terms that could be used. But if you really look at, it was not predicting any fringe order along the

crack access, at least the modification by Irwin said that you need to have a constant fringe order along the crack axis, but what you see in reality?

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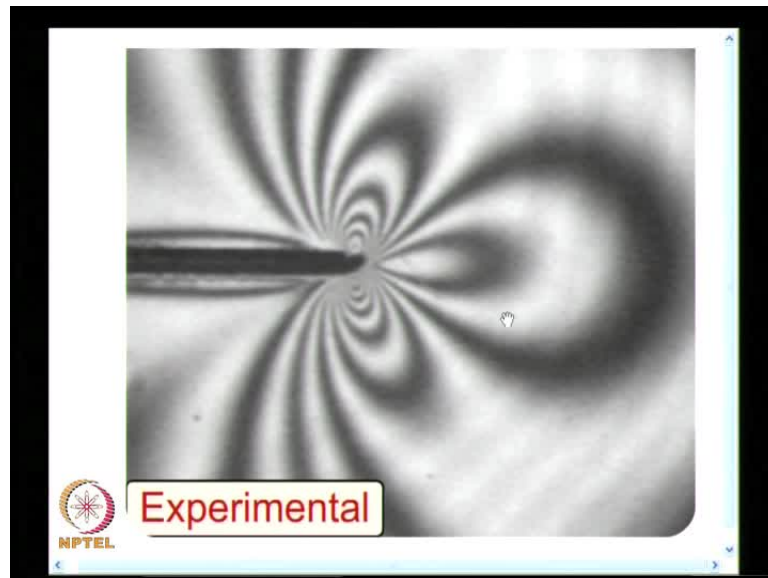
In certain situations, you see fringe patterns like this. So, what you find here is along the crack axis, you have many fringes; you have fringes forward till tilted as well as a frontal loop and this is obtained in an experimental scenario, unless your basic equations of stress field explains this phenomena, the solution is not contained. See if people have not looked at photo elastic fringes, they would not have raised a question like this.

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This shows something has gone wrong in our solution development. So, we have to re have a relook at it and what is the kind of a modification that you can think off?

Sanford introduced an additional stress function Y z to Westergaard stress function capital Z z to explain this behavior. You know this is very unusual; see you have taken a stress function, in most of the problems somebody gives you the stress function, you only ensures whether it is satisfy the boundary condition, whether it satisfy the bi-harmonic equation and then, you proceed with it; you never question, whether these stress function is complete or not in fact, you wanted tau xy to be 0 along the crack axis as stress function was giving it.

But I had emphasized in our earlier discussion, though we wanted tau xy to be 0 along the crack axis, without our emphasize, the solution what we got from Westergaard stress function was the maximum shear stress was also 0. In the case of Westergaard stress function as well as the modification by Tada, Paris and Irwin we did not want it; we have only specified tau xy to be 0; we did not impose the maximum shear stress is also should be 0, but what the solution you got was the maximum shear stress is also 0.

So, in the case of actual experimentation, you find there is a variation of maximum shear stress, so this needs to be explained. So, from this stand point, Sanford introduced additional stress function capital Y z, is it justified this also we have to look at it. And the Airy's stress function is modified in this fashion, you had earlier seen phi equal to real part of Z double bar plus y imaginary part of Z bar, this is what we had seen for the mode 1 problem. Sanford introduced an additional term y imaginary part of Y bar, where the stress function y is given as function of two other analytic functions psi and chi.

See the moment you introduce something like this, the proof of our introduction is what? It should explain what is seen in an experiment. So, when I have a very generic solution, particular cases should reduce to what are the simplified cases, which we had seen earlier; if it satisfies that, then we can accept, but whatever the solution that we are getting now is the most general form of stress field equations or most general form of stress functions for the given problem that kind of a understanding that we could develop.

And what Sanford did was, in theory of elasticity you have a very famous approach known as Kolosov Muskhelishvili approach for solving problems using complex variables. So, he used this approach to explain the additional term y imaginary part of Y bar.

