Tools in Scientific Computing
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Lecture - 27
Singular perturbation for ODE

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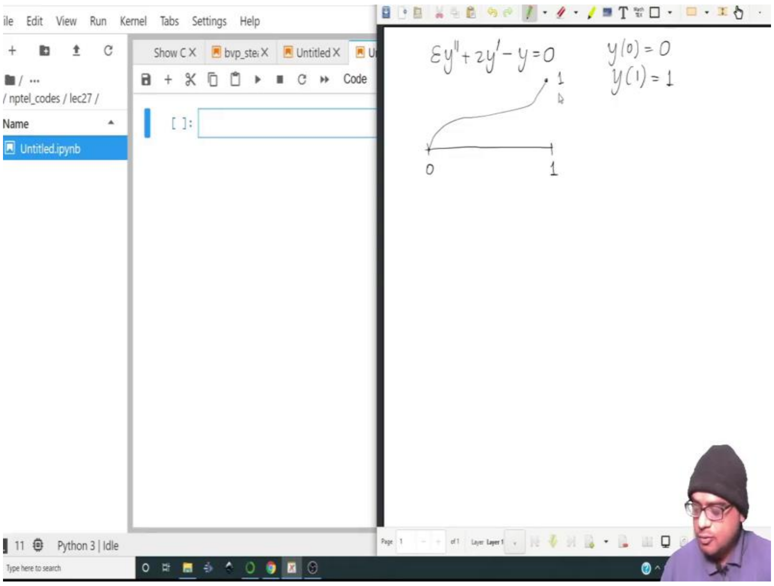

Hello everyone welcome to this lecture we are going to discuss about Singular perturbation. I have given a hint in the last class as to what premise of singular perturbation is and we will start with that premise, order where the highest derivative will be multiplied by a small number. So, let us consider the following differential equation. Now this is a second order differential equation and let us define the 2 boundary conditions alright.

So, we have a domain spans from 0 to 1 and $y(0)=0$ and $y(1)=1$ and there is some functional behavior something like this we do not know what it is. But we would like to find out what it is. So, before going into the approximate solution and it is worth while looking into the analytical solution for this and obviously the analytical solution does exist.

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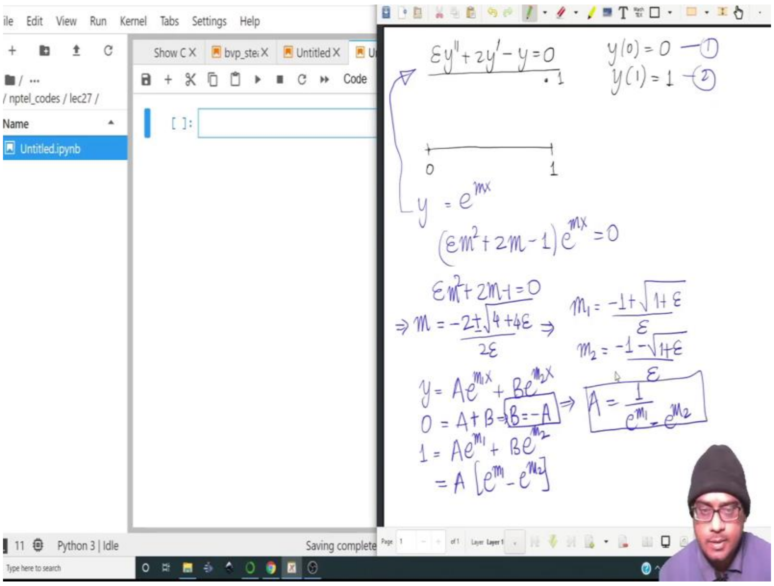

So, let $y = e^{mx}$. So, what do we have over here? We have $(\epsilon m^2 + 2m - 1)e^{mx} = 0$. And you will obtain this once you substitute this equation over here and now if this is equal to 0 then $\epsilon m^2 + 2m - 1 = 0$, which implies at $m = \dfrac{-2 \pm \sqrt{(4 + 4\epsilon)}}{2\epsilon}$ right.

These are the roots which imply that $m_1 = \dfrac{-1 + \sqrt{(1+\epsilon)}}{\epsilon}$ and $m_2 = \dfrac{-1 - \sqrt{(1+\epsilon)}}{\epsilon}$ by (Refer Time: 03:05). So, fine so now we can write down the solution $y = Ae^{m_1 x} + Be^{m_2 x}$.

Let us use the first boundary condition 0 = A + B fair enough and the second boundary condition will yield $1 = Ae^{m_1} + Be^{m_2}$ because x equal to 1. So now because B = -A, so this becomes $1 = A[e^{m_1} - e^{m_2}]$ and this implies that $A = \dfrac{1}{[e^{m_1} - e^{m_2}]}$ while B = -A. So, now with the help of this we can go ahead and make a plot of this function.

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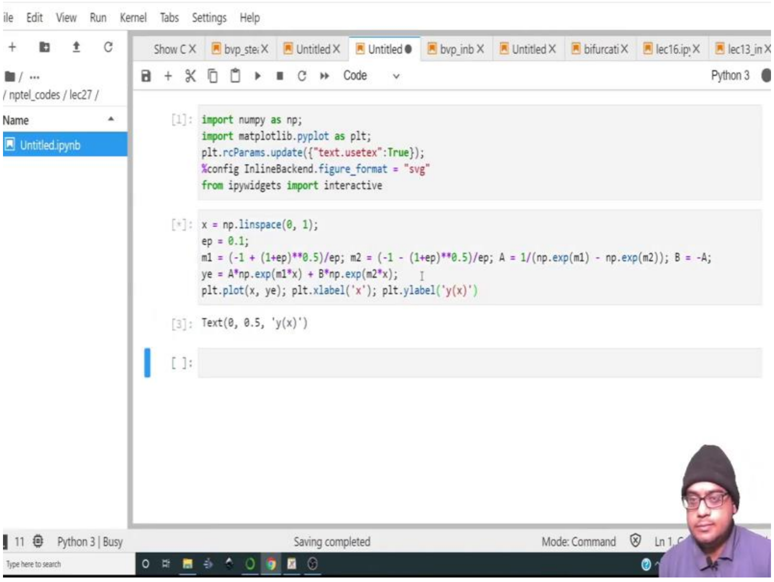

So, let me just grab the starting code let me define x then let me define ϵ 0.1. What else do we need? So, we need to define m1, we need to define m2, we need to define A, we need to define B. So, m1 is - 1 +. What is 1 + ϵ? This whole thing divided by ϵ well this bracket term is raised to the power 0.5 alright.

So, let me this should be a bracket. So, let me copy this m2 is equal to this, but the sign over here is negative; A = 1/(np.exp(m1) - np.exp(m2)) and B = -A right. So, y is or let me put it y exact isye = A*np.exp(m1*xe) + B*np.exp(m2*xe). So, then let us plot it ok.

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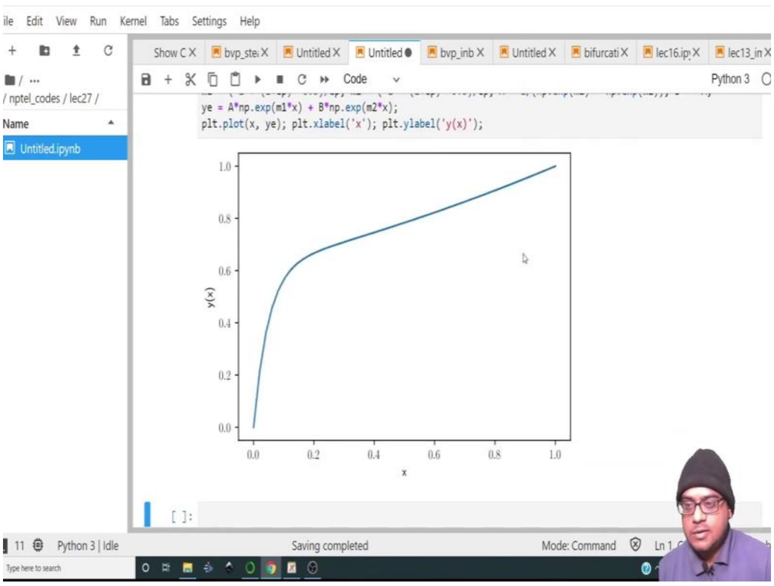

So, we have this kind of behavior and we see that. So, what do we observe over here? There is a zone near x = 0 not near x = 0, but its towards the x = 0 boundary. Where there is a fast variation in the solution after which it gradually goes.

