**Engineering Fracture Mechanics**
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**Module No. # 02**
**Lecture No. # 10**
**Energy Release Rate**

In the last class, we looked at the problem of a center crack in a tension strip. We raised the question - in the presence of a crack, how to find out the strain energy?

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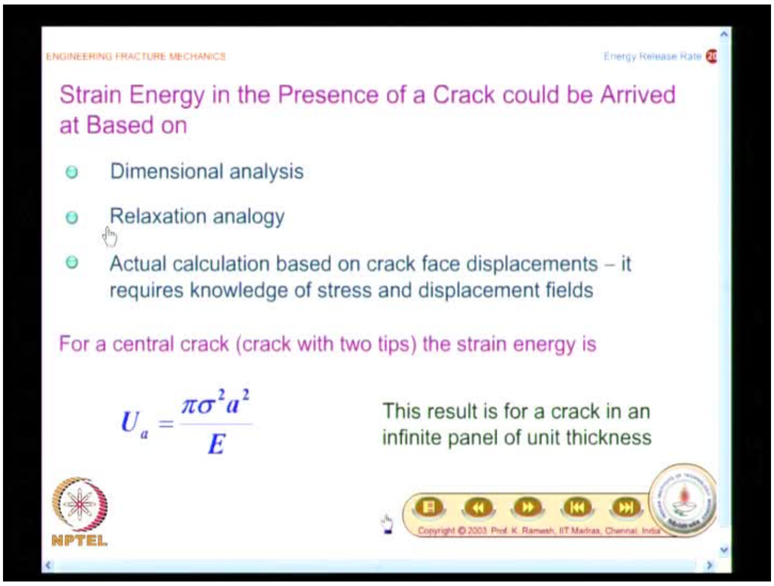


In fact, we listed there could be three possibilities. It can be done through dimensional analysis, relaxation analogy, and finally, actual calculation based on crack face displacements. For us, to do this, we need to know the knowledge of stress and displacement fields. For our discussion, what we did was, we took the available solution for a central crack and it is emphasized crack with two tips.

See, it may appear trivial. In fact, quite a bit of confusion was there in the initial stages of fracture mechanics for not recognizing this subtle point. In order to emphasize that, I am saying it again. We have taken the solution for the case of a central crack; it is crack with two tips you should recognize, and this is given as strain energy U a equal to pi sigma squared a squared by E. Another aspect also, I pointed out that this is obtained for an infinite panel of unit thickness.

See, if you look at books in fracture mechanics, you have equations given for unit thickness as well as for a finite thickness. So, when you look at the books, you should be able to recognize by looking at the expressions, whether it is derived for unit thickness or for a finite thickness b.

So, in all my discussions, what I have consciously done is to go back and forth on this; certain things we will develop on unit thickness; we will also look at when you have for a finite thickness.

In fact, to aid your appreciation of strain energy in the presence of a crack, we looked at relaxation analogy; you should keep in mind, these were not mathematically rigorous. While discussing the relaxation analogy, we considered a plate of thickness b so that you get trained, how these equations change when the thickness is brought into the development of these equations.

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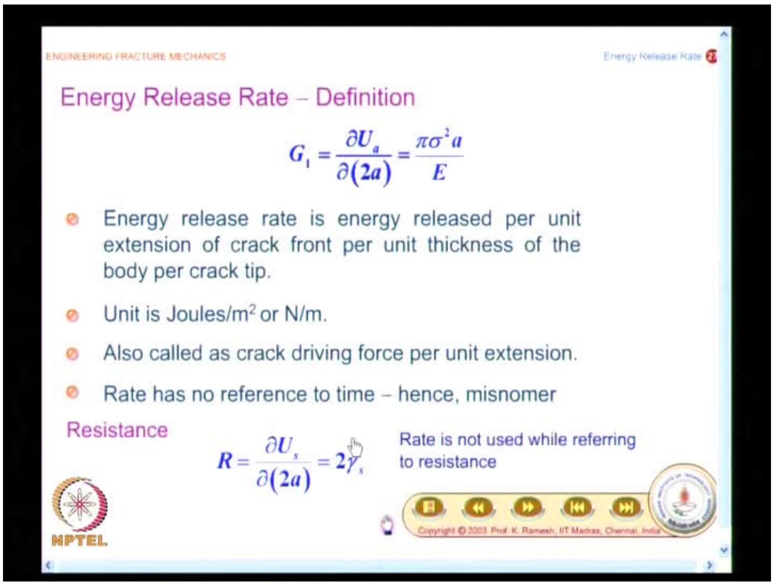


And now, we take up what is the definition of Energy release rate. Actually this is the continuation of the problem that we had discussed, and the result that we have got for this was pi sigma squared a divided by E, and this is for the specific problem of central crack in a tension strip. But whatever the points that are summarized, it is for any generic problem situation.

And what do you find here? Energy release rate is energy released per unit extension of crack front per unit thickness of the body per crack tip. Because we have taken a center crack, here the crack length is put as 2a. Do not take this as the generic expression of energy release rate; this is for a specific problem of a center crack; In fact, the thickness b is also hidden.

We would see a little while later, a generic expression; in that we would have this as 1 by b dou U a divided by dou a, and for a finite thickness plate, this will be modified to pi sigma squared b a divided by E. So, the b will get cancelled, and that is how you will get it. And the units for energy release rate is joules per meter squared, or in other words newton per meter, and this can also be called as crack driving force per unit extension. And a very important point to note is, whenever you have the terminology rate, you expect something to do with time.

Here, there is no question of time is brought in; so, calling this as energy release rate, in a sense - a misnomer. So, we have seen the energy release rate and you will also look at the resistance. They call it as resistance; they do not call it as resistance rate. In the literature, it is called only as resistance, and for the center crack, we have looked at this resistance in terms of the surface energy - gamma s. See, this is very important.

We have looked at the problem of a center crack at a tension strip and we have also looked at it in a particular fashion; this helped us. By phrasing the resistance in terms of the surface energy, we were able to compare what is the theoretical strength based on lattice calculation which also had a term involving surface energy. And what scientists were coining it as size effect, was identified by Griffith as indeed a crack size effect. And from this perspective, we were also able to dispel the paradox generated by Ingli solution. The length of the crack also plays a role - that was a key contribution by Griffith.

And another aspect what you will have to look at is - we have done an energy balance; we have also said, for the formation of two new surfaces, I need energy, and this energy comes from the strain energy of the system, and the resistance is inherent to the material.

And for brittle materials, we have looked at for the center of crack - R equal to 2 gamma s. Whether this remains a constant or whether it is the function of a, we will have to wait and see. Because only confining our attention to brittle solids will not be sufficient because we have to look at what happens in high strength ductile alloys; how Irwin was able to modify for such a situation? So, we have to wait and see, and he plays with the definition of R.