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$$\phi = \operatorname{Re}\bar{\bar{Z}} + y\operatorname{Im}\bar{Z} + \jmath$$

$$\text{where, } Y = f(\psi.$$

Sanford used Theory of E
Muskhelishvili to explain the add

For Further details see:
R. J. Sanford, A Critical re - examination of Westergaard method for solving opening mode crack problems. Mech. Res communication, 6(5), 289 - 294, 1979

In fact, you could see some of this in his publication. This was published in 1979 by Sanford; the paper was a critical reexamination of Westergaard method of solving opening mode crack problems appeared in mechanics research communication.

So, you could go and have a look at the paper and before we get into this Kolosov Muskhelishvili approach, we will also look at the relevance of these fringe patterns, why this fringe patterns was taken as the candidate to verify the inclusion of the stress function Y?

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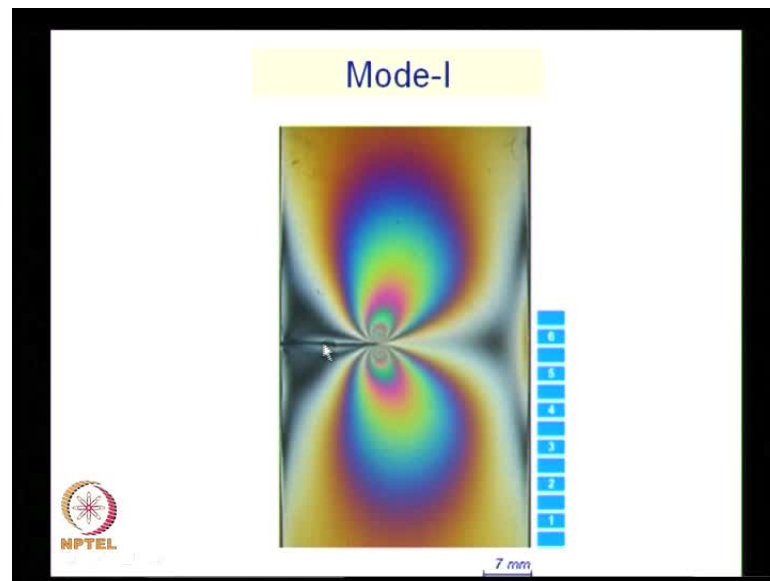
Let us now look at the typical mode 1 fringe patterns. If you look at here, you see the boundary of this specimen and I have a crack coming from one of the edges and this is known as a single edge notched specimen. I have a crack from this edge, I have the other edge shown here and if you look at crack is somewhere in the middle or slightly less than the middle. So, the a by w ratio would be something around 0.4 it is not 0.5, but a by w is around 0.4.

And you could also have a situation, where the crack length becomes longer and longer and the crack becomes closer to the free boundary. So, what people have noted is, when the crack goes very close to the free boundary, you have the frontal loops develop like what we had seen earlier.

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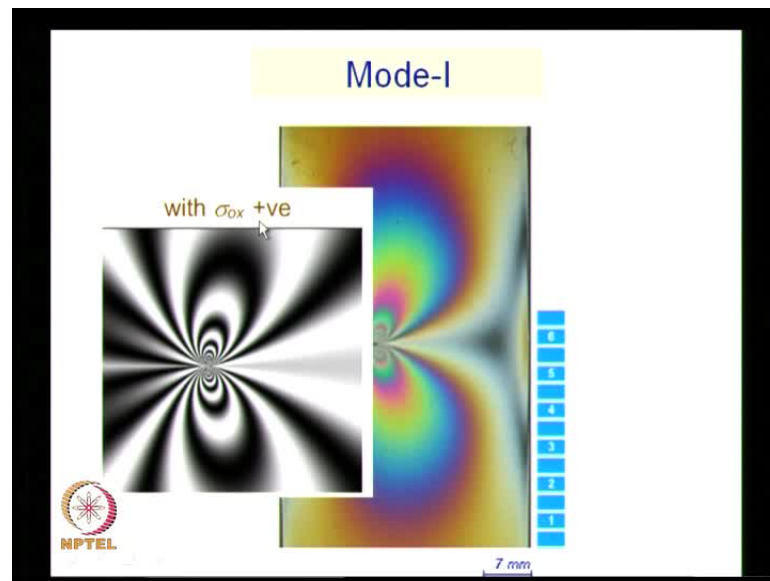
And if you look at the Irwin modification, for a crack which is short enough, the fringes are forward tilted that aspect is captured by Irwin solution and it showed a constant fringe order along the crack axis.

