Tools in Scientific Computing
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Lecture - 26
Regular Perburbation for ODE

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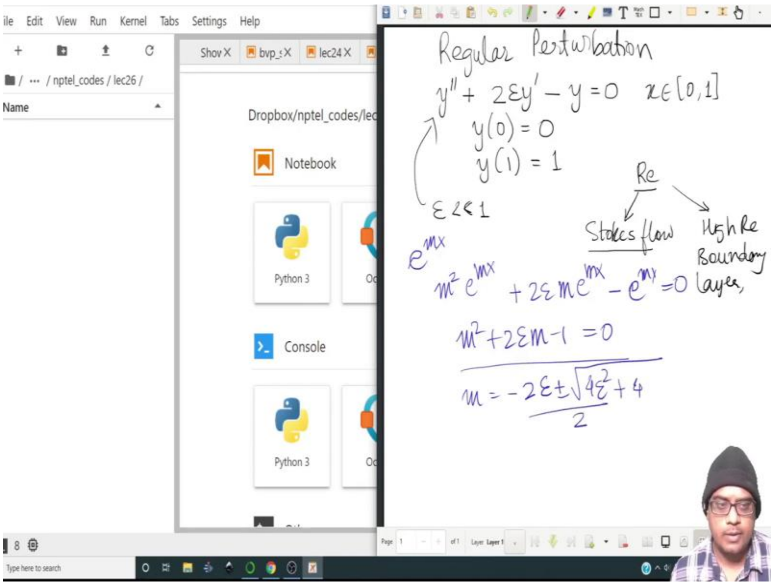

In which, we are going to study about Regular Perturbation. As applied for boundary value problems. Now, obviously this technique has a whole lot more applicability than just two point boundary value problems, but in this particular lecture we are going to focus on using regular perturbation for bvp. Let us consider this particular bvp. So, $y'' + 2\epsilon y' - y = 0$ subjected to $y(0) = 0$ and $y(1) = 1$. So, it is defineid on the domain 0 to 1 i.e. $x \in [0,1]$.

Now, we see the presence of a small parameter ϵ. And usually physical problems are defined in terms of various dimensionless parameters. For example, if you are studying fluid flow you will end up with Reynolds number, it may either be high, but it may either be low and depending on what the magnitude of Reynolds number is.

We know physically that you will either have what is called a Stokes flow or more generally called as higher Reynolds number flow which encompasses boundary layer theory or turbulence and all these things.

So, these this kind of a physical bifurcation if you like because of the presence or rather characterization of the equation through a parameter is quite common in various aspects of physics.

And so, if ϵ is small can we do something about this equation? Well in this particular case the equation is not at all difficult to solve in fact, we can try to solve this directly. So, what is the solution for this? Because it is a homogeneous equation we can assume the solution is of the form e^x, but in this particular case we can choose the solution to be $\sinh x$ and $\cosh x$ that is a linear combination of $\sinh x$ and $\cosh x$.

In terms of the D operator I could have written it as I mean, for the d operator I could have simply used a solution of the form e^{mx}. So, what do we have? So, the derivation the double derivative will give me $m^2 e^{mx} + 2\epsilon m e^{mx} - e^{mx} = 0$. So, we have $m^2 + 2\epsilon m - 1 = 0$ right. So, $m = \frac{-2\epsilon \pm \sqrt{(4\epsilon^2 + 4)}}{2}$ that is the root. So, these are the two roots.

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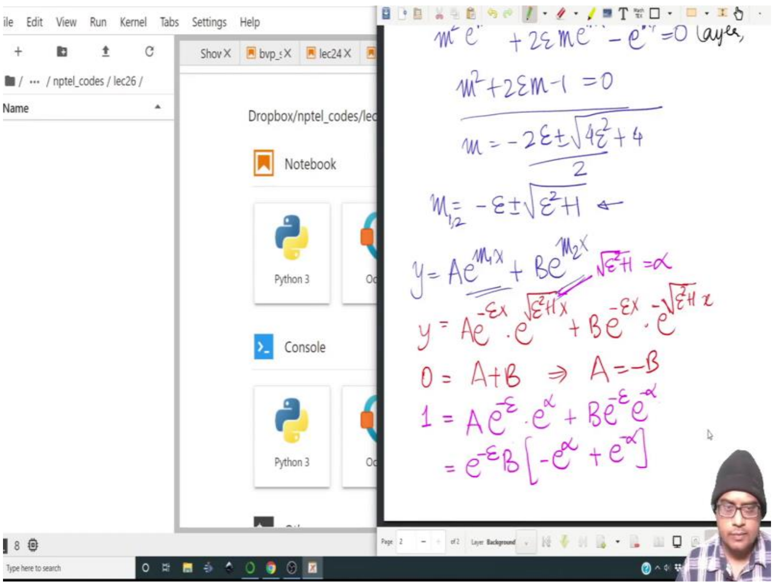

So, m is $m = -\epsilon \pm \sqrt{(\epsilon^2 + 1)}$ I mean, m is either the plus root or the - root.

So, you could write the solution y as Ae to the power. So, if I call this as m_1 and m_2, so $y = Ae^{m_1 x} + Be^{m_2 x}$. And we know that if a solution is of this particular kind we could also write it in terms of sinh x and cosh x because a linear combination of these two terms is also equivalent to a linear combination of those two terms ok.

So, let us simplify this further. So, this becomes $y = Ae^{-\epsilon x} \cdot e^{\sqrt{(\epsilon^2+1)}x} + Be^{-\epsilon x} \cdot e^{-\sqrt{(\epsilon^2+1)}x}$. We have these two terms and now we can utilize the boundary conditions in order to find out the constants.