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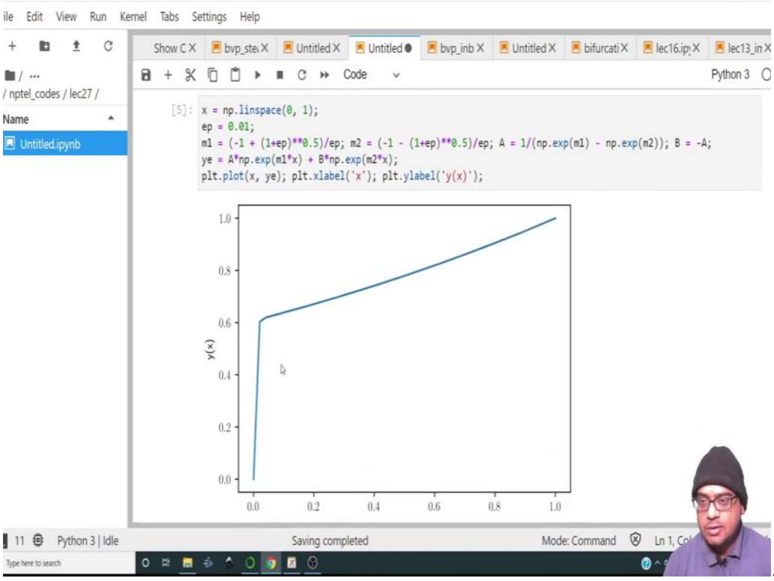

And let us change the value of ϵ to see whether I mean what the effect is. So, when we make it 0.1 the variation appears to be sharper, in fact we need to have a final resolution for x in order to view that rapid change.

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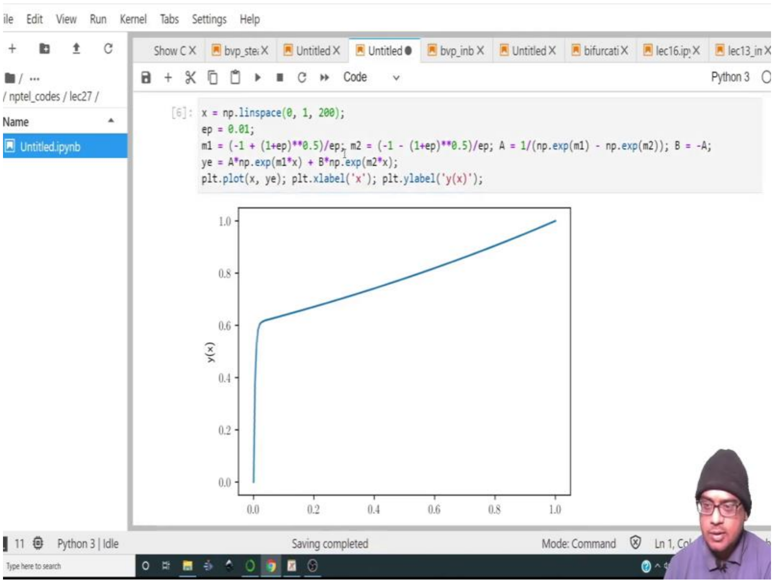

So, let me take 200 points. So, you have a fast variation over here and then a smooth variation something like this and this kind of thing is called as a boundary layer. So, boundary layer refers to that zone in the domain where there is a very rapid change in the variable and the term boundary layer is very closely associated.

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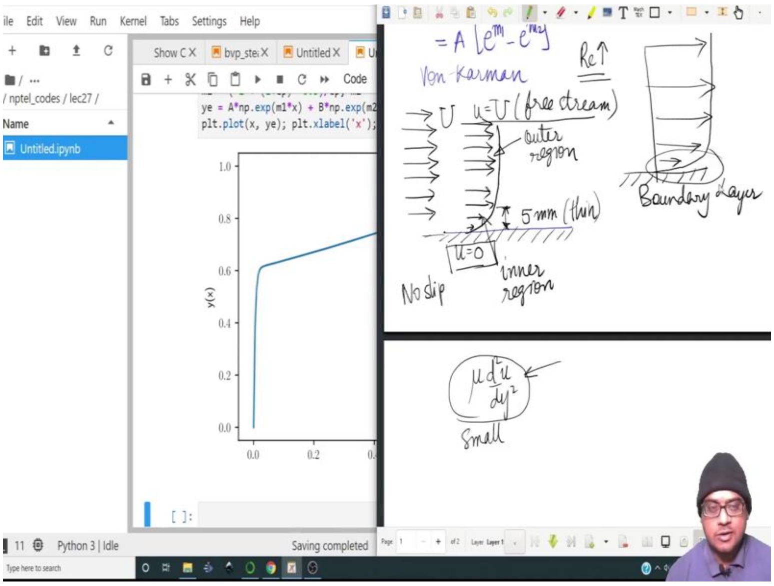

In fact, Von Karman is sort of credited for assigning this name. So, very near a wing of an airplane what happens? So, there is fluid or rather air which is relatively flowing into the very higher velocity, because the plane is flying at a very large velocity. So, very near the wall the effects of viscosity cause the very first layer of fluid to slow down.

So, the fluid which is away from or rather air which is away from the wing it does not really feel the effect of this velocity, but air very close to the wall will feel the effect. So, there will be some region near the wall which feels the effect of the velocity, but the outer region does not feel the effect of the wing. So, this is the outer region.

Whereas the region in which effect of viscosity is felt the effect of viscosity is manifested as a no slip boundary condition and deplete and so this region is the inner region. And typically this thickness the thickness of the region over which the change happens is maybe 5 millimeter its quite small compared to the scale of the wing. For example, the wing could be several meters wide, so its quite thin.

So, if we now plot the velocity profile at some location we will have a rapid change in velocity and then it will remain more or less constant right. This region near the wall is termed as the boundary layer and such boundary layers exist for Reynolds number quite large. So, Reynolds number for those of you who are from a fluid mechanics background, a higher Reynolds number indicates that the flow is dominated by inertia rather than viscosity, but despite that dominance alright.

So, those of you who are from fluid mechanics background they will realize that viscosity manifests itself in the momentum equation, through a term which looks something like this.

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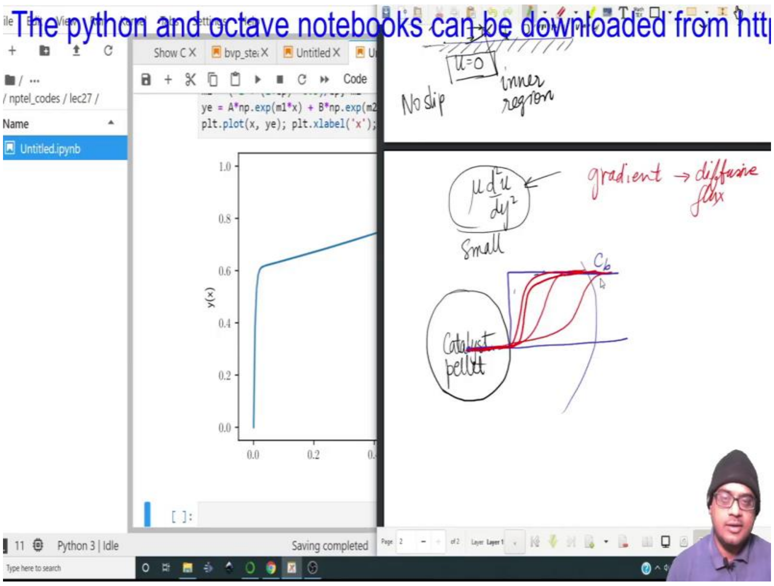

And when the flow is dominated by inertia this particular effect is quite small effect of viscosity is quite small. But if we neglect this term and completely the equation the momentum equation instead of being a second order equation boils down to a first order equation.

And a first order equation would not be able to satisfy 2 boundary conditions simultaneously, unless the 2 boundary conditions are consistent that is a trivial case near the wall the velocity is 0. But far away from the wall the velocity is equal to the free stream velocity. It has to satisfy these 2 boundary conditions, if viscosity is not there then u = 0 is no longer the boundary condition which the flow would satisfy, we cannot enforce no slip in the absence of viscosity.

And it is this viscosity in such inertia dominated flows that allow for a small region near the wall of the wing or plate or whatever it may be to sort of have that viscous effect beyond which the effect of viscosity does not dominate anymore. For those of you who are from a chemical engineering background.

Consider a catalyst pellet, so this catalyst pellet suppose there is a fluid over here alright, there is some fluid and the catalyst reacts with the fluid. So, imagine I draw a transect like this and when the catalyst has not yet begun reacting I can plot the concentration as something like this it is 0 inside the catalyst and suddenly it has some concentration in the bulk and this is the bulk concentration.

Now, the catalyst starts reacting with the fluid maybe it makes some product we do not care about that. So, there will be depletion of concentration near the surface of the catalyst. So, it will be still 0 and then it will be something like this, this there is a sharp gradient going from the catalyst to the fluid and a sharp gradient leads to diffusion gradient leads to diffusive flux.

But if the diffusivity is quite small this profile that I am drawing it will be slowly propagated towards the bulk, the diffusivity is quite small and the reactivity of the pellet is quite high the profile would look something like this. It will be depleted the fluid concentration will be depleted or rather the reactant concentration will be depleted near the catalyst pellet.

But fluid has not yet or rather the concentration from the bulk has not yet been able to replenish that lost reactant near the surface of the pellet. If the diffusivity were to be high then the reactant would rush in from the bulk towards the pellet something like this and we would have mild gradients in concentration.