So, this modification will not be applicable when that crack length increases further and comes closer to a free boundary.
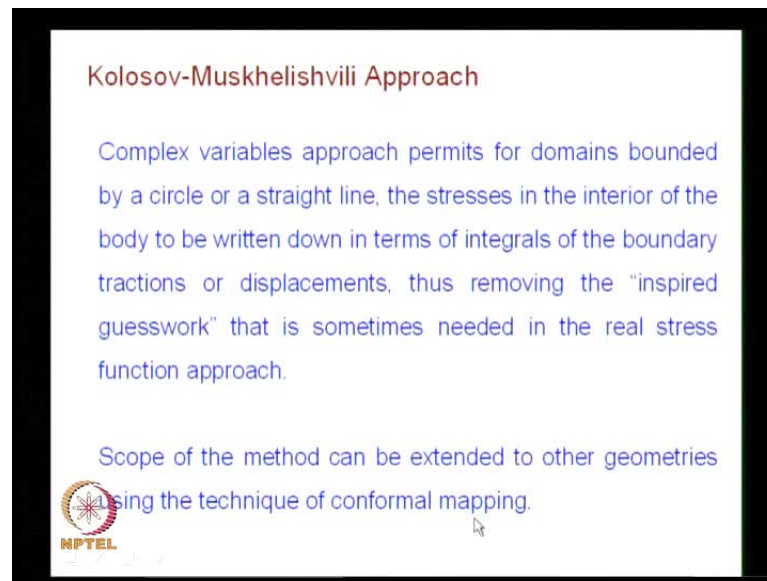
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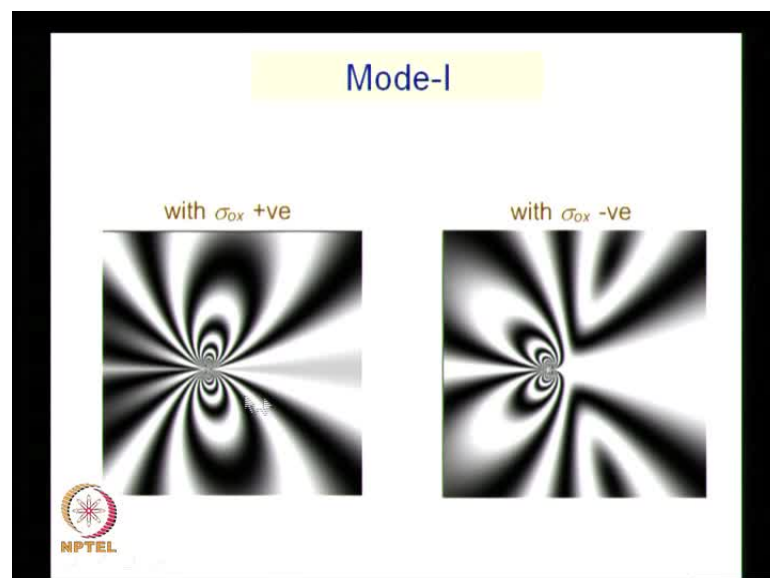
See if you recall, we in fact got the frontal loop situation for the case of a crack in a pressurized cylinder, the crack was long enough; it was close to one of the boundaries of it and I also mention it is in a stress concentration field. So, in that case you got a frontal loop. People also have noticed when that crack length is longer in the case of SCN specimens, you have a frontal loop and we also noticed earlier by modifying what is sigma naught x the sign positive or negative, the characteristic fringe patterns you see near the crack-tip changes totally; when it is negative, mind you in my solution I have taken minus sigma naught x; in that minus sigma naught x when sigma naught x is positive or sigma naught x is negative is what is given in these interpretations. So, only those type of representation should be looked at only then this positive and negative has significance.