So, $y(0) = 0$ and $y(1) = 1$. So, substituting x = 0 we have 0 = A+B. So, this is everything becomes this implies A = - B and the second term. So, substituting x = 1, $1 = Ae^{-\epsilon} \cdot e^{\alpha} + Be^{-\epsilon} \cdot e^{-\alpha}$. So, substituting A = - B, so what do we have? $1 = e^{-\epsilon}B[-e^{\alpha} + e^{-\alpha}]$.

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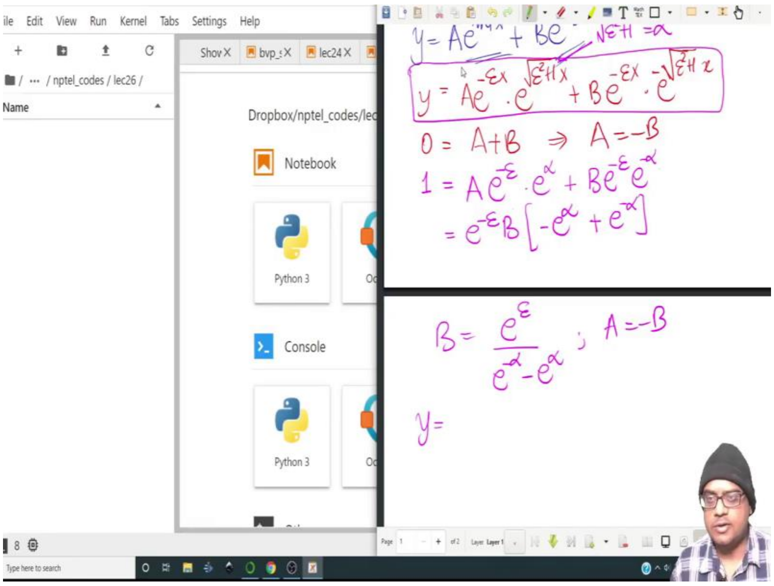

So, we obtain $B = \dfrac{e^{\epsilon}}{e^{-\alpha} - e^{\alpha}}$ and $A = -B$. So, the analytical solution that we obtain is y = simply this expression with the appropriate constraints.

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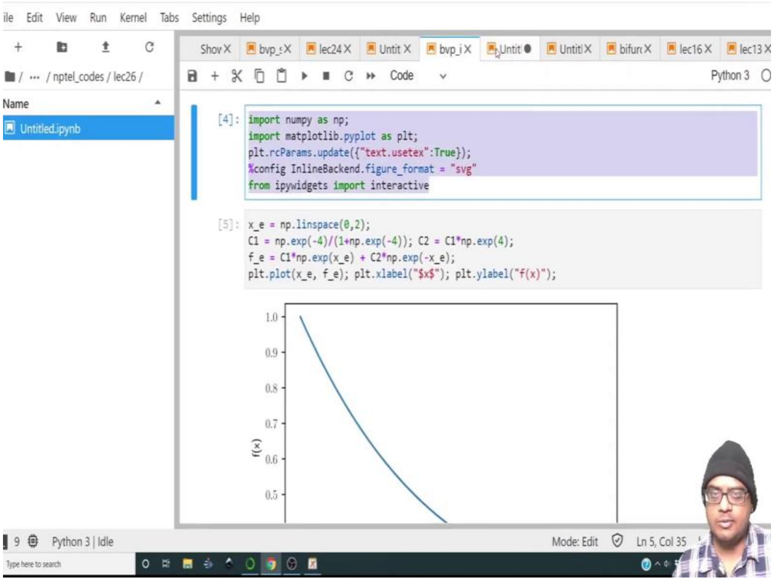
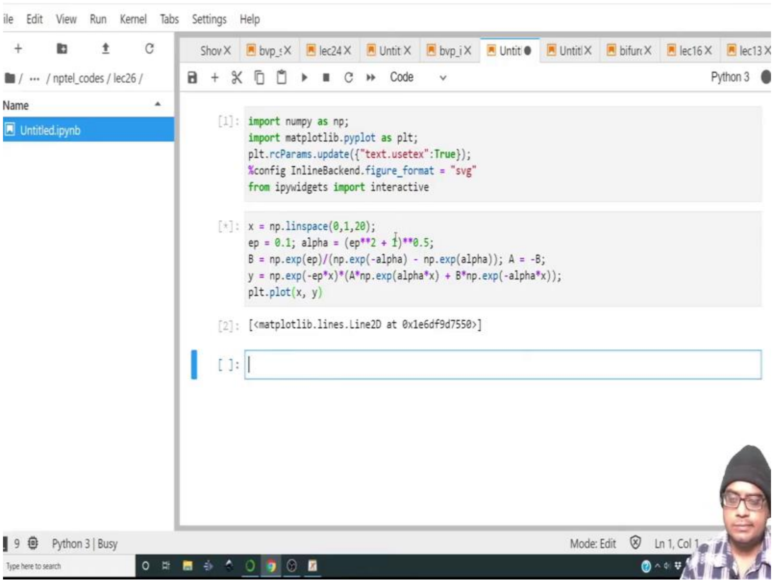


So, let us look at let us go towards plotting this. (Refer Slide Time: 06:59)



So, let me create a new file, let me copy the usual modules we will need. So, let me create x = np.linspace(0,1,20), let me define A as rather let me first define ep = 0.1 small number, then what do we have? We have B.

So, let me define alpha = (ep**2 + 1)**0.5. Let me then define B = np.exp(ep)/(np.exp(-alpha) - np.exp(alpha)). Alright and A = -B.

So, the solution y will be y = np.exp(-ep*x)*(A*np.exp(alpha*x) + B*np.exp(-alpha*x)) alright.

So, now let us plot this, ok.