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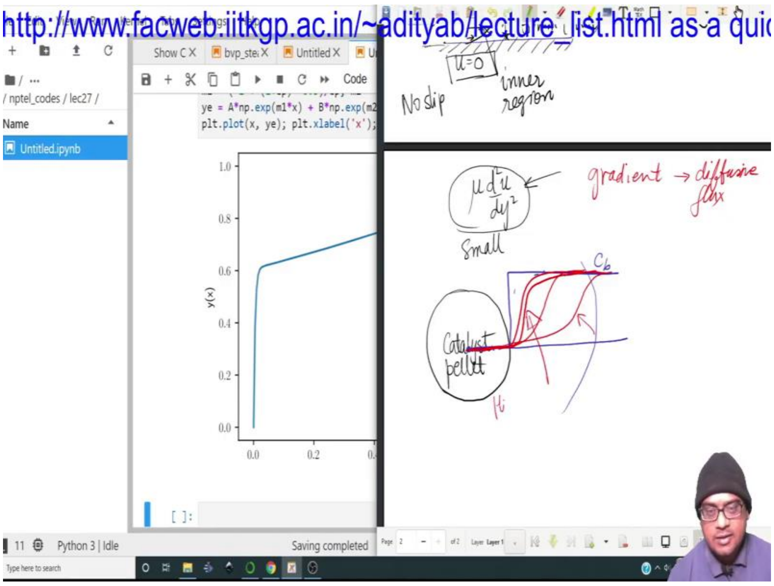

So, this case where the reactivity is high, but the diffusivity is low right.

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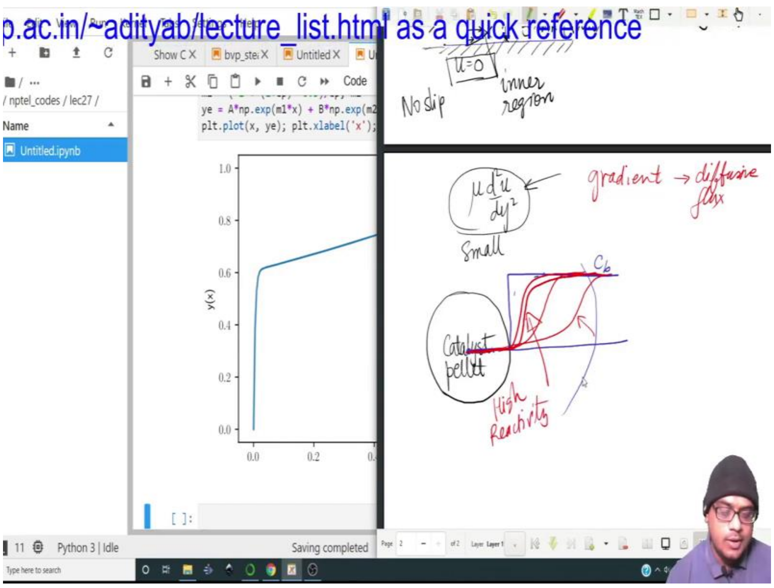

Even that leads to the formation of a concentration boundary layer near the surface of the pellet. So, with these 2 applications in mind let us now look at how we can analytically solve at least this synthetic problem. You can of course apply these techniques to various problems that you might encounter. But the point that I am trying to make is they will have the same hallmark.

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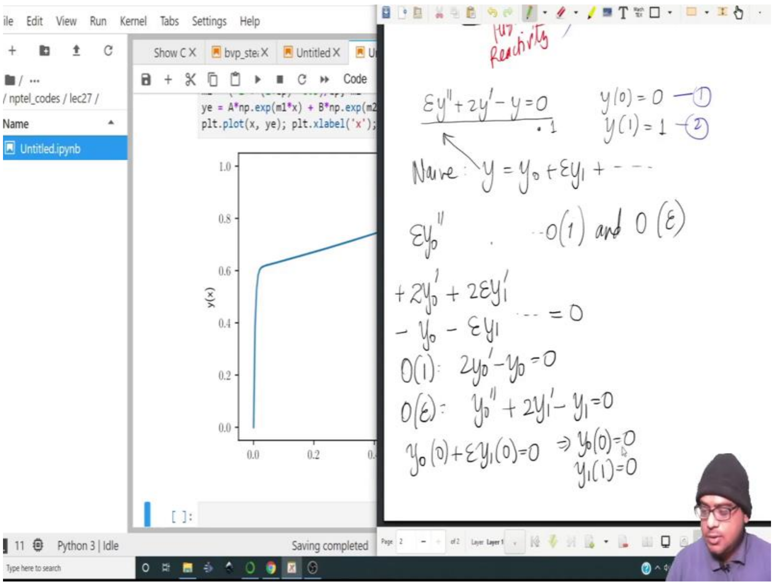

That is there will be some parameter which will be multiplying a higher order term and dropping that term would lead to reduction in the dimensionality of the problem. If I want to cast it in that particular form, so let us do the Naive perturbation that we did in the case of regular perturbation let me consider $y = y_0 + \epsilon y_1 + \ldots$ so on. In the hope that this particular sequence converges let me substitute this over here.

So, what do we have $\epsilon y_0'' + 2y_0' + 2\epsilon y_1' - y_0 - \epsilon y_1 + \ldots$ and so on. So, I am just writing the maximum order as ϵ^2, but we do not want to go so far.

So, we can drop this as well, so I am just writing order 1 and order ϵ terms. So, now if we isolate the various terms of order ϵ, so at order 1 we would have $2y_0' - y_0 = 0$ at order ϵ we would have $2y_0'' + 2y_1' - y_1 = 0$. And now what are the boundary conditions at $y_0(0) + \epsilon y_1(0) = 0$ which implies $y_0(0) = 0$ and $y_0(1) = 0$ fine.

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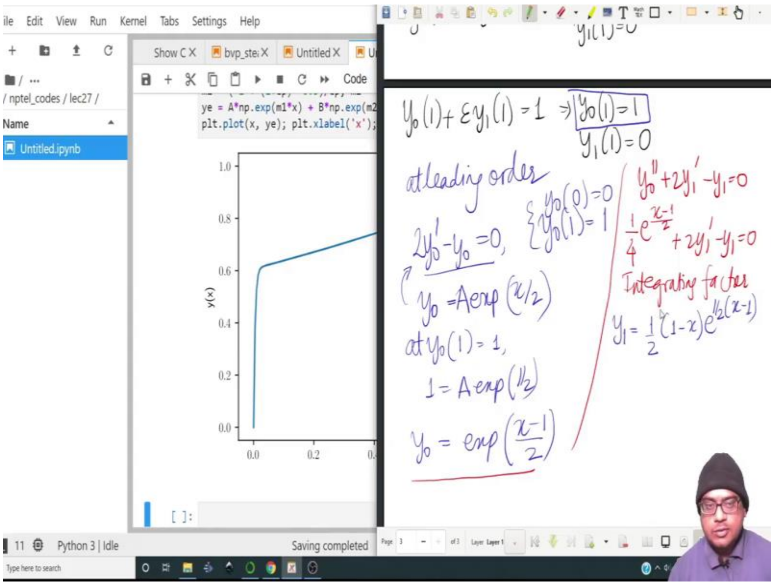

What about the second boundary condition? So, $y_0(1)+\epsilon y_1(1)=1$ which implies $y_0(1)=1$ and $y_1(1)=0$. But look at the leading order we have a first order differential equation, but we are presented with 2 boundary conditions not this, but the 2 boundary conditions are this and this.

So, at the leading order we have $2y_0'-y_0=0$; $y_0(1)=0$ and $y_0(0)=0$, we cannot satisfy 2 boundary conditions simultaneously for a first order differential equation. So, we must take a call there must be only 1 boundary condition which this can satisfy and we already know from the graph that we plotted that y_1 we must have this being satisfied right.

So, let us write down the solution $y_0=\exp\left(\frac{x-1}{2}\right)$ right this is the solution for this equation. So now, let us choose that at $y_0(1)=1$. So, using this boundary condition we have $1=A\exp\left(\frac{1}{2}\right)$ and thus we have $y_0=\exp\left(\frac{x-1}{2}\right)$ so that is the solution.

Let us now write down the first order equation which is $y_0''+2y_1'-y_1=0$. We already have a solution for y_0 and so a second derivative of this will be $\frac{1}{4}e^{\frac{x-1}{2}}+2y_1'-y_1=0$. So, this equation can be integrated with the help of an integrating factor alright.

But I am going to write down the solution directly you can try to find it out. So, the solution for y_1 turns out to be $y_1 = \frac{1}{2}(1-x)e^{\frac{x-1}{2}}$. So, this is the solution for y_1. So, let us do it I mean let us construct the outer solution.