So, here I have cracks tilted backward and this kind of a situation is seen in the case of a different type of specimen known as rectangular double cantilever beam specimen. We have already looked at double cantilever beam; we had very thin top and bottom portion, suppose I have that broad enough and I have a specimen long enough like this, in that situation if you have a crack and if you have the loading is applied, you find the fringes are backward tilted.

So, people are happy with Irwin solution that by changing the sigma naught x suitably as positive or negative, they could analyze short cracks in SCN specimens and they could analyze short cracks in the case of RDCB specimens. Suppose in this specimen also if the crack advances and comes closer to the boundary, imagine that this is the boundary of the specimen, then again experimentally it is recorded, and the fringes have forward tilted loops and a frontal loop.

So, people have seen when the crack-tip is closer to a boundary and it is long enough, you invariably have frontal loops that means, maximum shear stress varies along the crack axis. So, this needs to be captured in your representation of the stress field; if it is not captured, then you are not able to satisfy what is observed in experiment, only here the approach of Kolosov Muskhelishvili has helped. We will see what it is.
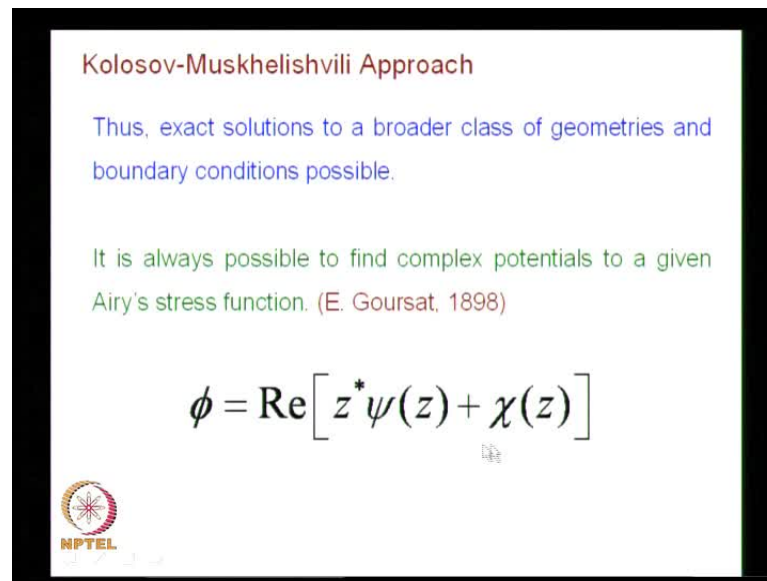
Kolosov Muskhelishvili approach uses complex variables and what is the advantage is, this approach permits for domains bounded by a circle or a straight line, the stresses in the interior of the body to be written down in terms of integrals of the boundary tractions or displacements, this is a key point.

If you specify the boundary conditions clearly, on that basis it is possible to find out the stresses and what way this has benefited? This has benefited by removing the inspired guesswork that is sometimes needed in the real stress function approach.

The idea is this, in our stress function approach, we have always said how do you coin the stress function is important. We have said, we would coin a stress function and go and investigate which problem it represents; we are not really looking at the problem from satisfying the boundary conditions, you are not evaluated the stress functions; we are having only a semi inverse approach.

So, if the stress function is given, what problem it solves that is the way we are looked at it and how do you arrived the stress function? We have said it is by intuition, it is by trial and error; there is no specific methodology, whereas the complex variables approach provides a methodology. But people have reported the methodologies are quite involved that is also a recorded by people. But the advantage here is the scope of the method can be extended to variety of geometries using the technique of conformal mapping.

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Kolosov-Muskhelishvili Approach

Thus, exact solutions to a broader class of geometries and boundary conditions possible.