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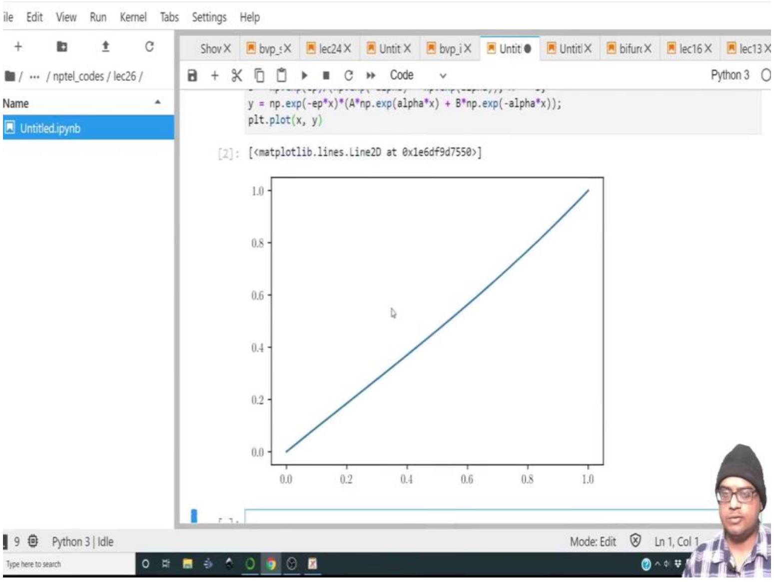

So solution looks something like this. Let me wrap everything inside a function which we can call later on.

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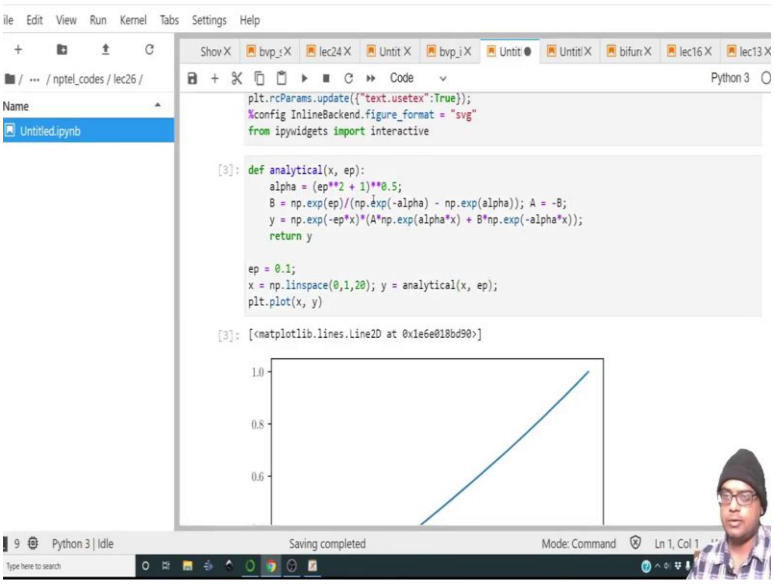

So, I will just take this whole lot def analytical and it will take the input x, ϵ and α and it will evaluate all this and finally, it will return y.

So, ϵ we will pass be α also we will pass. In fact, we just need to pass ϵ because α we can evaluate inside straight forward. So, outside the function we will just define this and we will say y = analytical(x, ep) alright. Yeah great.

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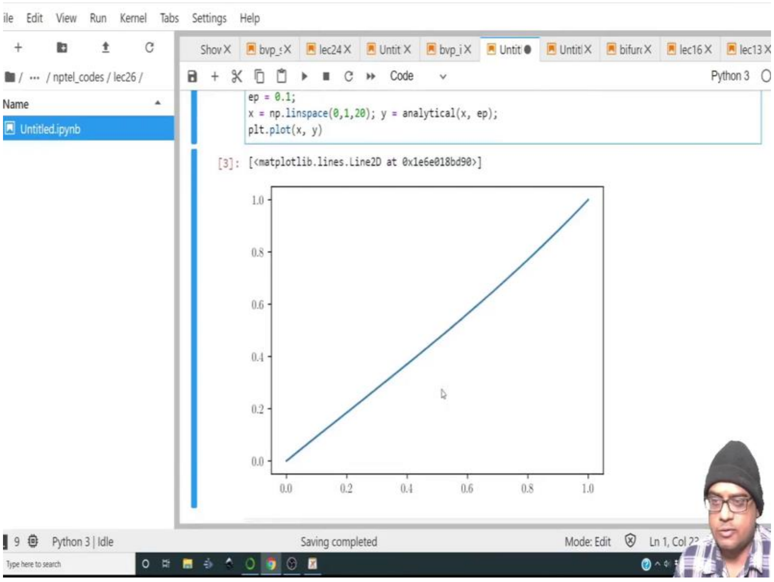

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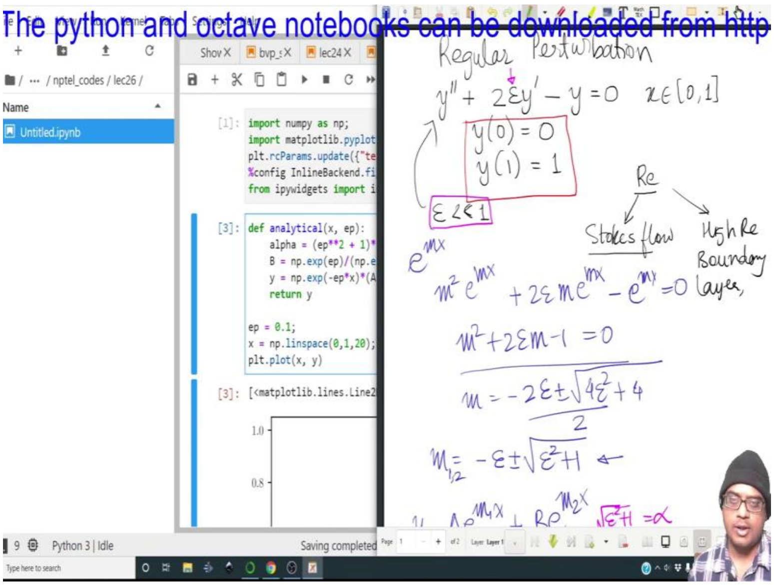

So, this is how the plot looks like and well in this particular case if the equation was rather simple and we could find analytical solution, but usually such kinds of parameters which we call as a perturbation parameter. And the reason why we call it a perturbation parameter is because it is magnitude is usually small.