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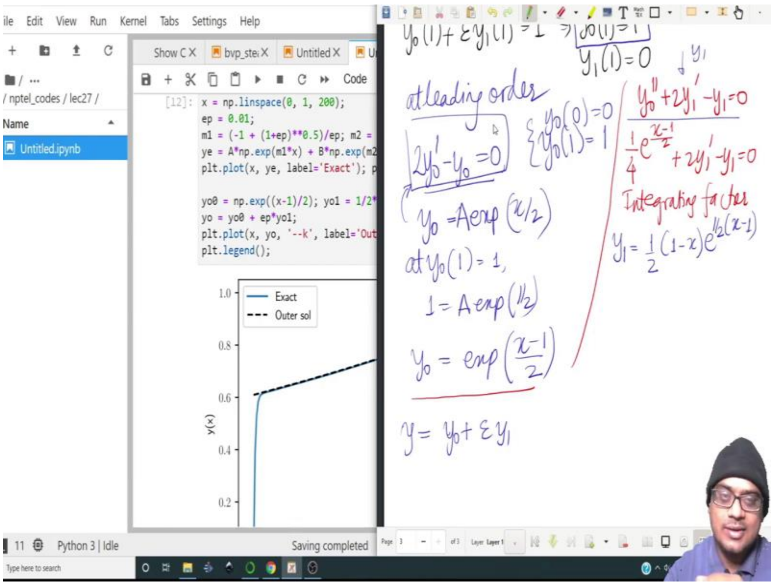

I mean I am calling it outer, but from what we have discussed so far, we can write the solution as $y = y_0 + \epsilon y_1$ let us do it.

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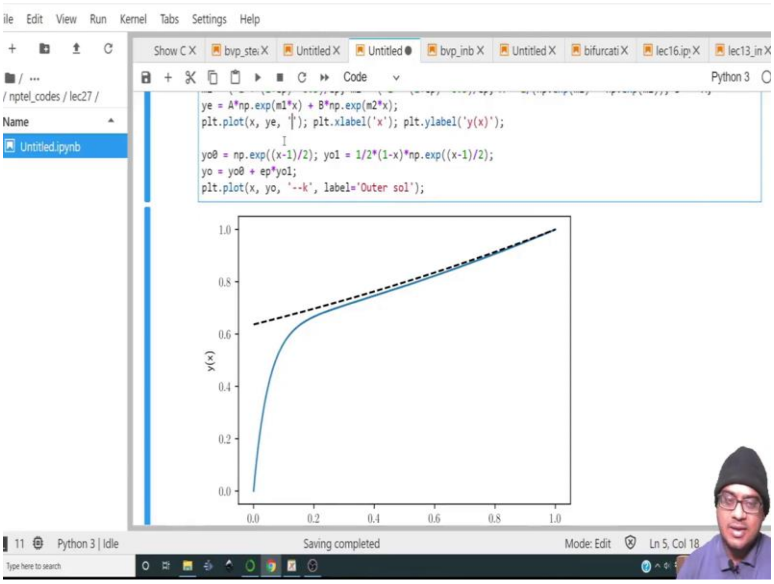

So, let me write down y0 or yo = yo0 + ep*yo1; yo0 = np.exp((x-1)/2) andyo1 = 1/2*(1-x)*np.exp((x-1)/2). Then we simply plot let us give it a label I will call it outer solution. In fact, let me make it as attached line style. So, let me run this and see what happens excellent.

So, let me reduce this parameter, but rather let me increase the parameter ok. So, it is clear that the approximate solution that we have obtained which we are calling as the outer solution it does match quite well with the behavior of the equation in the non boundary layer part and hence it is called as the outer solution; much like the region away from the wing it is the outer region right sorry.

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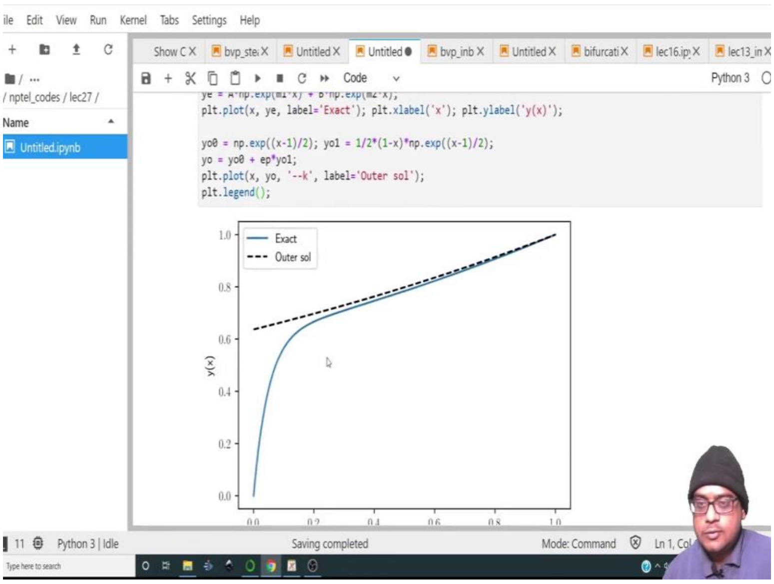

So, it is satisfies and as epsilon becomes smaller the approximation becomes better.

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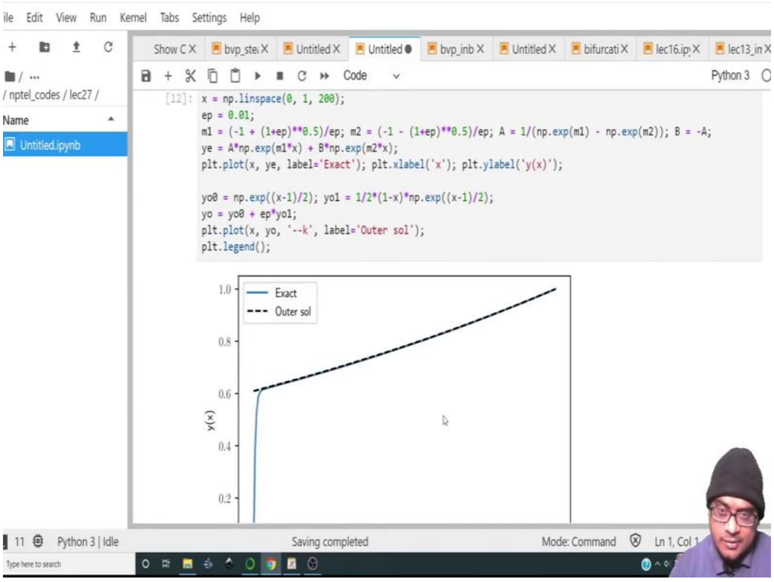

But what is the caveat? Obviously, the outer solution has been found out using only 1 boundary condition that is the boundary condition at x = 1. We have nowhere utilized the other boundary condition that is at x = 0 and in fact at each hierarchy alright at. So, this is the first equation the second equation at each hierarchy this is the equation for actually y_1 it is not an equation for y_0, because y_0 has been already obtained using the leading order equation.

So, each hierarchy the each equation appearing in the hierarchy will be first order. So, you will never be able to satisfy the boundary condition at x = 0. But there must be a way there is obviously some functional form which is different than the outer solution that is happening inside the inner region. So, how do we find that out? That is the question well?

So, far we have not done anything which is different from this from the regular perturbation business. So now, we will do rescaling the coordinate.

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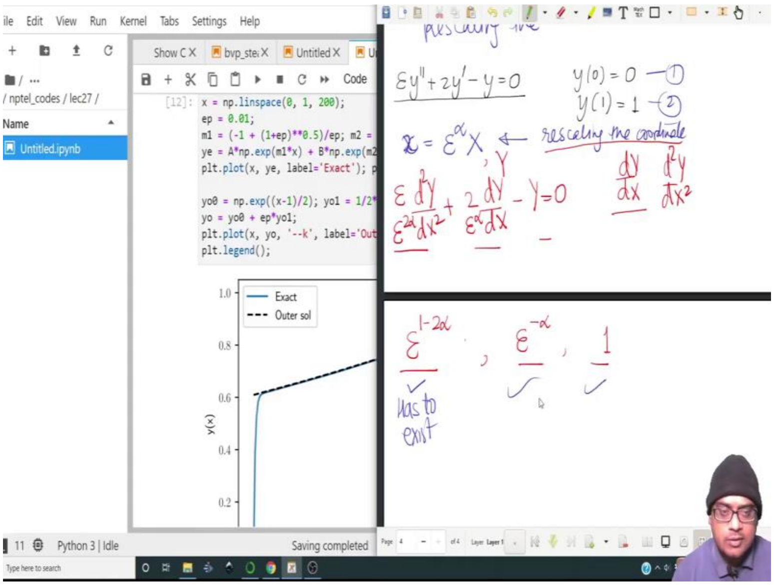

So, we must now rescale the coordinate, so that we can have some region in which this particular term will be important let us see how we can do that. So, we have the equation over here. So, let us consider $x = \epsilon^{\alpha} X$. So, essentially what we are doing here is we are rescaling the coordinate.

So, depending on the smallness of epsilon we must then appropriately choose x to be quite large rather X to be quite large right. That should come out naturally as a consequence of the analysis that I am going to show, after substituting this let us see what happens. So, let us denote the solution of y in the rescaled coordinate system as Y.

So, what do we have $\frac{d^2Y}{dX^2} + \frac{2}{\epsilon^{\alpha}}\frac{dY}{dX} - Y = 0$. Let us now analyze the different orders of magnitude of the 3 terms. So, the first term has an order of magnitude $\epsilon^{1-2\alpha}$, second term has an order of magnitude as $\epsilon^{-\alpha}$, third order third term has a as a term which is order 1.