It is always possible to find complex potentials to a given Airy's stress function. (E. Goursat, 1898)

$$\phi = \mathrm{Re}\left[z^{*}\psi(z) + \chi(z) \right]$$

So, if I have a complex boundary value problem, if you are able to identify a conformal mapping, for example, plate with an elliptical hole, they could find a conformal mapping and then, put that assessor and then, solve the problem. So, conformal mapping goes hand in hand with complex variables approach and what is the advantage? The advantage is you will be in a position to get the exact solutions to a broader class of geometries and boundary conditions. So, it has really enlarged the hope of problems that you can solve, that is a way people have looked at it complex variables approach is bit involved, but it provides you a methodology by which you can attack the problem systematically.

And you know there was also a very interesting result obtained by Goursat in 1898. What he obtained was, it is always possible to find complex potentials to a given Airy's stress function.

You know it is a very important statement, this was made in 1898, then only other developments came. So, whenever you have an Airy's stress function, it can always be represented in terms of complex potentials, how they are represented?

You have phi equal to real part of z star and z star denotes your complex conjugate. So, z is x plus i y z star would be x minus i y multiplied by an analytic function psi of z plus chi of z and mind you, here I have on the left hand side Airy's stress function; on the right hand side in general I have two analytic functions, you have to keep that in mind. In the case of Westergaard approach what we saw? We have represented phi in terms of capital Z alone, but in a generic situation, you find phi is expressed as function of psi as well as chi; so there are two analytic functions involved.

And another advantage of this Kolosov Muskhelishvili approach is, this method provides a simpler way to calculate the displacement. For any given Airy's stress function if I identify the complex potentials, I could find out the displacement in a straight forward manner.

You know I had said we have stress formulation as well as displacement formulation by which you can solve stress field, displacement field, strain field so on and so forth. Though we take stress formulation many times, we find we just solve the stress field problem and go to the next problem, rather than looking at the strain field and the displacement field. For the case of mode 1, what we did? We looked at stress field, we look at the strain field and we also moved ahead and found out the displacement field; while finding of the displacement, we had to make long bending arguments why the

integral functions f of y and G of x should in general be 0, we said that they represent rigid body translation and rotation.

We had to do a circuitous approach to get the displacement field. On the other hand, one of the advantages is of Kolosov Muskhelishvili approach is, once you find out psi and chi, I can write the displacement field in a very comfortable and straight forward fashion and how it is written? 2G multiplied by u plus iv equal to 3 minus nu divided by 1 plus nu multiplied by psi of z z psi star prime.

So, it is a complex conjugate of it, which is the function of z star chi star prime z star, this is for plane stress and these equations were obtained by Kolosov in 1905, this was part of his doctoral dissertation.

Now, you can think in those days, doctor thesis are very simple; it is not so and that point in time arriving at an ideal like this and establishing it was quite difficult. In fact, people raise several questions on whatever the equations that he proposed, only after a debate these equations are recognized as derived with certain amount of mathematical rigor. And I have these expressions for plane stress for plane strain replace nu equal to nu by 1 minus nu.

See in the case of Airy's stress function what we did? Once Airy's stress function is determined, stress field could be express in terms of Airy's stress function; we have sigma x equal to dow squared phi by dow y squared and sigma y equal to dow squared phi by dow x squared. But we never wrote how to write the displacement components in terms of stress function that is the difference. In fact, when you look at Kolosov Muskhelishvili approach, they first give the advantage in terms of determine the displacements only then, they go and write the stress field in terms of psi and chi; the first quantity they write is the displacement.

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Kolosov-Muskhelishvili Approach

This provides a method for generating the displacements to a given Airy's stress function – which is not always straightforward in the real stress function formulation.

$$\sigma_x + \sigma_y = 2\psi'(z) + 2\psi''(z^*) = 4\,\mathrm{Re}\,\psi'(z)$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2\left[z^*\psi''(z) + \chi''(z)\right]$$

Once you have determined psi and chi for a given problem, you could also write the stress components and you have this as sigma x plus sigma y equal to 2 psi prime z plus 2 psi star prime z star equal to 4 times real part of psi prime z and I have sigma y minus sigma x plus 2 i tau xy equal to 2 times z star psi double prime z plus chi double prime z.

