

Advanced Dynamics
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Module No # 03
Lecture No # 14
Impulse- Momentum Relation – II

We will carry forward our discussion on impulse momentum relation.

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Moment of linear momentum

- First moment of Newton's second law
- Angular momentum equation
- Angular impulse-momentum relation
- Conservation of angular momentum

We are going to look at the first moment of the Newton's second law and define a new quantity called the angular momentum. Then we will derive the angular momentum relation and see how angular impulse is related to the angular momentum. Finally, we will discuss the conservation of angular momentum.

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Angular momentum

Newton's 2nd law: $\frac{d\vec{G}}{dt} = \vec{F}$

$\vec{G} = m\vec{v}$ linear momentum

Taking moment of 2nd law about a fixed point O

$$\vec{r} \times \left(\frac{d\vec{G}}{dt} = \vec{F} \right) \Rightarrow \frac{d}{dt}(\vec{r} \times \vec{G}) = \vec{r} \times \vec{F}$$

$$\Rightarrow \boxed{\frac{d\vec{H}_0}{dt} = \vec{M}_0} \quad \vec{H}_0 = \vec{r} \times \vec{G} \begin{cases} \text{Moment of linear momentum about O} \\ \text{Angular momentum about O} \end{cases}$$

Diagram illustrating the derivation of angular momentum. A particle is at position \vec{r} from a fixed point O. A force \vec{F} is applied. The linear momentum is $\vec{G} = m\vec{v}$. The diagram shows the vector cross product $\vec{r} \times \frac{d\vec{G}}{dt}$ and its expansion into $\frac{d\vec{r}}{dt} \times \vec{G} + \vec{r} \times \frac{d\vec{G}}{dt}$. The first term is zero, leaving $\vec{r} \times \frac{d\vec{G}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{G}) - \vec{v} \times \vec{G}$. The term $\vec{v} \times \vec{G}$ is circled in blue.

Let us start with Newton's second law: the rate of change of linear momentum \vec{G} is equal to the force \vec{F} , as shown above. Consider a fixed point O as shown above about which we take moment of Newton's 2nd law to write

$$\vec{r} \times \left(\frac{d\vec{G}}{dt} = \vec{F} \right) \Rightarrow \frac{d}{dt}(\vec{r} \times \vec{G}) = \vec{r} \times \vec{F}$$

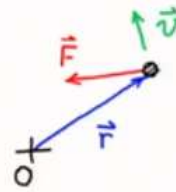
$$\Rightarrow \boxed{\frac{d\vec{H}_0}{dt} = \vec{M}_0} \quad \vec{H}_0 = \vec{r} \times \vec{G} \begin{cases} \text{Moment of linear momentum about O} \\ \text{Angular momentum about O} \end{cases}$$

\vec{H}_0 is the new quantity which we have now obtained by taking the moment of Newton's second law about the fixed point O. This is called the moment of linear momentum vector, or also known as angular momentum vector. It is very important now to attach this point about which we have taken the moment.

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Angular impulse-momentum relation

$$\frac{d\vec{H}_O}{dt} = \vec{M}_O$$
$$\Rightarrow \vec{H}_O(t_2) - \vec{H}_O(t_1) = \underbrace{\int_{t_1}^{t_2} \vec{M}_O dt}_{\text{Angular impulse about O}}$$



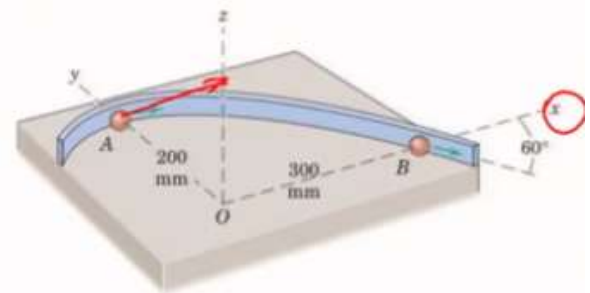
- If the moment of force on a particle about a fixed point vanishes, angular momentum of the particle about that point is conserved

Thus we have this relation: rate of change of angular momentum of a particle about the fixed point O is equal to the moment about O of the net force acting on the particle. Now if we just integrate over time, we get the difference in angular momentum about O at 2 time instants is equal to the integral of the moment about O over this time interval. This is the angular impulse momentum relation. Now, it straight forwardly follows that if the moment of the force on a particle about a fixed point vanishes, then angular momentum of the particle about that point is conserved.

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Problem 1:

A small 0.1-kg particle is given a velocity of 2 m/s on the horizontal x - y plane and is guided by the fixed curved rail. Friction is negligible. As the particle crosses the y -axis at A, its velocity is in the x -direction, and as it crosses the x -axis at B, its velocity makes a 60° angle with the x -axis. The radius of curvature of the path at B is 500 mm. Determine the time rate of change of the angular momentum H_O of the particle about the z -axis through O at both A and B.



Source: Dynamics, Meriam and Kraige

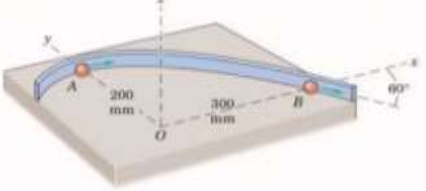

We consider the above example.