Now, how do we know that these are the orders it is because we are going to assume that upon rescaling the $\frac{dY}{dX}$ or $\frac{d^2Y}{dX^2}$ they will behave more uniformly over the domain over the compressed domain that is they will now be assumed to be order 1 and thus the order of that particular term will be decided purely by the pre factor that is these terms.

Let us now assume see we know for a fact that the second derivative has to exist in order for us to somehow satisfy the condition at the left boundary. So, this term definitely has to has to exist, but now in order for that to exist it must coexist with one of these 2 terms.

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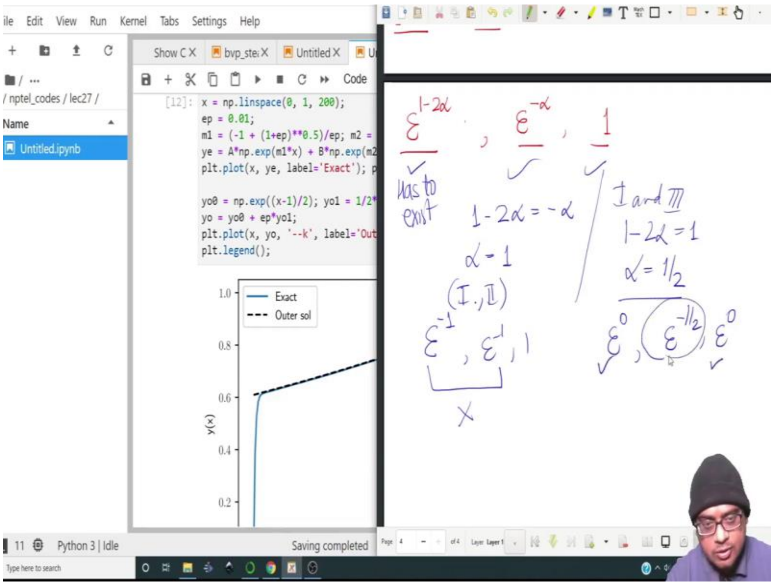

So, suppose this exists with this term. So, what do we have $1-2\alpha=-\alpha$ that yields α = 1. So, this is balance of 1 and 2 terms, the balance of 1 and 3 yields $1-2\alpha=1$ which implies $\alpha=\frac{1}{2}$. So, if we go by α = 1 then what do we have the orders of magnitude will be ϵ^{-1}, ϵ^{-1} and 1.

So, even in this case what is going on these 2 terms of the same order of magnitude, but they are smaller than the order of magnitude of the third term which is simply 1. So, this is clearly not allowed, because we are essentially then dropping another order of the equation, we cannot have that we need to somehow keep the second order derivative as the dominant term.

What about this when $\alpha=\frac{1}{2}$, this order of magnitude becomes ϵ^{0}, this order of magnitude becomes $\epsilon^{-\frac{1}{2}}$ and this becomes ϵ^{0}. So, now these are the highest order terms while this term is smaller, so we have the last term as order 1.

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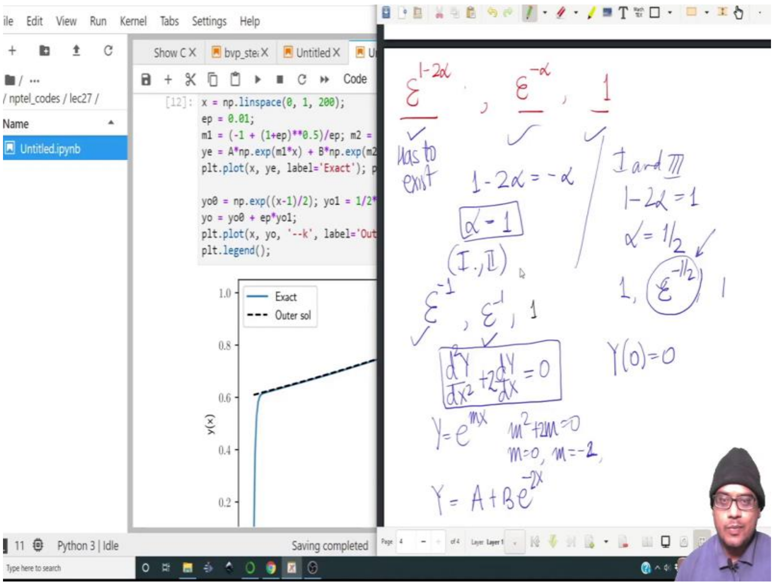

While if $\alpha = \frac{1}{2}$, what are the different terms this particular term will be order 1, this particular term will be $\epsilon^{-\frac{1}{2}}$, this particular term will be order 1. So, over here what do we have ϵ^{-1}+ when ϵ will be small this is $1/\epsilon$, so this term and this term will dominate. So, essentially the higher order term is the yeah the second order term is being retained over here.

But when we have this is the largest order term because there is $\epsilon^{-\frac{1}{2}}$ and ϵ when it is small it is going to be much larger than 1. So, here we are actually again losing out on that higher order derivative. So, this has to be the correct scaling. So, $\alpha = 1$ is the correct scaling alright. So, if $\alpha = 1$ is the correct scaling.

Then what do we have? In that case the governing equation will look something like this plus $\frac{dY}{dX} = 0$ and so this equation can also be solved by assuming that $Y = e^{mX}$ right. And this gives us $m^2 + m = 0$ which implies m = 0 and m = -1. Therefore, the solution will be Y = A + Be^{-X}. I think I am missing a factor of 2 there is a factor of 2 over here which I missed.

So, there will be 2 and so on this will be - 2 and this will be minus 2x alright. So, now we must find out the 2 boundary conditions that this will satisfy ok. Now this is a second order differential equation this has to satisfy 2 boundary conditions, so obviously the first boundary condition will be $Y(0) = 0$ and this is the left boundary which we have not utilized in finding out the outer solution.

But what about the inner solution; rather what about the other boundary condition? Now obviously, we have done something to make this second order derivative exist in this equation and that thing is finding out a rescaled coordinate. So, that the second order derivative is now relevant in the problem unlike the first approximation where it completely went to 0.

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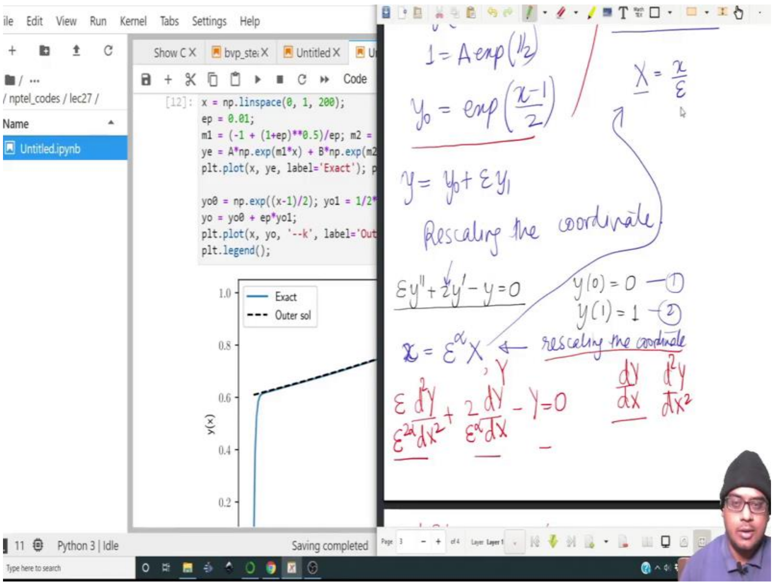

So, $\alpha = 1$. So, let us see what happens to the coordinate in that case. So, when $\alpha = 1$ this rescaled coordinate becomes $X + \frac{x}{\epsilon}$. It means that for small values of x also and I will obtain a large value of X because ϵ is small and so that implies that let us look at this variation I mean I can explain it using this variation.

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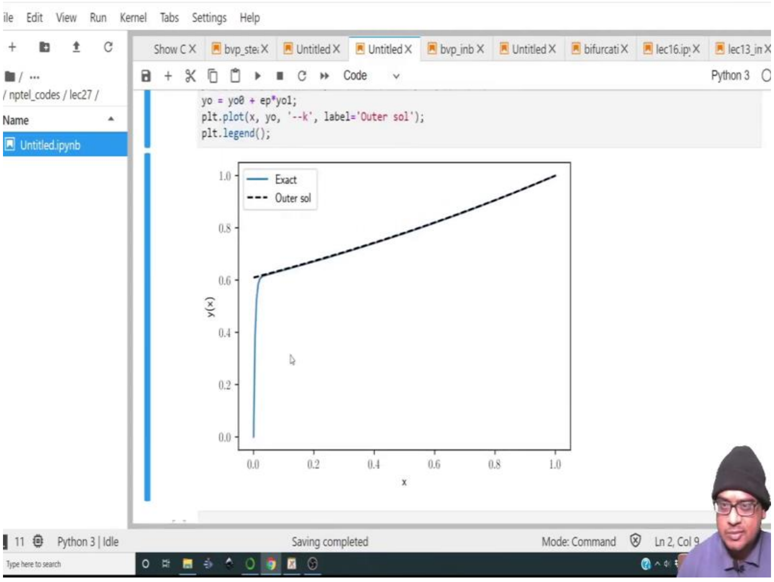

So, we have a fast change and then smooth change.