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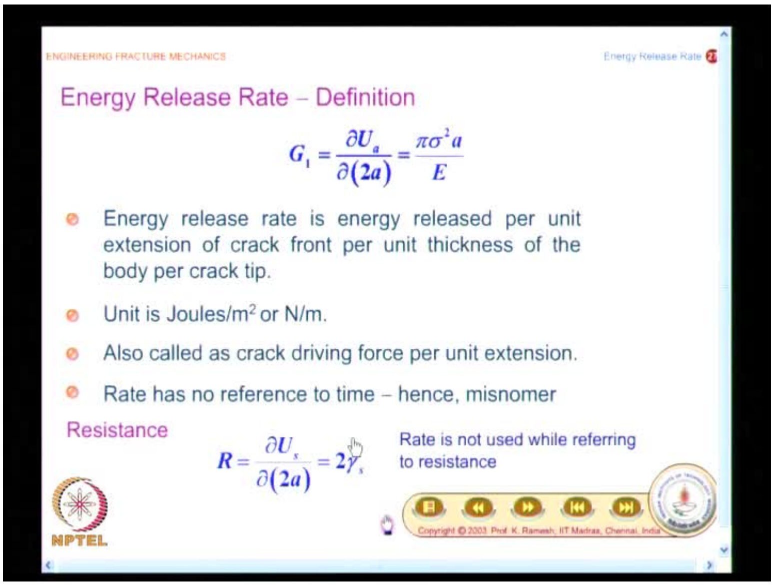


So, in all our future discussions, we will simply keep this as R; whether it is constant, whether it is a curve - all that, we look at it. Another aspect is, we have sent for the formation of two new surfaces; I need energy to be available. See, in any mathematical developments, when you say a condition, you have to qualify whether it is a necessary condition or sufficient condition, whether the condition is both necessary and sufficient - these issues, we will postpone it for the time being; we would really look at fracture instability in greater detail, and qualify what is the necessary condition, what is the sufficient condition; all along, we have only looked at energy balance; that is how this problem was handled.

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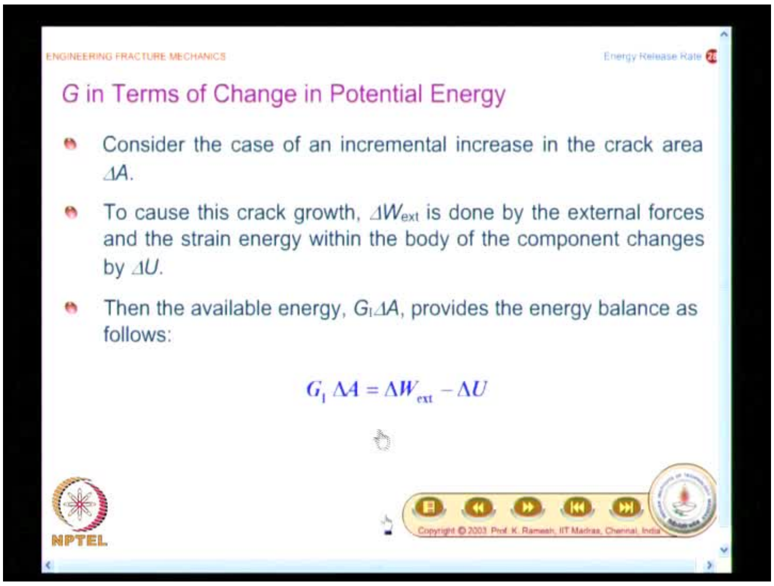


Now, we will try to go and generalize, and list the energy release rate in terms of change in potential energy. And when I take up this topic, I am going to consider a single crack having one crack tip; it will not have two crack tips. if you look at the examples that we have taken for illustrating constant load and constant displacement, we had the double cantilever beam specimen; that had a single crack; only one crack tip was available; either it could be of that category or a plate with the edge crack. And what we would look at is - the crack has an incremental increase in area; that is given by delta A.

And we want to find out how this could be related to the work done or strain energy, and the energy release rate. In a generic situation, you will have change of work done given by delta W external as well as change of strain energy. In a particular situation, you may have only change of strain energy. So, you should not confuse these two when we are trying to write a generic expression which will have components from external work done as well as change in strain energy.

Since we have already looked at what is energy release rate, I could write the energy balance in this fashion; if I do not know what is energy release rate, we would write this on the left hand side and write this on the right hand side. Since in our development, we have already looked at what is energy released rate, we could write G 1 into delta A to denote that we are discussing the mode 1 situation; that is equivalent to change in external work done minus change in strain energy. And how we are justified in writing this energy balance?

We are considering ideally brittle solids. There is no dissipation of energy. One form of energy turns into another form, and here again, we look at the point when the crack is about to propagate; we are not looking at the instance after it has started propagating. We are writing only for that instant; this energy balance, what we have got, we would recast in convenient fashion.

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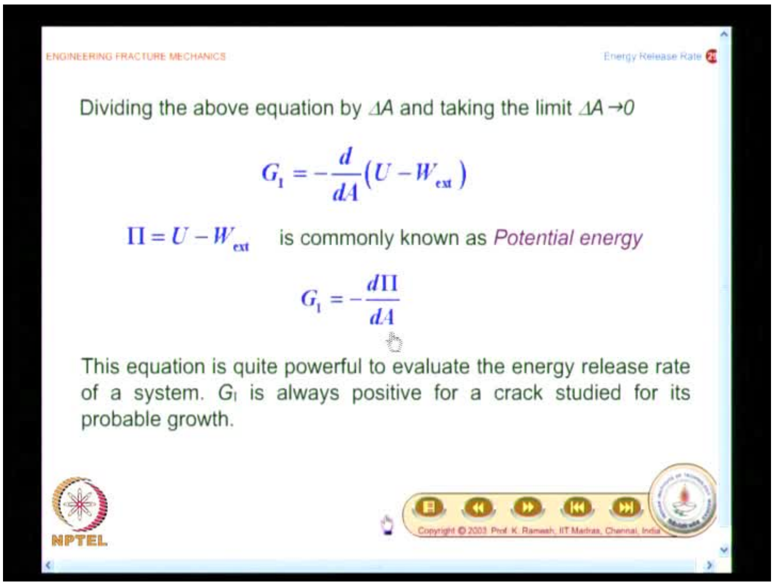


See, in theory of elasticity, people have a definition of what is potential energy. We would try to recast this expression in such a fashion, and that is what is listed in this slide. So, what we do is, we divide the equation by delta A. And note this step - and taking the limit, delta A tends to 0; this is the key point; we should not miss this. Whatever the incremental change, the incremental change is extremely small; that is what we are looking at. So, what I get is G 1 equal to minus of d by dA U minus W external. And in theory of elasticity literature, the combination of U minus W external is termed as capital pi; it is known as potential energy.

When you are trying to move away from continuum mechanics to fracture mechanics, people wanted to look at what parameters that continuum mechanics, how they look like when you come to fracture mechanics. So, you would like to express the fracture parameter in terms of the quantities that we are accustomed to. So, what we are trying to write is, we write G 1 as minus of d capital pi divided by d A. So, this is the generic expression.