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$\dot{\vec{H}}_O = \vec{M}_O$
 At A: $\vec{F}_A = -F_A \hat{j} \Rightarrow \vec{M}_O = \vec{OA} \times \vec{F}_A = 0$
 $\Rightarrow \dot{\vec{H}}_O = 0$
 At B: $\vec{F}_B = m \vec{a}_B$
 $\vec{a}_B = \cancel{a_t} \hat{e}_t + \frac{v_B^2}{\rho_B} \hat{e}_n$
 Energy conservation $\Rightarrow v_B = v_A = 2 \text{ m/s}$
 $\rho_B = 0.5 \text{ m}, \hat{e}_n = -\cos 30^\circ \hat{i} - \sin 30^\circ \hat{j}$
 $\Rightarrow \vec{a}_B = 4(-\sqrt{3} \hat{i} - \hat{j}) \text{ m/s}^2$
 $\vec{F}_B = 0.4(-\sqrt{3} \hat{i} - \hat{j}) \text{ N} \Rightarrow \vec{M}_O^B = \vec{OB} \times \vec{F}_B = -0.12 \hat{k} \text{ N}\cdot\text{m}$
 $\Rightarrow \dot{\vec{H}}_O^B = -0.12 \hat{k} \text{ kgm}^2/\text{s}^2 \sim \text{kgm}^2/\text{s}^2$

$\vec{H}_O = \vec{r} \times \vec{G}$
 $= m \times \text{kg m/s}$
 $= \text{kgm}^2/\text{s}$
 $= \text{Nms}$

$\Delta \vec{H}_O = \int \vec{M}_O dt$
 Nms

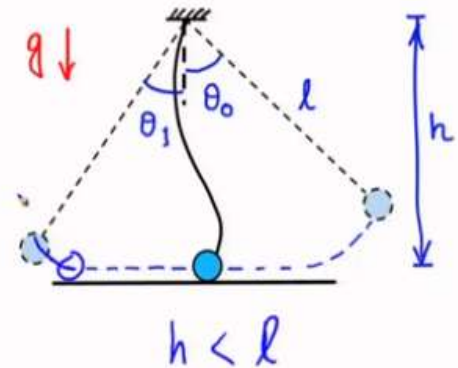



The detailed solution is presented in the slide above. It is important to note of the units of angular momentum as detailed above.

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Problem 2:

The bob of a pendulum released from $\theta = \theta_0$ hits the frictionless ground and moves straight till the string becomes taut again and lifts up to $\theta = \theta_1$. Taking the length of the pendulum to be l and the height of the support to be h , determine θ_1 in terms of the other parameters.



We consider the above problem next.

The problem is solved in 3 parts: (1) motion on a circular arc from the point of release to the first impact with the ground at A and subsequent horizontal frictionless motion on the ground, and (2) impulse from the string at B, and (3) motion on a circular arc from B to the highest point reached. These are shown in the following 2 slides.

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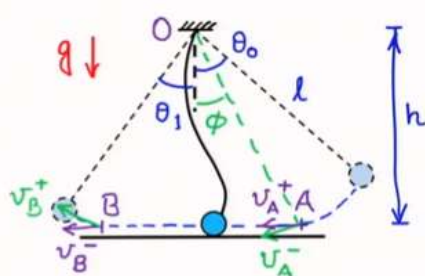
At A : conservation of horizontal linear momentum
(vertical impulse)

$$m v_A^- \cos \phi = m v_A^+ \quad (\cos \phi = \frac{h}{l})$$

Energy conservation

$$\frac{1}{2} m (v_A^-)^2 = m g (h - l \cos \theta_0)$$

$$\Rightarrow v_A^- = \sqrt{2 g (h - l \cos \theta_0)}$$

$$\Rightarrow v_A^+ = \frac{h}{l} \sqrt{2 g (h - l \cos \theta_0)} = v_B^-$$


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At B : conservation of angular momentum about O
(impulse along string)

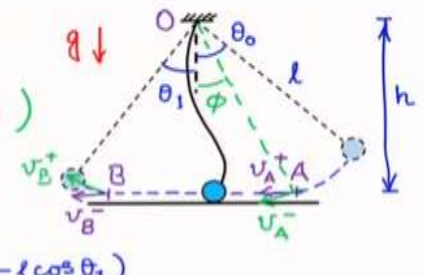
$$h(m v_B^-) = l(m v_B^+)$$

$$\Rightarrow v_B^+ = \frac{h}{l} v_B^- \quad (v_B^- = \frac{h}{l} \sqrt{2 g (h - l \cos \theta_0)})$$

Energy conservation

$$\frac{1}{2} m (v_B^+)^2 = m g (h - l \cos \theta_1)$$

$$\Rightarrow h - l \cos \theta_1 = \frac{h^4}{l^4} (h - l \cos \theta_0)$$

$$\Rightarrow \cos \theta_1 = \frac{h}{l} \left[1 - \frac{h^3}{l^3} \left(\frac{h}{l} - \cos \theta_0 \right) \right]$$


The final answer is shown above.

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Summary: Moment of linear momentum

- First moment of Newton's second law
- Angular momentum equation (vector equation)
- Angular impulse-momentum relation
- Conservation of angular momentum
- Applications

The summary of the discussions is provided above.