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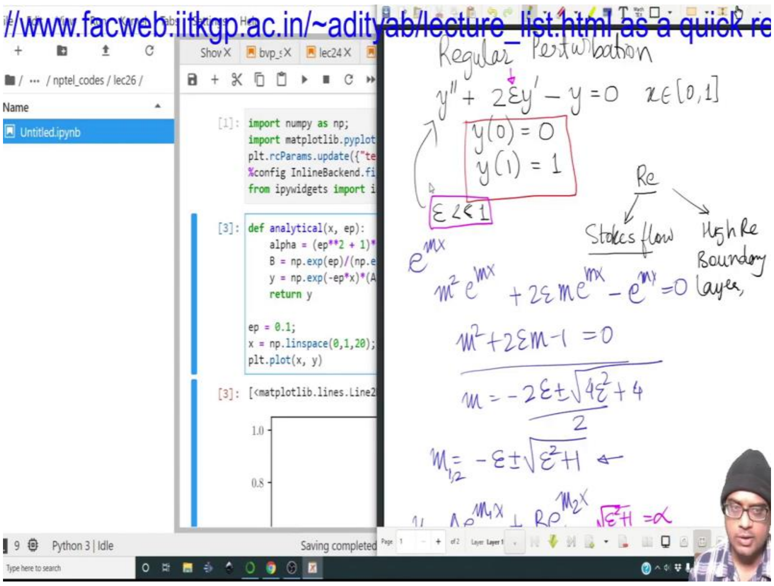

And the effect of that particular term is like a perturbation to some kind of a base term ok. And it will be clear why it behaves like a perturbation on top of some base solution once we do the expansion.

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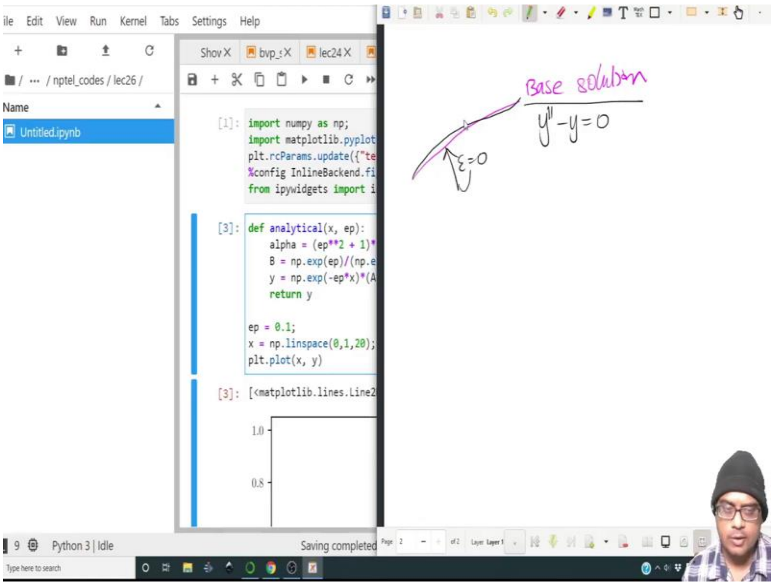

But ok so, for now we need to remember that you will have some kind of a base solution. And if that perturbation parameter were to be 0 ok, if this particular parameter epsilon it were to be 0.you would obtain $y''-y=0$ and so, the base solution is sort of $y''-y=0$. But the moment you have a non zero ϵ, but small the solution will be sort of some correction to this base solution. Because the limiting condition of zero ϵ you will have a solution which is like the base solution.

So, I am not going to do this specifics of regular perturbation, but in general what I am about to show it will work for a host of equations and eventually you will see that you can avoid doing very complicated analytical solutions, but obtain very nice approximations to the scientific solutions, ok.

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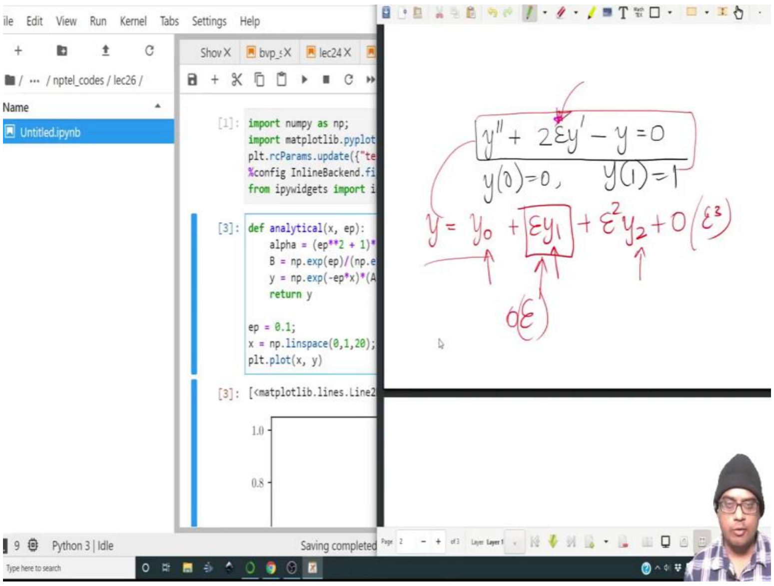

So, let me grab this equation. Let us begin with this. So, this is subjected to $y(0)=0$ and $y(1)=1$. So, let us make an assumption that we can represent the solution y as a series of $y=y_0+\epsilon y_1+\epsilon^2 y_2+O(\epsilon^3)$. Meaning each successive term is of a decreasing order of magnitude. And the fact that we could write the solution in this particular form, automatically assumes that each of these terms y_0, y_1, y_2 they are of order 1.