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Typical Stress Functions

Rectangular plate subjected to uniform tensile force of intensity q in the y-direction:

$$\psi(z) = \frac{q}{4}z; \qquad \chi'(z) = \frac{q}{2}z$$

Rectangular plate subjected to uniform shearing forces of intensity q in the x-and y-direction:

$$\psi(z) = 0; \qquad \chi'(z) = iqz$$

Now, the problem is reduced to what? You have to find out psi and chi for a given point; if psi and chi are determined, everything about the problem is solved. And we look at typical stress functions in terms of a complex functions and we look at for rectangular

plate subjected to uniform tensile force of intensity q in the y direction, this is like uniaxial loading; you have this as psi of z equal to q by 4 multiplied by z and chi prime z equal to q by 2 multiplied by z.

So, you have psi and chi prime for a uniaxial loading. See if you really look at polynomial functions, suppose I have the polynomial is like ==x square== some a x square and what is that it represents? When you say x squared, it automatically satisfies the bi-harmonic equation, because you have dou power 4 phi by dow x power 4 and so on.

So, a second degree polynomial will automatically satisfy this. So, if you take that, you have sigma y as given as dou squared phi by dou x square.

So, if I have a uniaxial stress field in the polynomial function, you will write it as ==x square== a x square or some such type of function. For the same thing, if you want to look at in complex quantities, you define psi and chi prime like this. Suppose I want to find out stress function for q in the x direction, psi z remains same chi prime z changes to minus q by 2 z.

Suppose I have a bi-axial loading situation, I can add these two stress functions. So, if I have bi-axial loading what happens only psi exists, chi vanishes.

Now, you take another problem, where I have a rectangular plate subjected to uniform shearing forces of intensity q in the x and y direction and the function psi and chi prime are given like this, psi of z equal 0 and chi prime z equal to iqz.

See even if you go back and look at a reinvestigation on what we have got the solution as Westergaard solution, when you compare the fringe pattern from the singular solution of Westergaard with experiment, it matched well.

So, if you look at shearing stresses that infinity, you could have just one function, because you have psi z 0; you have only chi is available. So, in the case of bi-axial loading, you have only psi is available

So, only if I have both of them, psi and chi, you have more flexibility in modeling a given problem; you can have a hint site of that. You will also look at two more cases; you know these are all simple and famous problems, suppose I want to look at what is the stress function for rectangular beam in pure bending.

The functions psi and chi prime are given like this, psi z equal to i into M divided by 8I z squared or I is the amount of inertia chi prime z equal to i into M divided by 8I z square both are equal; here it is psi of z, here it is chi prime z.

And another problem is whole of radius a in a tension strip. You have the relevant function psi and chi, and psi is given as 1 by 4 sigma z plus 1 by 2 a squared by z and chi of z you have to notice, till now we have been looking only at chi prime. For this problem, chi z is directly given and chi z equal to minus half of sigma z minus 1 by 2 sigma a squared divided by z plus 1 by 2 sigma a power 4 divided by z cube.

See what is the focus here is, for a variety of problems, you could identify the analytic function psi and chi; so, bottom line is any Airy's stress function can be expressed as a combination of two analytic functions.

So, this is most general that is the kind of argument that which Sanford put forth, while introducing an additional stress function capital Y to the Westergaard stress function. So that means I have more flexibility in defining the problem, because we wanted only tau

xy to 0 on the x axis; it so happen the solution was restrictive, it also made tau max 0, whereas it is not 0 in actual experimental situations; so we need to relax that. So, I need little more flexibility in defining the stress field.

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So, now we will come back to the crack problem; before we get into that, we will just look at the Airy's stress function is given as phi equal to real part of z star psi z plus chi z and we have also look at sigma x plus sigma y is 4 times real part of psi prime z and sigma y minus sigma x plus 2 i tau xy equal to 2 z star psi double prime plus chi double prime.