What will have to keep in mind is G is always positive for a crack studied for its probable growth. So, what you will find is, the whole expression will become positive for given specific instances. And what is delta A? Delta A is equal to B into delta a of the crack extension, and we are looking at only one crack-tip. So, the crack extends by delta A.

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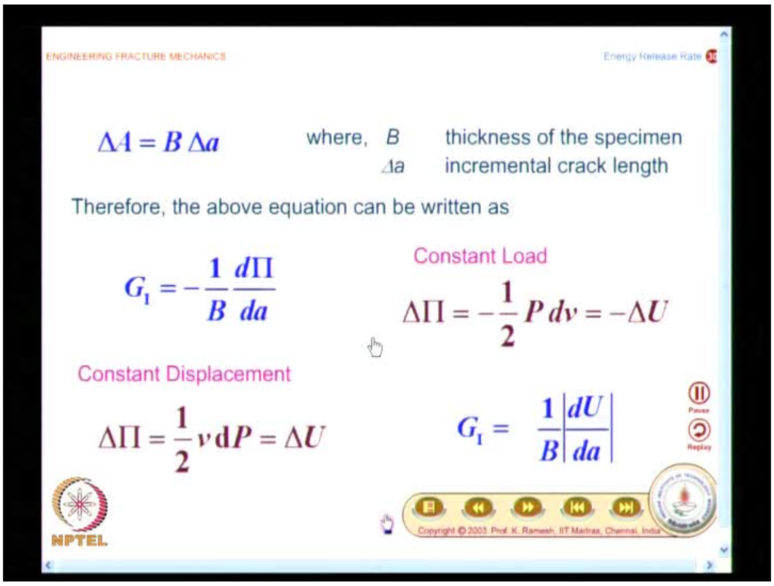


So, using this, the equation can be re written as minus of 1 by B d capital pi divided by d a. So, this is the most general expression for strain energy released rate. So, you will have to calculate the term d capital by da for a given situation. And we would look at what is the result for a constant load as well as constant displacement.

For a constant load, delta pi s given as minus 1 half P dv; that is equal to minus of delta U. So, minus of delta U, when you put it here, minus of minus becomes plus; so finally, G is positive. And in the case of constant of displacement, I have this as 1 by 2 v into dP, and I get this as delta U; but this delta U is negative because we have seen already, we will also look that up again, that minus of minus will become positive, or in other words, we can also write G simply as 1 by B dU by da; either you can look at it in terms of potential energy, or you can also look at in terms of the strain energy.

By looking at the second expression, people also called it as strain energy release rate, but that is not a generic definition. Though I have 1 by B dU by da mathematically, if you look at from physics of the problem, in the case of a constant load, we would see the energy requirement for the formation of two new crack surfaces has actually come from the external work. So, we will retain the definition as energy release rate; we will not call this as strain energy release rate; that kind of thinking was also there in the development of fracture mechanics.

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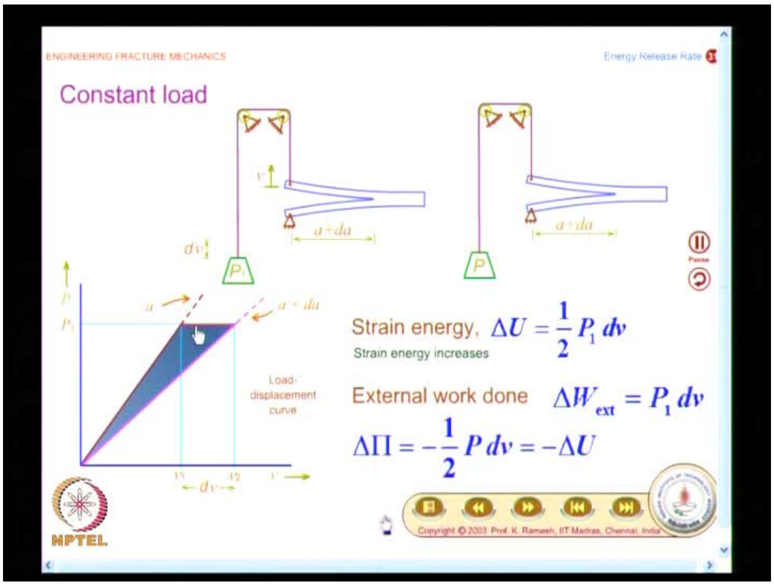


Now, we again go back to our constant load. We will reinforce our understanding. So, what you find here is, you draw the graph between load versus displacement for a specimen with crack of length a, for another specific length with a crack of length a plus da; obviously the stiffness as reduced; so the graph is below the graph drawn for crack length of a.

And just look at these energies (Refer Slide Time: 17:40); So, we will retain the definition as energy release rate; we will not call this as strain energy release rate; that kind of a thinking was also there in the development of fracture mechanics.(Audio not clear 17:41 to 17:46)

We have a change in strain energy. We find out what is the final strain energy; then, we look at what is the initial strain energy, and the difference in strain energy is seen. Then, you also see the external work done. The subtle point here is it is P 1 into dv; it is not half of that because the load has remained constant for the displacement dv. So, it is a rectangle. And eventually what you get as the change in potential energy of the system is given by this triangle (Refer Slide Time: 18:34).

This is very important. We are looking at from the point of view of graphical approach; it gives you a good amount of understanding.

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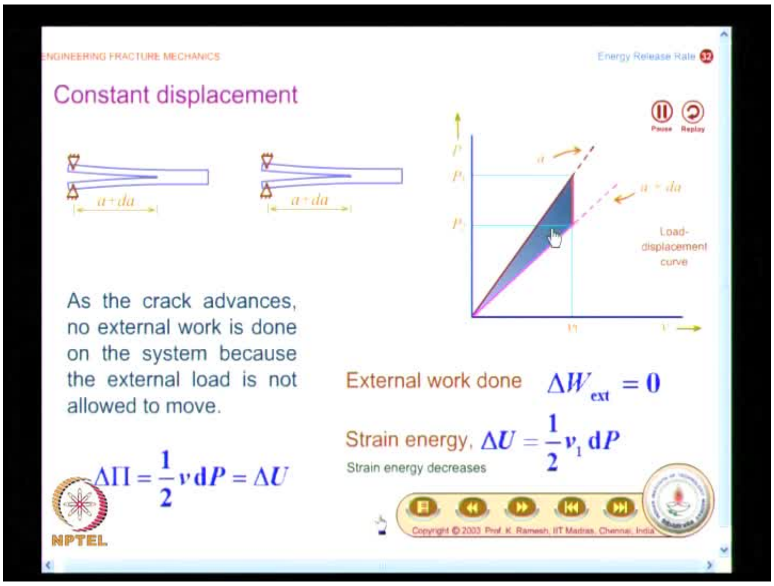


Then we look atp constant displacement. So, here, again you have two systems; one in which crack was initially a; it has propagated to a plus delta a, and you have these graphs and the point of application do not know; it is called fixed grips. In view of this, external work done is 0. And you get the change in strain energy by looking at the final strain energy and the initial strain energy, and you find strain energy has decreased in the process, and the potential energy what you get is equal to delta U. But this change is negative. And what is the energy availability? Again, this triangle.