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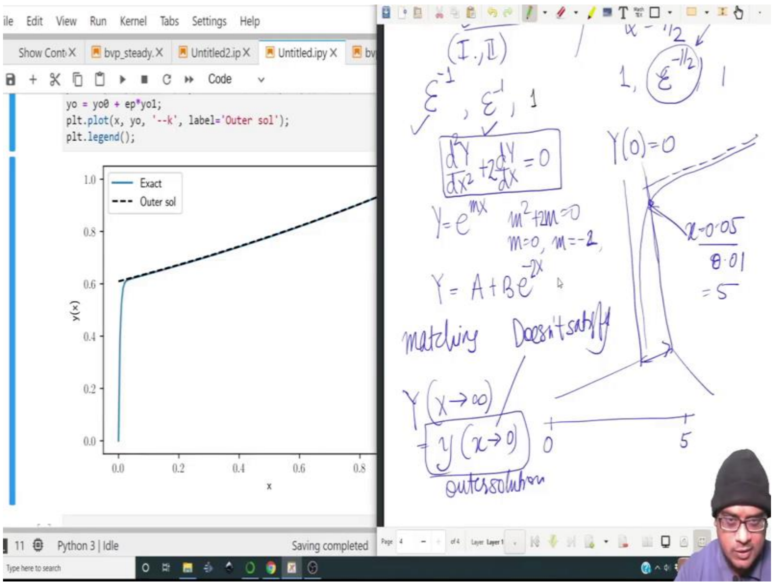

So, we are taking this domain where there is a fast variation and we are magnifying it. So, that the 0 boundary still remains 0, but suppose this change happens at 0.05 ok this change is happening at x = 0.05. But I am dividing it by epsilon meaning I am magnifying at 0.05 to something very large ϵ in this case was 0.01.

So, 0.01 and this becomes 5 essentially this boundary becomes magnified it becomes large. So, the other boundary conditions comes from the principle of matching that is Y for large X large values of X meaning x $\rightarrow \infty$ should match with y for x $\rightarrow 0$ that is this is from the outer solution.

So, x $\rightarrow 0$ for the outer solution means for the dashed line over here whatever limit it is going. So, this dashed line is sort of going to this. So, this boundary is like x $\rightarrow 0$ for the outer solution and why is it true because, obviously it cannot satisfy the x = 0 boundary conditions. So, as x $\rightarrow 0$ this obviously does not satisfy the left boundary condition.

And hence we must merge these 2 solutions 1 solution from the inside layer 1 solution from the outside layer and we must see how we can match them. Well we are not going to match them completely, but I am going to show you the approximate form of the solution in the inner layer. So, the solution was this.

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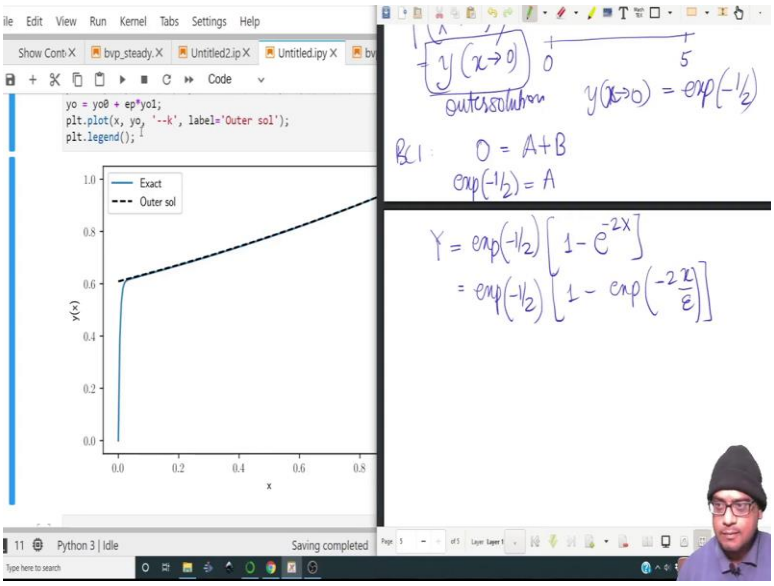

So, what is the limit as x $\rightarrow 0$ Y as x $\rightarrow 0$ x $\rightarrow$ 0 is $\epsilon^{-\frac{1}{2}}$ alright. So, Y so this is the solution and the first boundary condition is this. So, Y (0) = 0 so obviously A will be 0, because once we substitute x = 0 this sorry A + B will be equal to 0 not equal ok. So, first boundary condition is 0 = A + B second boundary condition $\epsilon^{-\frac{1}{2}}$ = A.

So now, x $\to\infty$ will make this to 0 is equal to A. So, $A = e^{-r}$ $p = e^{e^{-1}}$ ok, because when x $\to\infty$ this term will go to 0 alright. So, $Y = e^{-\frac{1}{2}}[1 - e^{-2X}]$. Now let us write this expression in terms of the original coordinate system.

So, this is $Y = e^{-\frac{1}{2}}[1 - e^{-2\frac{x}{\epsilon}}]$, because this is how we found out the scaling. Let me now plot this particular function and see what happens.

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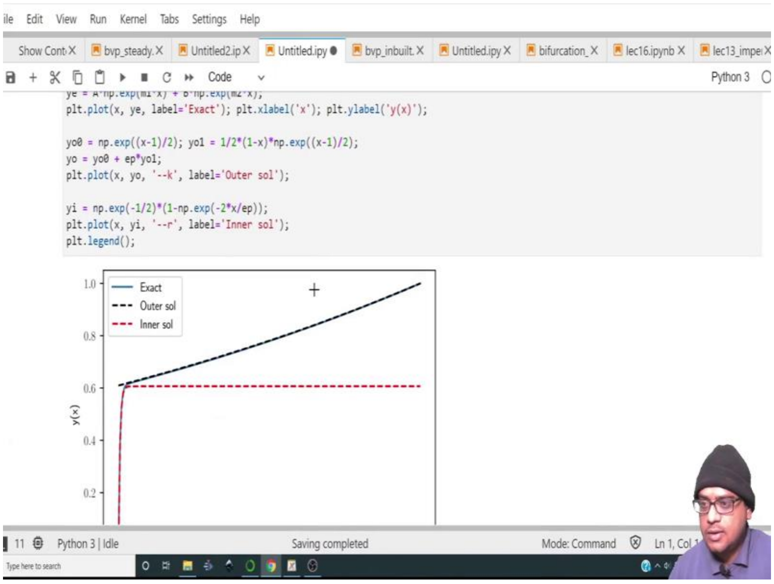

So, I am going to call it yi. So, its yi = np.exp(-1/2)*(1-np.exp(-2*x/ep)) ; plt.plot(x, yi, '--r', label='Inner sol').

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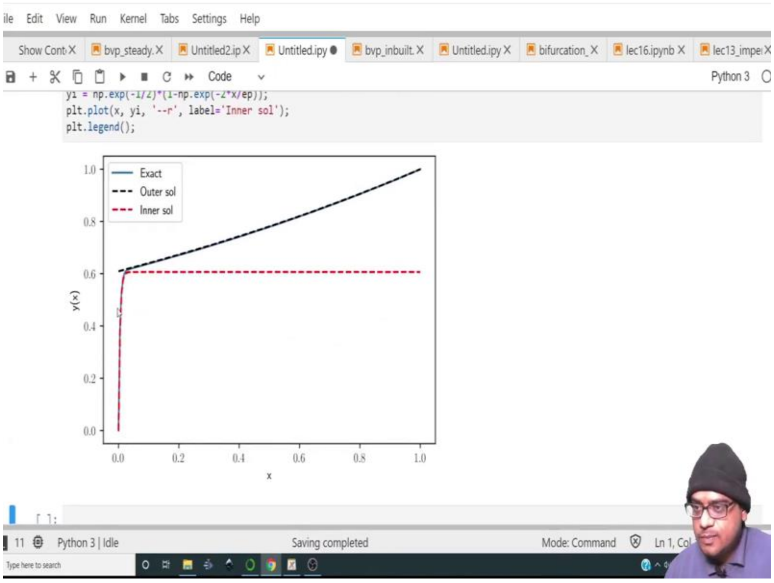

So, let us plot them and see ok. So, the inner solution obviously matches quite well in the inner zone that is for the scale zone outside it does not match. But look the outer limit that is this flat limit it matches with the inner limit of the outer solution. So, the leftmost limit of the black dashed line matches well with the rightmost limit of the red dashed line ok and that is what we imply by matching.