This could be further expanded, you know it is a very simple arithmetic; I can write what is expression for sigma x sigma y and tau xy, it is very simple and straight forward make an attempt.

You need to make an attempt, because that will keep you attentively in the class and will also help you to go through the notes comfortably in any case, I will show the expressions which you can verify.

So, I have this sigma x, sigma y and tau xy; sigma x is given as real part of 2 psi prime minus z star psi double prime minus chi double prime, and sigma y is real part of 2 psi prime plus z star psi double prime plus chi double prime; tau xy is given as imaginary part of Z star psi double prime plus chi double prime. Some of the expressions you are

going to see in this class are very long, be patient to write it; these are all culled out from such papers.

See once Airy's stress function is given, I could simply differentiate and then get the expression for sigma x, sigma y, tau xy. But what is attempted here is an indirect justification for existence of stress function capital Y, from the Kolosov Muskhelishvili approach that is why we are seeing both of them together; if somebody as given the stress function, we never ask how the stress function was derived; here we are going into that aspect also.

We are trying to find a justification why this stress function capital Y is added. So, we would try to find out in terms of the Kolosov Muskhelishvili formulation, that how you can look at capital Y in terms of psi and chi.

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So, now, we have to look at the shear stress because that is what as precipitated at this kind of a discussion. So, we can write the expression for shear stress in long hand. So, tau xy becomes x imaginary part of psi double prime minus y real part of psi double prime plus imaginary part of chi double prime. And we have really looking at what happens on y equal to 0 on the axis of symmetry tau xy should be equal to 0.

So, when I do this, this term automatically get knocked off and we also want to have this should go to 0. So, I get a requirement x imaginary part of psi double prime plus imaginary part of chi double prime should be equal to 0.

Now, let us define capital Y as z psi double prime plus chi double prime. There is mathematical jugglery that you are using it, so do not get annoyed. Now, we would like to have an indirect justification for the stress function y that is what we are doing it here. So, from the condition tau xy equal to 0, you are able to write down this x imaginary part psi double prime plus imaginary part of chi double prime should be 0 and we define capital Y in this fashion and this will turn out nothing but imaginary part of Y.

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And what was the contribution by Sanford was, by selecting appropriately the stress function Y, he could show the results that you get from Westergaard, Irwin's modification as well as generalized solution. Any general solution in a particular case should reduce to the earlier simplified cases that you are looked at. So that aspect was also satisfied by introducing capital Y that was the success.

You know it is a very, very subtle point, because you had the advantage of looking at the fringes in fact, Sanford was the collaborator with Irwin and they had done several experiments related to photo elasticity. So, they had these results before them, so those results really prompted them to think and ask this kind of question and also find a

solution. Now, you can replace chi double prime in terms of psi double prime and y the equations are recast and the equation appear like this, sigma y minus sigma x plus 2i tau xy equal to 2 into z star minus z multiplied by psi double prime plus capital Y.

So, once you define this, we can find out sigma x, sigma y and tau xy in terms of the stress function psi and y. So, I could write this and this is simplified and the final result is given. So, you have sigma x sigma y tau xy which is given as sigma x equal to 2 times real part of psi prime minus 2y times imaginary part of psi double prime minus real part of Y.

As a very small change between the sigma x term and sigma y term, you have only the sign changes in the second term from minus to plus; similarly, in the third term from minus to plus and you have the expression for tau xy is given as minus 2y real part of psi double prime plus imaginary part of Y.

If you can go back and look at what was the Westergaard stress function for tau xy, it will have only one term involving capital Z. So, I could find the identity between psi and capital Z that way; you had only this term minus 2 y real part of psi double prime was available; this is an additional term, imaginary part of Y is an additional term.

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So, on the axis of symmetry, shear stress is 0. By defining Y as above, the condition reduces to because the first term what you have is y times the function comes. So, when

y is 0, that term automatically goes to 0; so shear stress being 0 along the crack axis reduces to imaginary part of Y z equal to 0.