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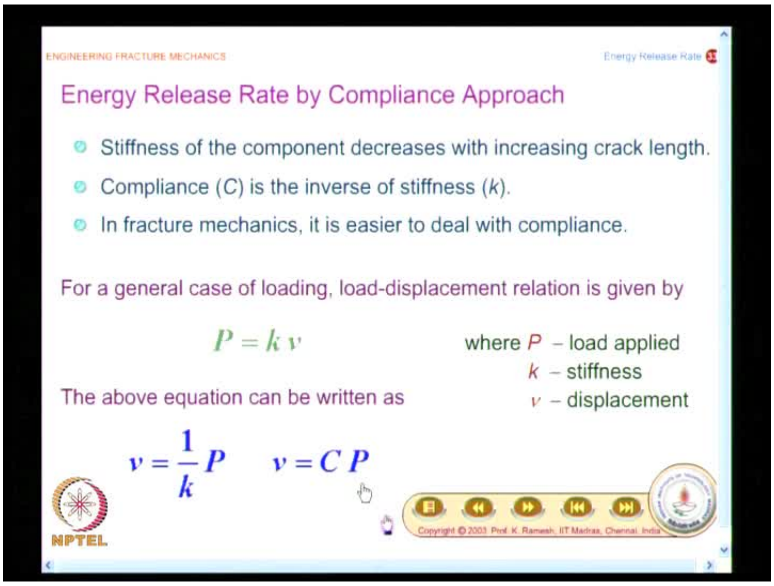


See, we will now take up another approach to energy release rate. See, for a given practical problem, finding out the energy in the presence of a crack is a challenging task. For the center crack, we took up the available solution. We have not done a rigorous mathematics in our development till now. We will be able to do that only after our discussion on crack-tips stress and displacement fields.

So, what Irwin suggested was - if you look at the compliance of the system, it is lot more easier to calculate in a realistic scenario. You can simply perform an experiment and find out the compliance in the presence of a crack. So, what we will now do is - we would obtain expressions for energy release rate in terms of compliance.

We have already noted stiffness of the component decreases with increasing crack length; this is the known fact. And we also know what is compliance, from your solid mechanics understanding; it is the inverse of stiffness, and compliance is given by the symbol capital C, and the stiffness is small k. So, if you look at historically, it was Irwin who suggested - it is easier to deal with compliance, as for as fracture mechanics problems are concerned. So, we would recast the expressions.

For a general case of loading, you could apply the load displacement relation as P equal to k v; do not read this as nu; it is appears in the italic font; appears as nu, but read this as k into v. I would like to recast this expression in terms of compliance. We have already seen compliance is inverse of stiffness. So, when you substitute for k, I would get the expression v equal to C P. These are all very simple expressions; in fact, we would go and do the calculations for constant load and constant displacement. We will have to play with this expression v equal to C P, and what we want to do is, instead of d capital pi by da, we would like to replace it in terms of dC by da; that is the focus, and I would like you to do this derivation yourself. I would take you half a way, give you some time, and play with these equations, and try to get the expression.

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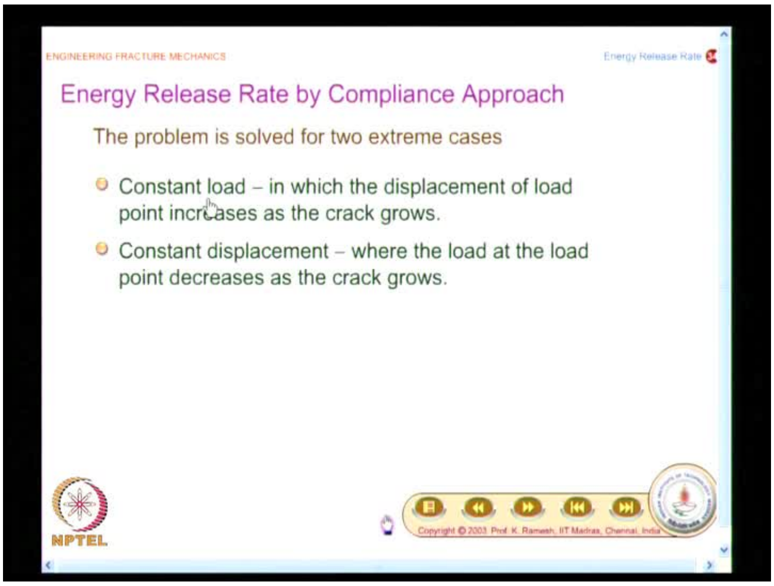


As I mentioned, we would solve this for two extreme cases: one is a constant load; another is a constant displacement. Why we look at these? They are convenient from the point of view of writing out your analytical expressions as well as performing experiments. We have already noted, as the incremental changes become closer to 0, the energy availability in constant load as well as constant displacement is identical.

In fact, if you have looked at carefully, for both the cases of constant load and constant displacement, we finally got the energy release rate as 1 by B dU divided by da. So, mathematically, we have already shown, the energy availability is same we have looked at that from the point of view a potential energy or strain energy.

Now, what we want to do is - in order to apply the concept of energy release rate to practical problems, Irwin suggests that, let us look at compliance; we will get this for constant load as well as constant displacement.

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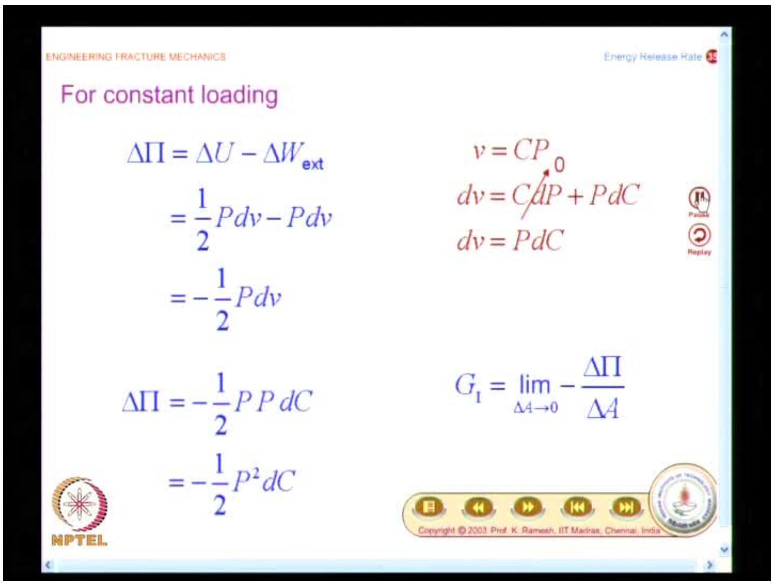


Now, let us look at for constant load. So, this is where we have got previously, we have come up to minus 1 half of P dv; this you know.