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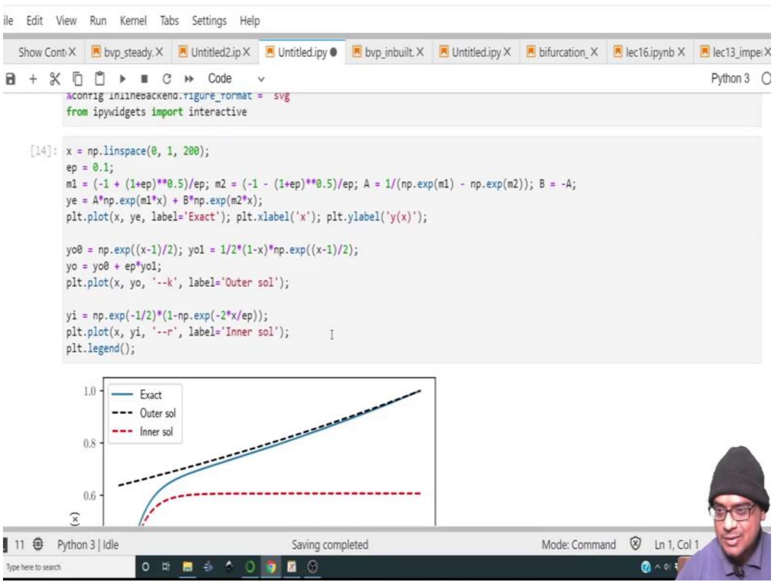

And in fact, let me reduce epsilon a bit that should make it a bit more obvious great.

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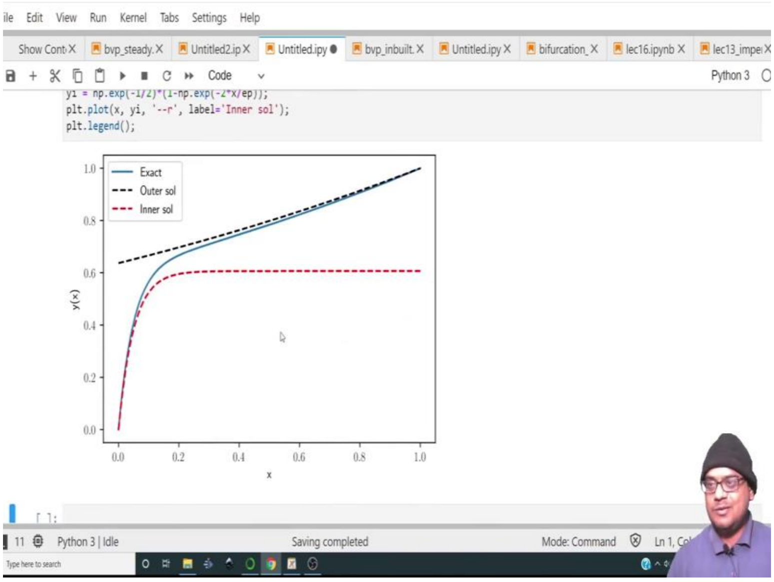

So, inside this red dashed line matches outside this black dashed line matches and this is how we can. So, this is by no means a complete discussion of a singular perturbation method, but I at least hope this gives you an idea of how to go about things. You have 2 distinct behaviors one inside the boundary layer one outside the boundary layer and you can find out approximations to those functions.

In fact, you can find a composite expansion which can give you one single function which will go across both the domains, but that is beyond the scope. But, so let me just show you how we can use some of our old code to solve that equation with the help of psi pi.

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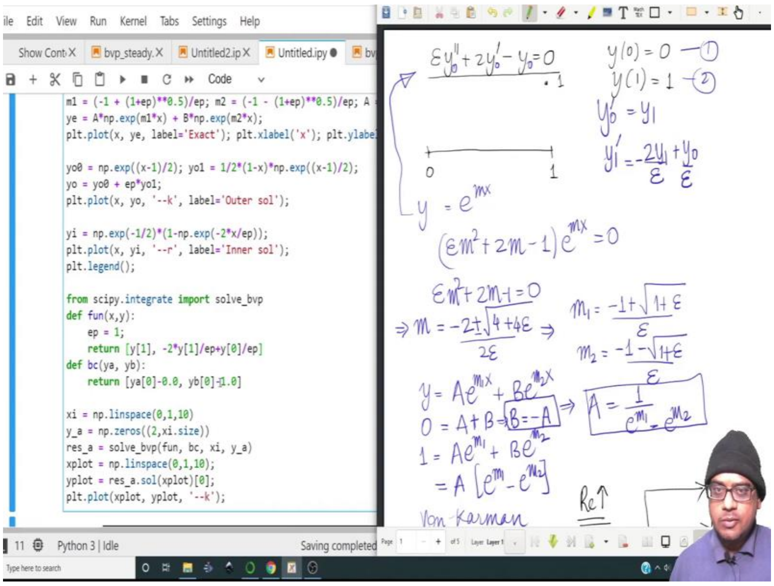

So, we are going to import solve bvp and it is a function we have to. So, one moment what was it? So, y' so let me call this as y_0, because I want to cast it in the Python form. So, let $y_0{}' = y_1$ and then $\epsilon y_1{}' + 2y_1 - y_0 = 0$. So, I must divide everything by ϵ.

So, now I will change the return value of the function. So, y_1 will stay this will be $-\frac{2y_1}{\epsilon} - y_0$ other $\frac{y_0}{\epsilon}$. So, essentially what I am doing is I have taken both these terms on this side of the equality. So, this becomes equal to - this + this that is all I have modified over here um.

And the boundary condition at the left boundary it is 0 and at the right boundary it is 1. So, nothing changes over here and xi is the initial guess its going from 0 to 1 only 10 points guess value residue i.e. xi = np.linspace(0,1,10) then for plotting I am defining a space x plot let me take 100 points.

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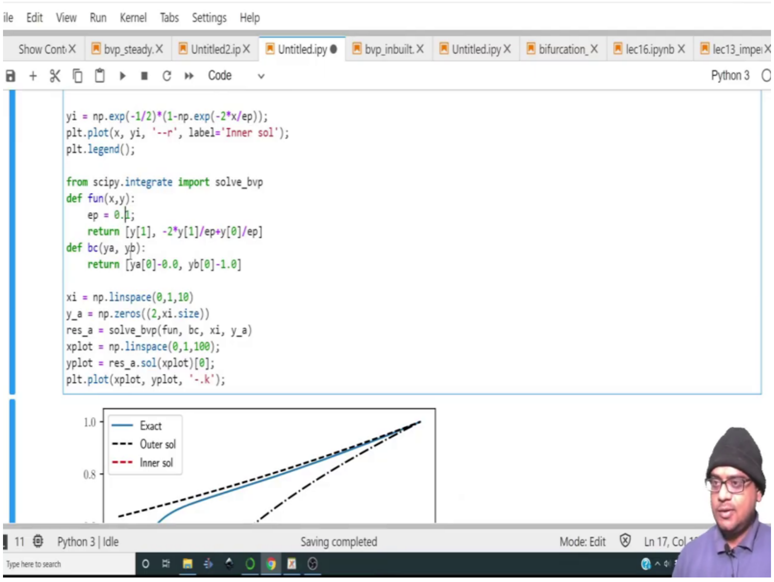

Let me make it dash dot. So, let me run this and see what happens, there appears to be some error let us see what happens I mean maybe if you have written something incorrectly, this has to be 0.0 so we were using 0.1 ok.

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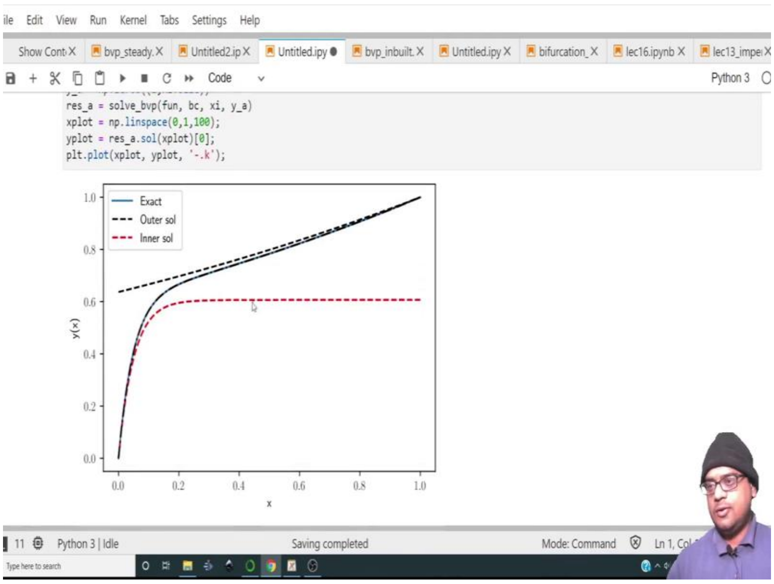

So, let me run this again ok there you go it matches so well that its on the its on the blue line. So in fact let me put the analytical solution with less number of points.