See if you really recall the way that we satisfy the boundary conditions when we looked at the mode 1 situation, we looked at what happens for the first set of boundary conditions, what happens at the crack phases and the second set of boundary conditions what happens at infinity, but the third condition we do not do anything, because we wanted on the crack axis shear stress to be 0; we did not even have any option for us to look at how the boundary condition behaves, it was automatically satisfying.

We had no difficulty at all; apparently we thought we had no difficulty, but when you compare the result with the experiments, it was found that it was lacking in something.

So, now, what we will see is the condition is modified as imaginary part of Y z equal to 0. So, by changing what way we take capital Y, we would have control on how this condition is going to dictate the problem.

Suppose I take capital Y equal to 0 which is very similar to what Westergaard did. No function like Y exists at all, and then I get Westergaard stress functions. The expression of stress field would be same as what you see in the Westergaard solution and you will also have to set 2 psi prime equal to capital Z.

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Then I get the conventional Westergaard solution, you have sigma x sigma y tau xy given; sigma x is nothing but real part of capital Z minus y imaginary part of Z prime, and sigma y equal to real part of capital Z plus y imaginary part of capital Z prime and your tau xy is nothing but minus y times real part of capital Z prime and this is straight forward, you know you are setting y as 0 and how do you get Irwin's modification? We finally want imaginary part of Y z equal to 0, one can also a set Y as a real constant.

When you say Y as a real constant, it is imaginary part is 0. So, what will have to look at is, the condition tau xy equal to 0 is still satisfiable by taking Y appropriate. This kind of a luxury we did not have; when you had only one stress function, we were getting the solution; we carried on with it because Westergaard in his paper, he also solved the variety of problem for which on y equal to 0 tau xy was 0.

So, people never looked at closely what is the kind of difficulties one would see in the case of a crack problem; when people compared the result with the experiment, they felt something more needs to be done.

So, one of the earliest modification was done by Irwin. So, if I have to get the Irwin's solution, you take capital Y equal to A and set 2 psi prime equal to capital Z minus A, then one get us sigma x sigma y tau xy in this fashion.

So, I get real part of capital Z minus y imaginary part of capital Z prime minus 2A. This will explain for your additional stress term sigma naught x and sigma y and tau xy, they are very similar to what you had obtained in the conventional Westergaard solution.

Now, what I am going to do is I am going to have Y present, but we will only ensure that the imaginary part of Y z equal to 0, this is what we are going to in emphasize. So, if I do this and if I take 2 psi prime equal to capital Z minus Y, I get the generalised Westergaard equations.

I have sigma x sigma y tau xy this is given as sigma x equal to real part of capital Z minus y imaginary part of capital Z prime minus y imaginary part of capital Y prime plus 2 times real part of Y and sigma y is given as real part of Z prime real part of Z plus y imaginary part of Z prime y imaginary part of Y prime

So, this is an additional term in comparison to Westergaard solution and finally, you get tau xy equal to minus y real part of z prime, then you have two additional terms minus y real part of Y prime minus imaginary part of Y.

No, we have gone through a circuitous route; we looked at Kolosov Muskhelishvili formulation; we provided a justification in a most general case an Airy's stress function is represented in terms of two analytic functions from that argument, the Airy stress function for Westergaard problem also should have two stress functions. So, you can have capital Z and capital Y as a candidate, we have still not looked at what is the form of capital Z and capital Y; I could also get this expression directly from differentiating

the Airy stress function, whatever the result that I have got here could also be obtained from this expression of Airy stress function.

So, in this class we had looked at what is the stress and displacement field in the case of mode 3, I again emphasized these are valid very close to the crack-tip.

Then we moved on to raising the fundamental question, the fringe patterns that has seen in a photoelastic experiment show that maximum shear stress varies along the crack axis, whereas the conventional Westergaard and modified Westergaard do not capture this phenomena.

To capture these phenomena, Sanford came forward and introduced an additional stress function y and whatever the stress field you get in terms of capital Z and capital Y, he termed it as generalised Westergaard equations. From the generalized Westergaard equations, by appropriately choosing the function y, you could get Irwin's modification as far as the basic Westergaard solution.

 Thank you.