Now, I would like you to take a minute or two in replacing the quantities here, in terms of the compliance suitably, and you have to bring in, in one case - constant load, in another case - constant displacement; a suitable kind of mathematical simplifications; fairly simple. And what you would have to get finally? Whatever the expression for energy release rate that I get in constant load should be same as constant displacement; that is the anticipation. I would like you to work it out. Please take your time to work it out. Then, I will help you; it is fairly simple.

And you look at v equal to CP. And the key point here is, we are going to look at incremental change. So, when you differentiate, I will have this as C into dP plus P into dC, and you have to look at, we are discussing constant loading; under constant loading, dP goes to 0. So, when I write energy release rate, I would replace this in terms of the compliance; fairly simple. You have to recognize that, when you differentiate we should write this as CdP plus PdC although you have learnt this mathematics, you may over look this.

So, what I get here? So, delta capital pi becomes minus one half of P P d C which reduces to minus one half of P square d C.

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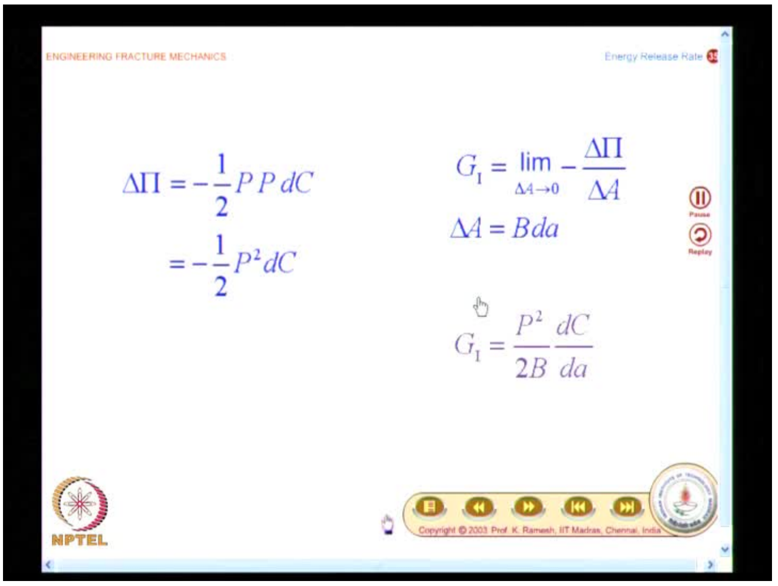


So, if I substitute it in the general expression, I get this as G in the limit delta A tends to 0. We had that as minus delta pi divided by delta A, and delta A is nothing but B into da, and we have already derived what is delta pi, and you get the expression for energy release rate as P squared by 2 B dC by da.

In fact, if you look at your assignment sheet, you have a problem from an experiment on the measurement of compliance. So, from the graphical approach, it is possible for you to collect the data of dC by da, substitute, and calculate the energy release rate.

So, it provides the via media to apply the concept of energy release rate to practical problems. So, when you solve the assignment problem, you will understand it better. When I move in for constant displacement, it is needless to say that you have to get the final expression of energy release rate as P square 2B dC by da. The reason is - we are looking at incremental changes which are very small; this is the key point.

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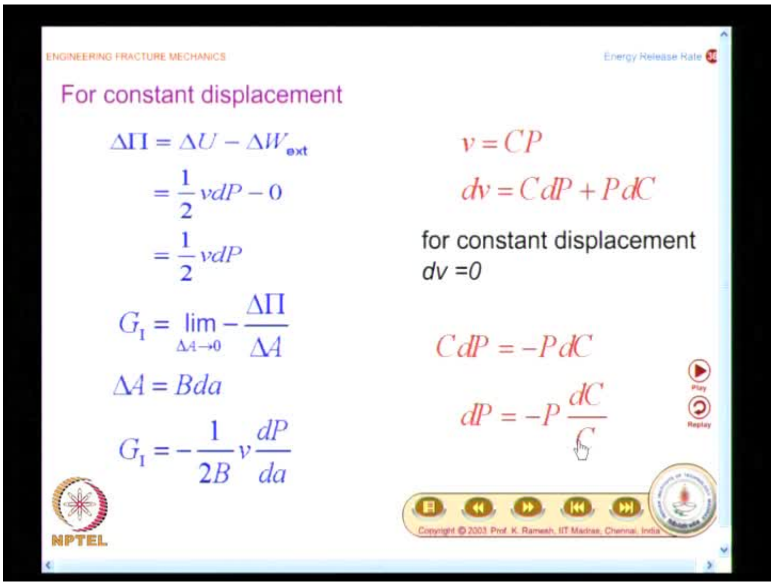


Here again, I will go half way. So, now, you have to recast this carefully. I have delta pi as 1 by 2 v into dP; again, go back and look at the basic relationship on compliance, differentiate it, and substitute it carefully. You need to work it out; take a minute or two; it is fairly simple because the final result; the result is known. So, you can always work backwards and fill in the blanks. Anyway, we will look at the mathematics. You have to recognize, for constant displacement, dv equal to 0.

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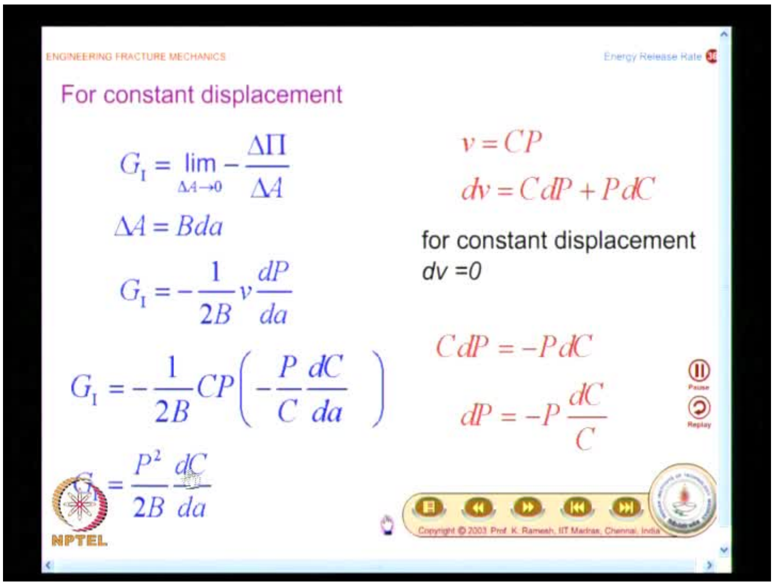

In the earlier case, when we looked at constant load, this term was going to 0; so, we simply wrote what is d v; now, dv is 0. So, you will get an inter relationship between the two. Only that, I will have to substitute it here. If I apply the basic definition of energy release rate, I get the expression as G equal to minus 1 by 2B v times dP by da. So, I will have to replace this dP and that I get from this. So, what I have is, when I put dv equal to 0, I get CdP equal to minus PdC. So, dP becomes minus P times dC by C. So, when I substitute it in this expression (Refer Slide Time: 30:41), I get the final expression as G 1 equal to P square by 2B dC by da. So, from a mathematical perspective, the energy availability in the case of both constant loading and constant displacement is same.