So, the order of this entire term is not governed by the order of y_1, y_2, y_3 and so on, but it is governed by the pre factor ϵ right. So, the order of magnitude of this is expected to be order ϵ not order ϵ^2 or something. If it were to be ordered ϵ^2 we would have to rescale y_1 so, that the orders of each successive terms are preserved.

And first of all it need not be even ϵ, ϵ^2 it could be $\epsilon \log \epsilon$ and there can be a whole variety of gauge functions, but in this particular case we observe that the equation does have a floating ϵ over here and in this particular it works out. But usually the choice of such an expansion has to be motivated to the physics of the problem.

And some intermediary scalings that may arise because of considerations from the governing equation or the boundary condition that will give rise to a natural sequence like this, but I digress in this particular case it will be quite straightforward. So, now let us take this particular expansion and substitute it over in this equation. So, what do we have?

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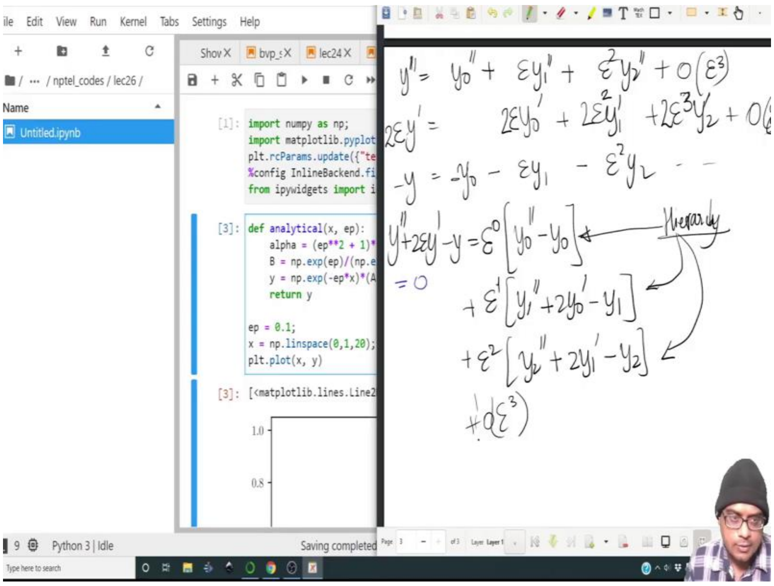

So, the first term is y''. So, $y'' = y_0'' + \epsilon y_1'' + \epsilon^2 y_2'' + O(\epsilon^3)$. What about the second term? So, it is $2\epsilon y'$. So, this will be what?

So, let me first write down what y' will be. It will be $y_0' + \epsilon y_1' + \epsilon^2 y_2' + O(\epsilon^3)$. But now, if you multiply everything by ϵ what will happen? Well it will look something like this $\epsilon y' = \epsilon y_0' + \epsilon^2 y_1' + \epsilon^3 y_2' + O(\epsilon^4)$.

So, in the term involving $\epsilon y'$ there is no term which is devoid of ϵ. This sequence naturally starts with the lowest order of magnitude being ϵ. So, now I need to multiply 2 as well that is straightforward alright. What about the last term? Is $-y = -y_9 - \epsilon y_1 - \epsilon^2 y_2 + ...$ and so on.

So, now let us add all of this. So, we have $y'' + 2\epsilon y' - y$ and this will be let us now, write the right hand side in terms of various terms collected in orders of magnitude of ϵ. So, $\epsilon^0[y_0'' - y_0] + \epsilon^1[y_1'' + 2y_0' - y_1] + \epsilon^2[y_2'' + 2y_1' - y_2] + o(\epsilon^3)$ and so on.

So, through this particular exercise we are able to form an entire hierarchy of equations of order 1 order ϵ order ϵ^2 and so on, right. We are able to form a hierarchy and because this will be equal to 0 each of these terms have to be also 0.

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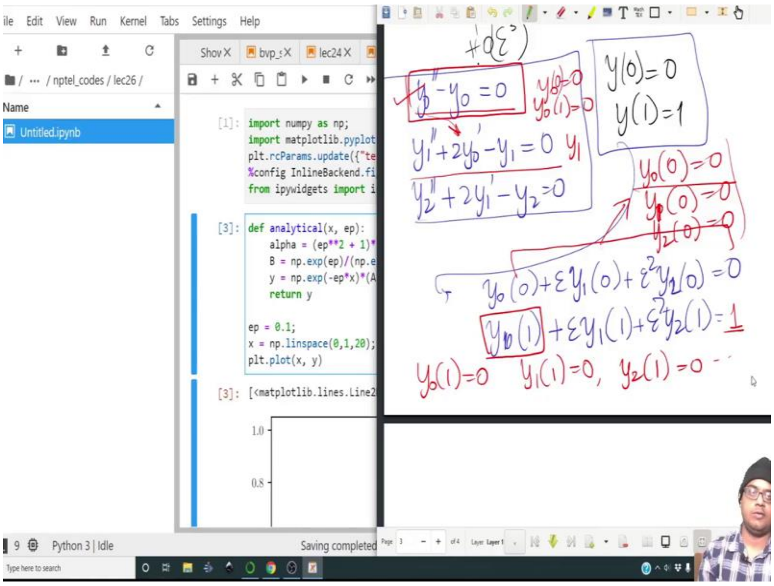

Meaning $y_0'' - y_0 = 0$, $y_1'' + 2y_0' - y_1 = 0$, $y_2'' + 2y_1' - y_2 = 0$. But what about the boundary conditions? Well that is also straightforward to do. Let us write down the boundary condition, so what do we have? $y(0) = 0$ and $y(1) = 1$.