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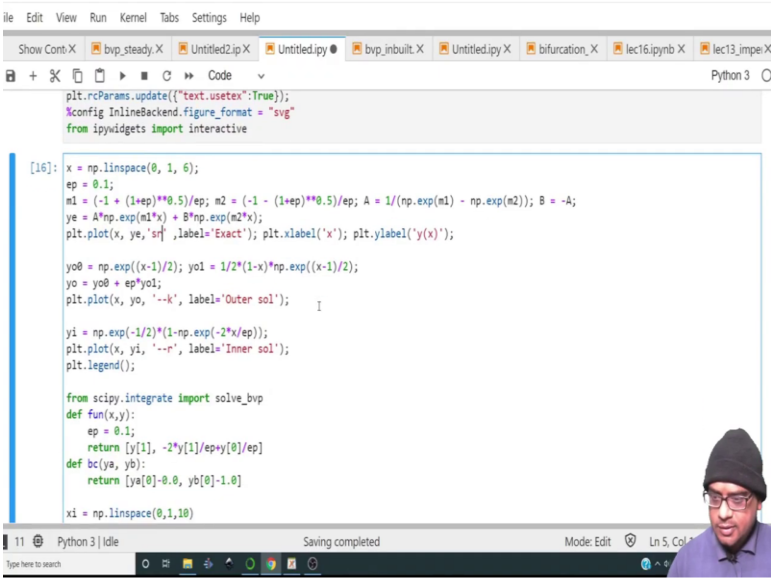

Let me just take.6 points and let me mark them with some kind of marker red marker square red marker let me rerun this ok.

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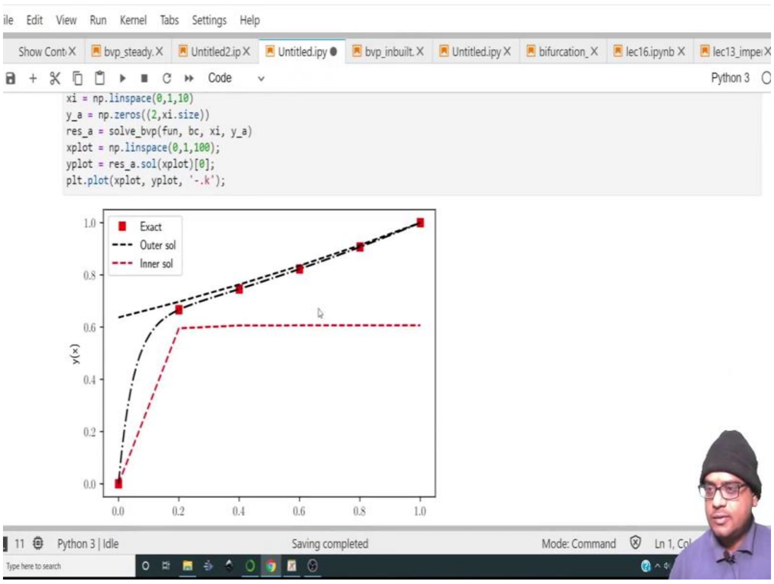

So, its affecting the other codes as well because I am deriving all the solutions with the help of this x. So, let me make this as 1000.

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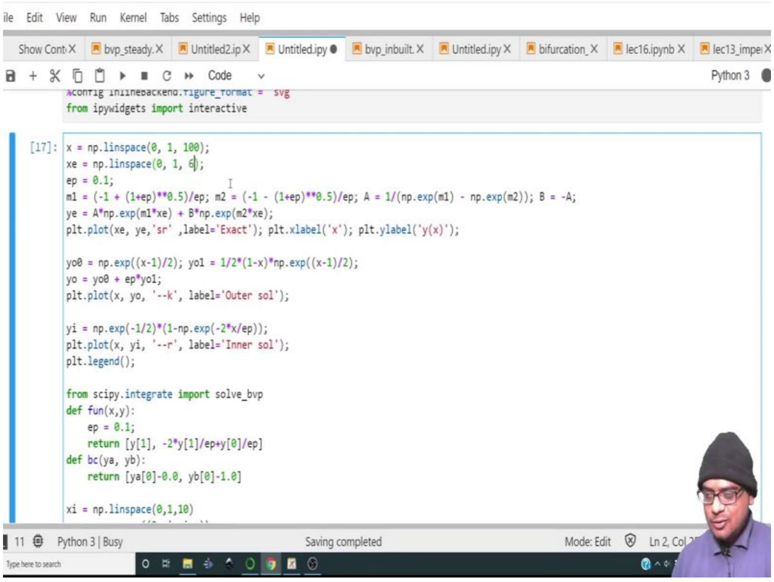

Let me make another copy let me call it xe and let me plot the exact solution with only 6 points, this has to change as well this is just small little bits. But it makes things look much neater ok.

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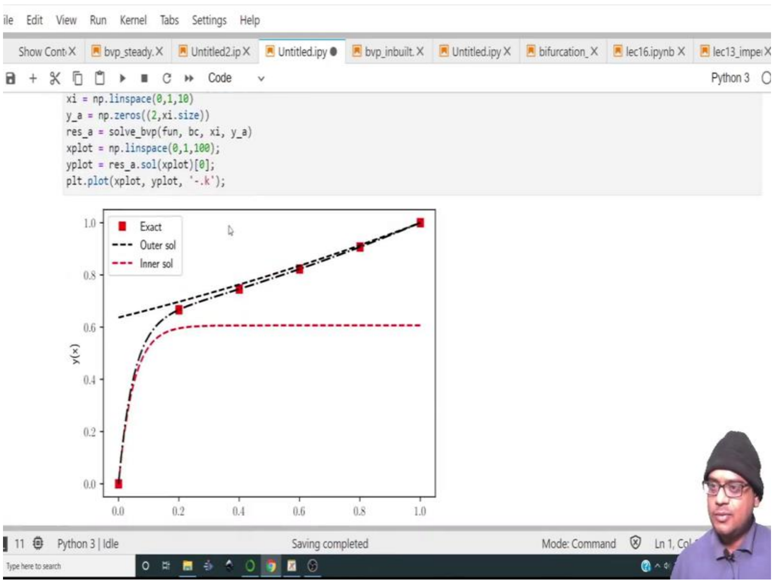

There you go let me put down a label for this as well.

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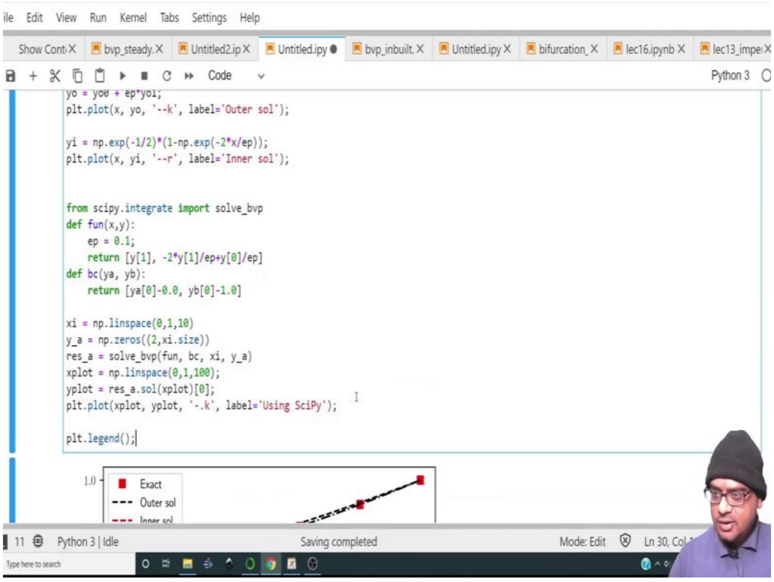

We must call the legend at the end so that it is updated properly, well there you have it.

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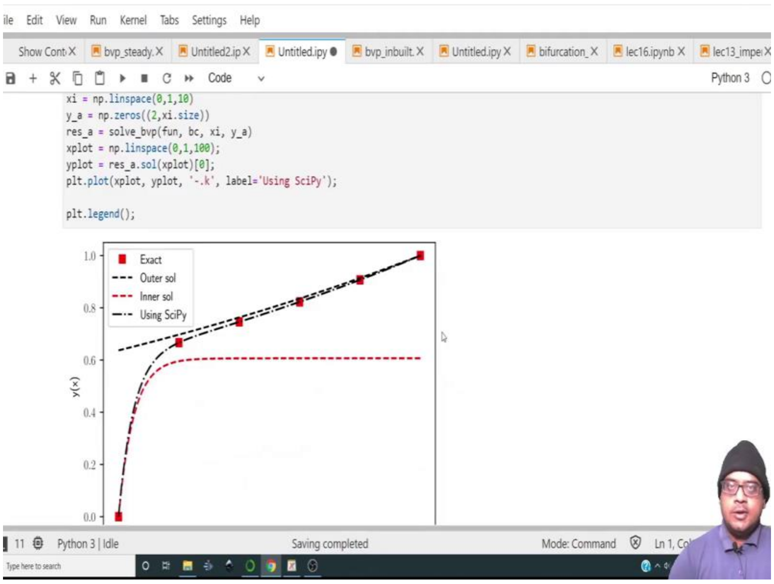

So, with this I am going to close this particular lecture I hope I have given you a very small introduction on regular perturbation singular perturbation, for more reading you can look at some of the reference books below some of the links below and with this I will end this lecture and I will see you next time with something new.

Bye.