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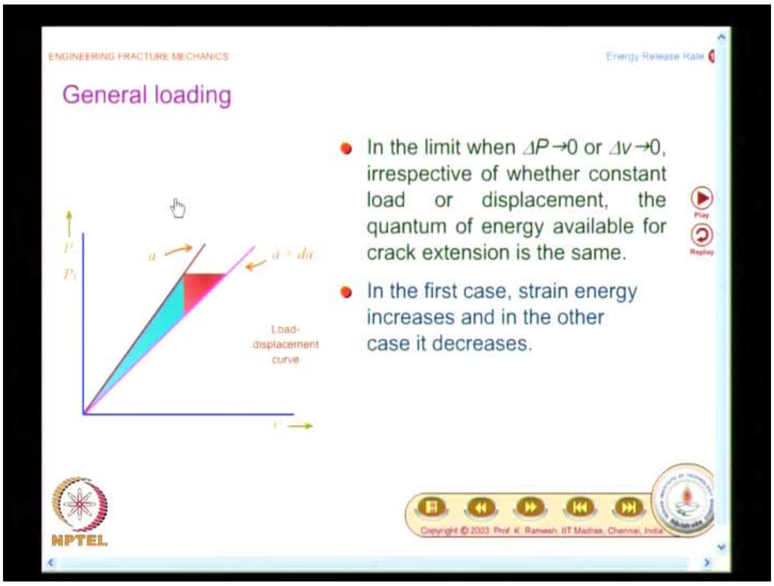


We would again go back and look at the graph just to re convince our self. You have the general loading and we put this as a red triangle, and the key point to note is, when dP or delta P tending to 0, or delta v tending to 0, this triangle also goes to 0; so, both from a graphical perspective as well as from the mathematical perspective, the energy availability is same in the case of constant loading as well as constant displacement. Why we switch between the two? The reason is certain mathematical developments are easier to do in one of these cases, and also, you could extend it for an experimental measurement. So, it provides a via media to apply fracture mechanics concepts to practical problems. So, this is the reason people have looked at it.

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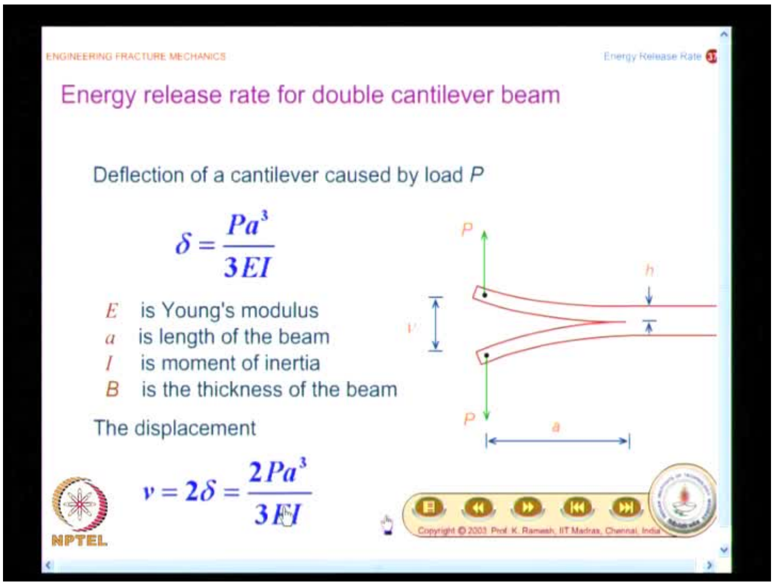


Now, what we will do is, we will try to evaluate the energy released rate for two examples. First, we will take up the problem of a double cantilever beam, and this is also known as, in fracture mechanics literature, d c b specimen. Now, once you say d c b, you should recognize you have a specimen of this nature, you have a crack, and the top and bottom portions behave like cantilevers. And from a simple course in strength of materials, you know what is deflection at the free end, and that is given as delta equal to Pa cube by 3EI; that is what is given here, where you have E is young's modulus; a is the length of the beam; I is moment of inertia; B is the thickness of the beam which is into the screen; the thickness is in the into the screen. This is what is the thickness and the height is taken as h, and the total displacement is v. So, if I have to write down v, it is nothing but 2 times delta. This is a free end deflection of a cantilever beam. I have one beam here, another beam here (Refer Slide Time: 34:00).

So, make a neat sketch of it and then write down what is v? So, the displacement what we get as v equal to 2Pa cube by 3EI, the moment you write force displacement relationship, it is easier for you to write what is the compliance. So, once you get the compliance, energy release rate is straight forward to apply. That is why I have taken up a simple problem; it is not only simple, the problem is quite useful. You will see how it is.

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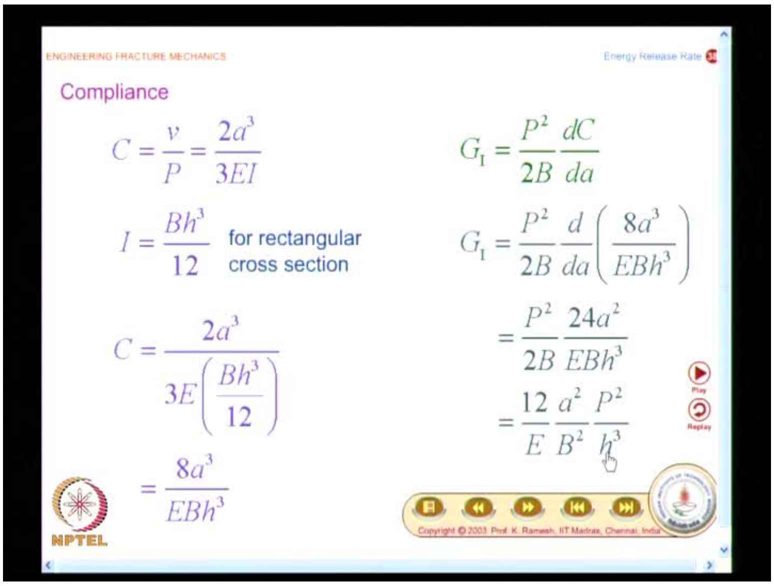


Currently, we will just go and look at what is the energy release rate for this problem, and what you will have to keep in mind is, the moment of inertia is Bh cube by 12, and from the expression for v, it is easy for you to write the compliance C as 2a cube by 3EI, and now replace I in terms of Bh cube by 12. You finally get the compliance as 8a cube divided by EBh cube. The moment you know the compliance, it is straight forward for you to get the energy release rate.

Here, again, do the simple calculation. See, if you do the simple calculation, in the class you feel satisfied, and also when you have to revise, it will not appear as Greek and Latin. So, do these simple calculations then and there, take a minute for substituting it in the relevant expression, and verify it with my result. So, this result is going to be quite useful for discussing certain aspects of fracture mechanics. We will now look at the final expression.