So, similarly we can write, over here $y_0(0) + \epsilon y_1(0) + \epsilon^2 y_2(0) = 0$ and we can write $y_0(1) + \epsilon y_1(1) + \epsilon^2 y_2(1) = 0$. So, now if each of this is 0; then we equate all the terms. So, this naturally implies. So, all this naturally implies $y_0(0)=0$, y $y_1(0) = 0$, $y_2(0) = 0$ and so on.

So, all the hierarchy of boundary condition, so this equation this particular equation is an equation for y_0, this particular equation is an equation of y_1. And why is it not an equation for y_0? Because once we have obtained a solution for y_0, we would substitute it simply over here right. And what are the boundary conditions which this equation is subjected to?

It is $y_0(0)$ = 0 and from this return this 0 by mistake has to be 1. So, this implies $y_0(1)$ = 1 but it implies $y_1(1)$ = 0, $y_2(1)$ = 0 and so on ok alright. So, that means, that this equation is subjected to $y_0(1)$ = 0. So, $y_0(0)$ = 0, $y_0(1)$ = 0.

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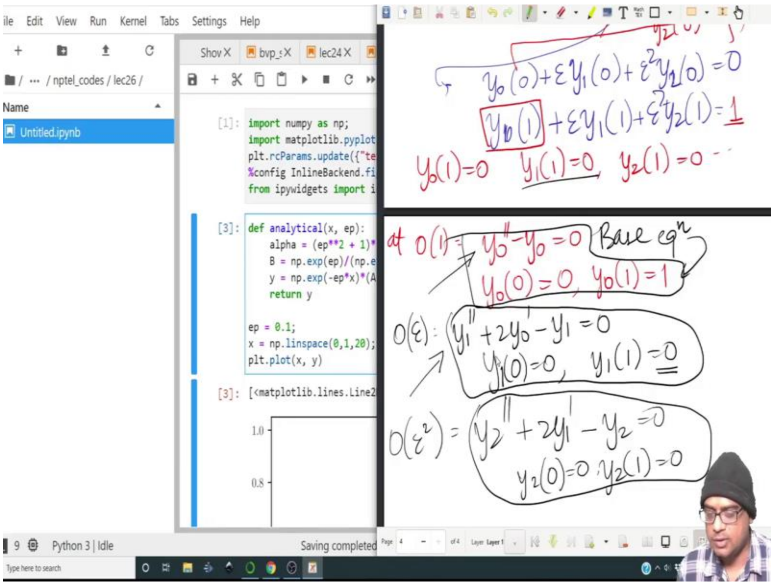

So, let me write it down once again over here. So, at order 1 we have $y_0'' - y_0 = 0$ subjected to $y_0(0)$ = 0 and $y_0(1)$ = 1.What about the equations at higher order ϵ? $y_1'' + 2y_0' - y_1 = 0$ subjected to now $y_1(0) = 0$ and $y_1(1)$ = 0. So, $y_1(1)$ = 0, $y_1(0) = 0$, $y_1(1)$ = 0.

So, this is called as the base equation which satisfies actually the base boundary condition, but the perturbation equation does not satisfy the base boundary condition. It satisfies equal to 0 because, it is like a perturbation to human equation and you can have a separate case where you are dealing with domain perturbation techniques, where you are actually not you are not motivated through the governing equation, but rather by the nature of the boundary condition, but that is a completely different story.

Here we have the governing equation which is split into the base equation and the perturbation equation and it is higher order perturbation equation as well. And this equation will also be satisfying this ok. So, the chain or the process hierarchy is first find out y naught using this then find out y 1 using this, and find out y_2 using this so on and so.

You can keep doing this. But usually you will see that after a few terms it itself you have obtained an approximate solution of sufficient accuracy. Well, how do you know it is of suffering accuracy? Alright you need to know asymptotic analysis, but over here we are going to take the leap of faith and hope that the solution satisfies the solution the approximate solution also satisfies the analytical solution.

So, let me oops yeah. So, let me encode this. Let us see how well this works. Well before going into coding we need to find out the solutions.

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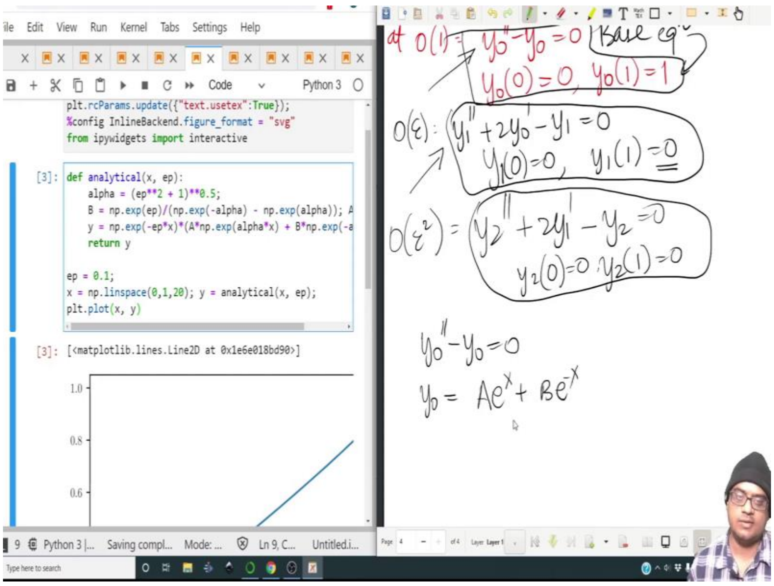

So, this is what $y_0'' - y_0 = 0$, the solution is; obviously, $y_0 = Ae^{x} + Be^{-x}$ or equivalently we could write the solution as sin h and cos h because, when the basis functions are e^{x} and e^{-x} you can equivalently cast it in the form of a linear combination of e^{x} and e^{-x} which is sin h and cos h.