We know that G is given as P square by 2B dC by da. So, when I substitute these expressions, when I differentiate 8a cube divided by EBh cube with respect to a, I get this as P square 24 a square divided by 2 B into EBh cube, and the expression is recast in this fashion. The energy release rate for a double cantilever beam specimen is 12 by E into a squared by B squared into P squared by h cube, and what we would do with this expression?

See, while developing fracture mechanics, I have said, whatever the observations they were trying to derive based on mathematics, they tried to look at in an experiment to whether verify the mathematical development has been consistent, free of any major problems. In fracture, we want to know how the crack propagates. We have also discussed situations where there could be study steady crack growth, fracture can be steady - this is one aspect.

Another aspect is - can you arrest a crack growth? That means the energy release rate should be different. We should be able to control, experimentally, what is the energy release rate. And one of the challenges in the early development of fracture mechanics was how to develop new test methods, and also new test specimens. So, one of the questions what people raised was after looking at this expression, is it possible to design a specimen which will have a constant G? It is possible to play with these expressions. Some of them can tweak it and create a specimen so that G remains constant as crack grows. And this is quite useful to verify some of the concepts developed in fracture mechanics.

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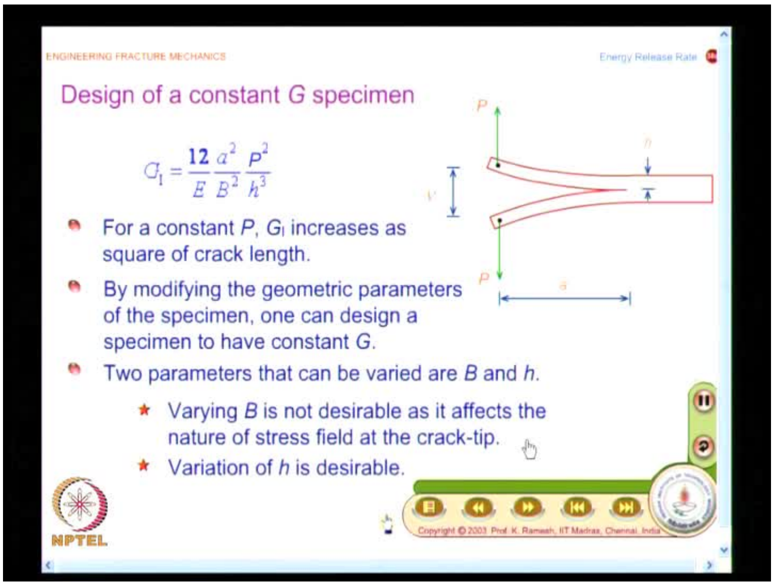


And that is what we are going to look at now - Design of a constant G specimen. And what do you find in this expression? G increases as square of crack length; G is a function of a square; by playing with the geometric properties of the specimen, it is possible to have a specimen of constant G. And what are the geometric properties that we can change? We can play with thickness B as well as h.

Suppose I vary the thickness, what would happen? You have a thickness keep on increasing as the specimen, along the length of the specimen. Then what would happen? We have been emphasizing in fracture mechanics; fracture mechanics is holistic; it gives you certain recommendation for thin specimens; it gives you another set of recommendations for thick specimens. So, by varying the thickness, you are varying the stress state in the vicinity of the crack. So, it is not a desirable. So, of the two parameters B and h, if you vary the thickness, the stress state itself changes at the crack-tip. So, it is not desirable to change B; on the other hand, you can vary the height.

In this fashion, we will not alter the stress field at the crack-tip; that is the advantage.

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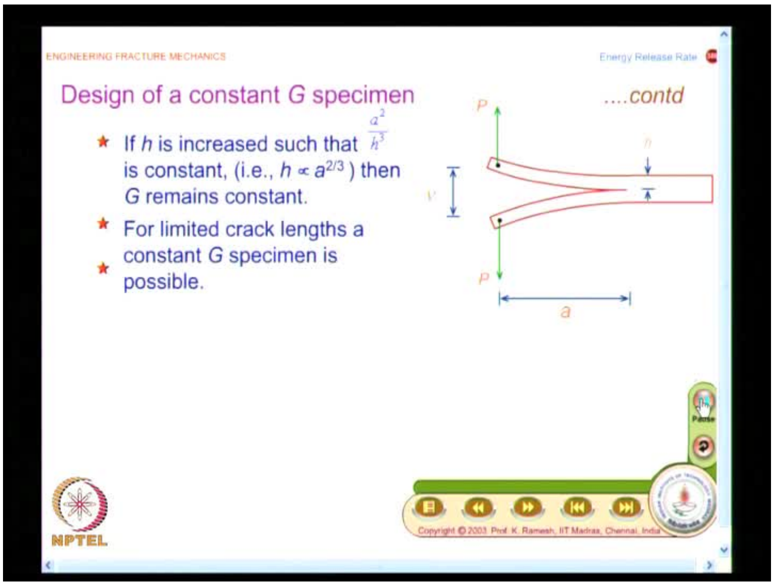


And we will also look at how to change it. What we should do is, for the expression to remains constant, that is G to remain constant, increase h in a fashion such that h is proportional to a power 2 by 3. So, I vary the h. Though, it appears like a non-linear expression, in reality, if you look at the dimensions and substitute, it is more or less like a straight line. Once you have a larger specimen, you will also have to look at certain other considerations. You will also have a brief look at that. So, what it shows? As I can make G constant by having a variable h, it is proportional to a power 2 by 3; in essence, I would maintain a square by h cube as constant.

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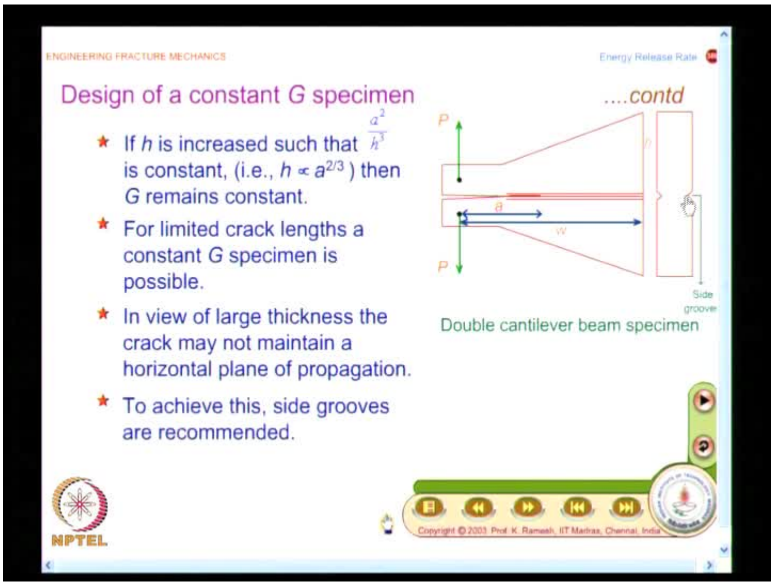


And for limited crack lengths, a constant G specimen is possible. I would like you to make a neat sketch of this. So, the specimen is shown like this, slightly different from what we had seen. So, I have a place where I can apply the load and the specimen is like this. In practice, this appears to be very close to a straight line. And when you have this height is varying along the crack length, one of the issues which you have to pay attention is, when you perform an experiment, see you would like the track to go in the same plane; you do not want the track to deviate; then the analysis becomes difficult because if I even if I achieve, a constant G specimen that shows whatever the development we have done on understanding energy release rate is valid. Because we have predicted for the specimen energy release rate is constant, and if we experimentally verify energy release rate is constant, then what do you find?

Our understanding of energy release rate is reasonably okay. But people have used such specimens and explode even crack arrest scenarios. People also a thought of G changes as a function of crack length. So, if the energy availability is less than for the formation of two new cracks surfaces, then crack has to eventually come to a stop; so, for all that, people tweaked on this. So, designing new specimens was equally challenging in early development of fracture mechanics. So, in order to prevent the crack to not to deviate like what is shown here, what is to be done? People suggested - provide grooves on the specimen, and these are known as side grooves, and you just watch the animation. You will find, in this case, the crack proceeds in the plane, and the side view is like this. So, you make over the thickness of the specimen, this central plane as v. So, you deliberately do that.

In fact, we would borrow this concept when we look at a specimen for fracture toughness test. There again we need to maintain the fatigue crack in a plane of our choice to conduct the actual experimentation. So, we would take this as an advantage. There also we would provide in some fashion the plane of our choice to be weakened.

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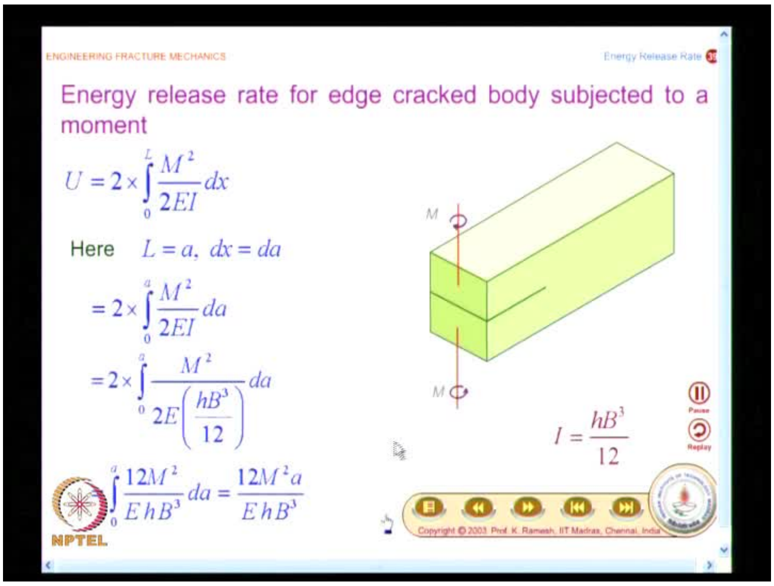


And we will also take up another example to see the utility of energy release rate. You make a neat sketch of the specimen. See, in the discussion of energy release rate, we have looked at the force and displacement. It could also be a generalized force and corresponding generalized displacement; it is equally applicable.

See, all along we have only looked at the energy release rate for the case of mode 1 loading; a similar approach could be extended for solving even a mode 3 problem or mode 2 problem, if the problem is post property.

And now, we take up a problem of a cantilever beam specimen. It is a double cantilever, but the loading is different. You apply a bending moment on this surface and this surface (Refer Slide time: 46:30), and it provides a tearing action.

It provides a tearing action. The moment you get a tearing action, this is the problem of mode 3, and we have already looked at for slender members, how to calculate the energy. So, what we recognize is the top portion is a beam subjected to bending, the bottom portion is another beam subjected to bending, and you can write the strain energy in the presence of a crack as integral 0 to L M square by 2 E I into dx, the whole thing multiplied by a factor 2 because we are look at the top beam as well as the bottom beam.

And what do you take here? The length of the crack is a; so, the beam length is taken as a, and dx is nothing but da. So, the integral you replace it as 2 into integral 0 to a M square by 2 E I into da, and you should also write this expression for I carefully. See, you should 0020recognize what is the plane of bending and then write the moment of inertia, and look at how it bends, how the dimensions h and B are shown. So, when you look at that, you get I as hB cube by 12, and just substitute it in this; fairly a simple exercise and you get the final expression which is given as 12 M square a divided by EhB cube.

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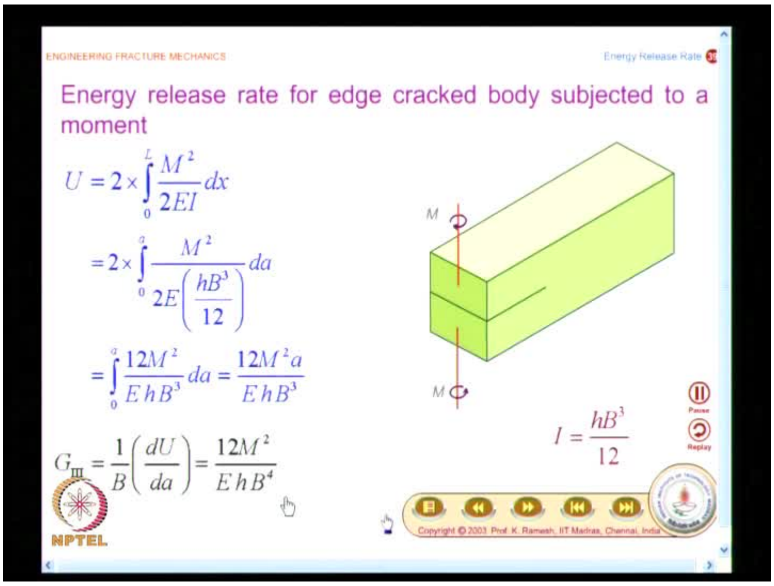


So, for this problem which is a mode 3 situation, the value of G is given as 1 by B dU by da; that gives you 12 M square divided by EhB power 4.

See, I have taken 2 problems: the first problem we solved by the compliance approach; the second problem we evaluated the strain energy and obtained the expression for G. And you have several problems in your assignment sheet where you could extend your knowledge of strength of materials to find out the energy stored or find out the compliance, and evaluate the energy release rate.

So, in this class, we have generalized our understanding on energy release rate. We have expressed energy release rate in terms of potential energy in some fashion. Then, for the problems involving crack, Irwin pointed out compliance is easier to handle; so, we also got the energy release rate in terms of compliance. To form up our knowledge of understanding, we have taken up two example problems and evaluated the energy release rate of this. One of the example problems provided a clue to develop interesting specimens for verifying concepts developing fracture mechanics. So, we saw the interesting case of constant G specimen.

Thank you.