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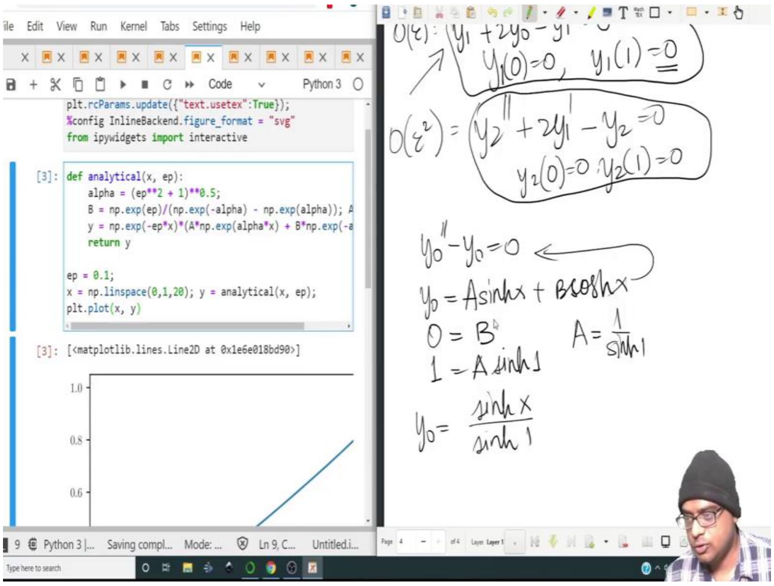

This is $A\sinh x + B\cosh x$ because ok. So, you can verify that this satisfies this equation. And now the boundary condition the first boundary condition is 0 = B and $y_0(1)$ = 1. So, 1 = A sin h 1 so, A = 1/sin h1. So, the solution for y_0 is going to be $y_0 = \dfrac{\sinh x}{\sinh 1}$. So, now let us go to the computer and try to plot the zeroth order solution.

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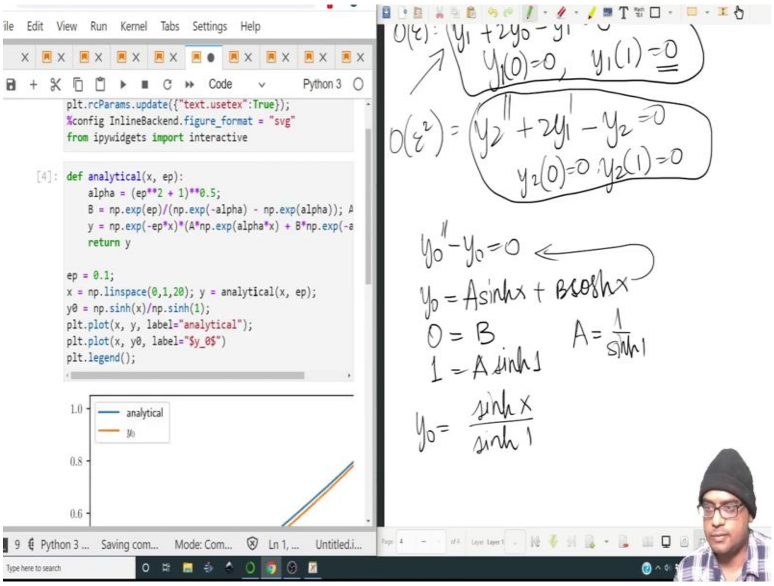

So, we have y as this y0 = np.sinh(x)/np.sinh(1). So, we are going to put a label analytical, alright.

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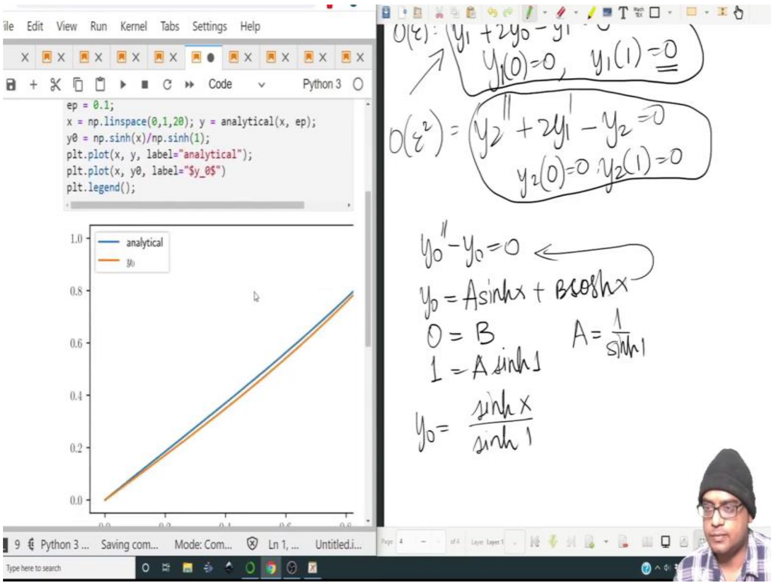

So, let us see what we have not bad.

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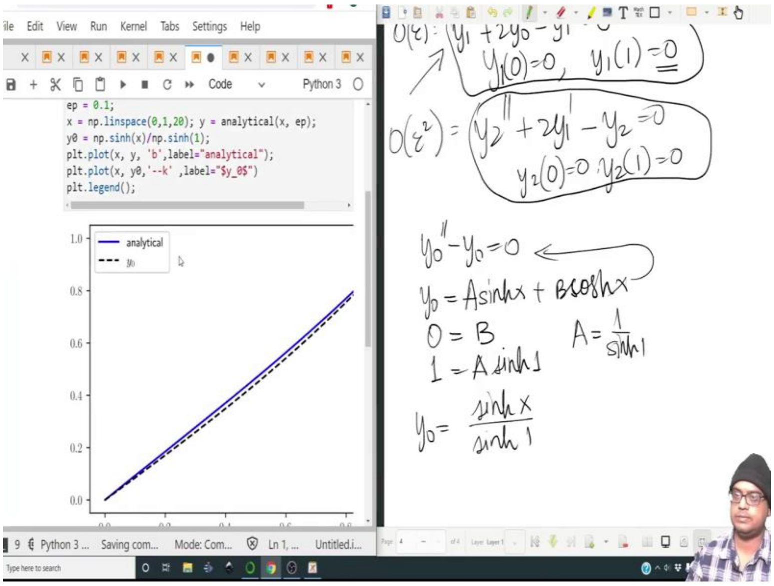

So, let me just change the line style to make it a bit more apparent. So, the leading order solution we going to show as a broken black line and this solution we going to show as a blue line.

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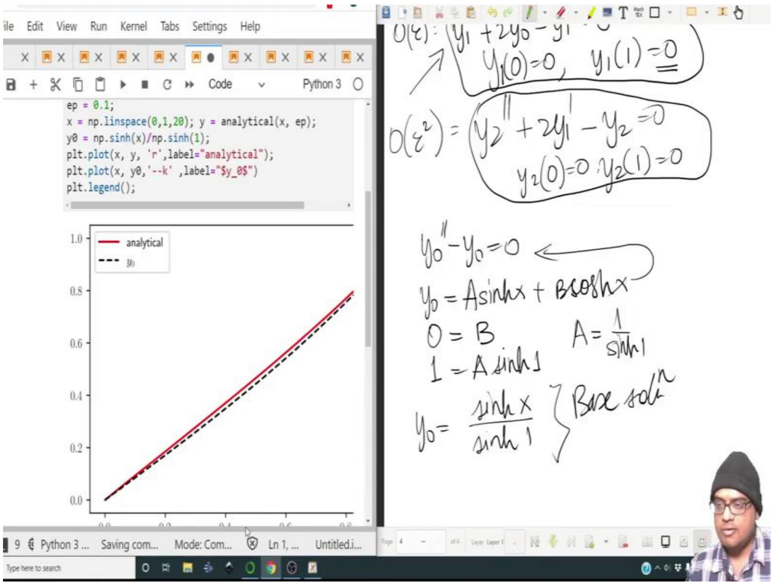

Well blue let me put as a red line ok. So, now, we see that the base solutions this is the base solution and the exact solution which is the red curve they are not far off.

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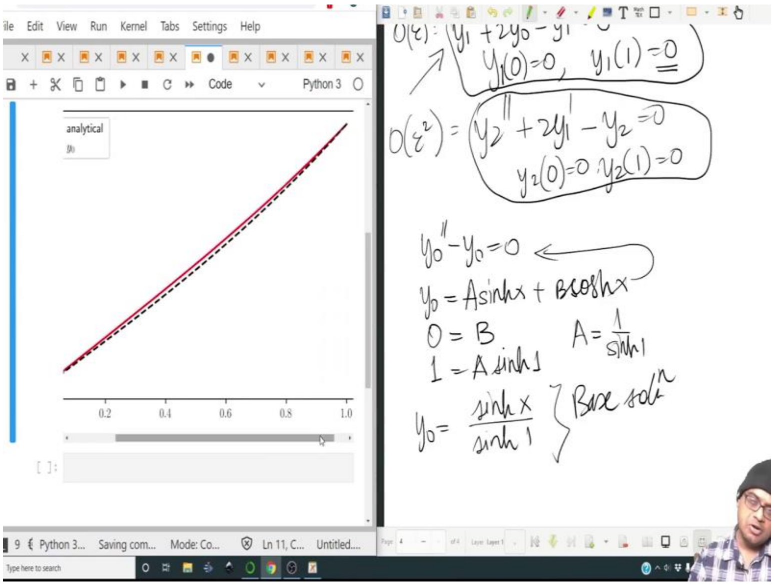

Meaning the overall behavior of this equation is quite accurately represented by the base solution for small values of the perturbation parameter epsilon and that is quite obvious.

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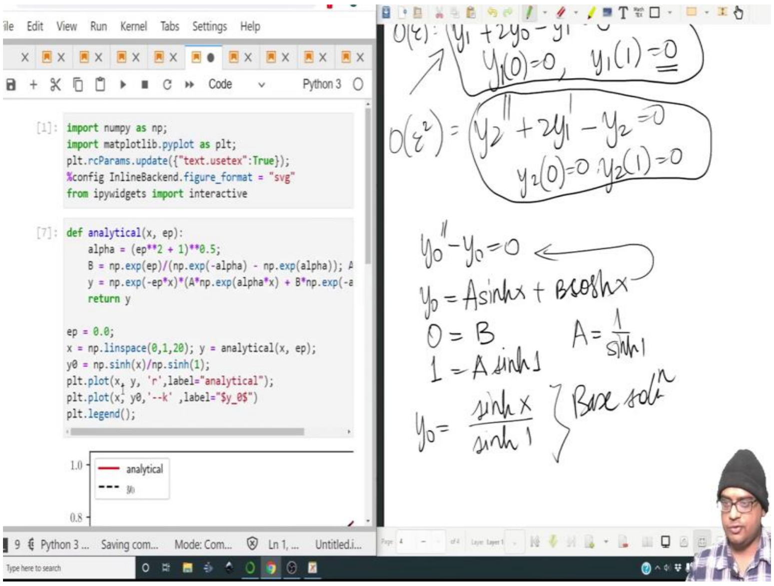

In fact, if I make it 0 they should exactly match great, they do exactly match.

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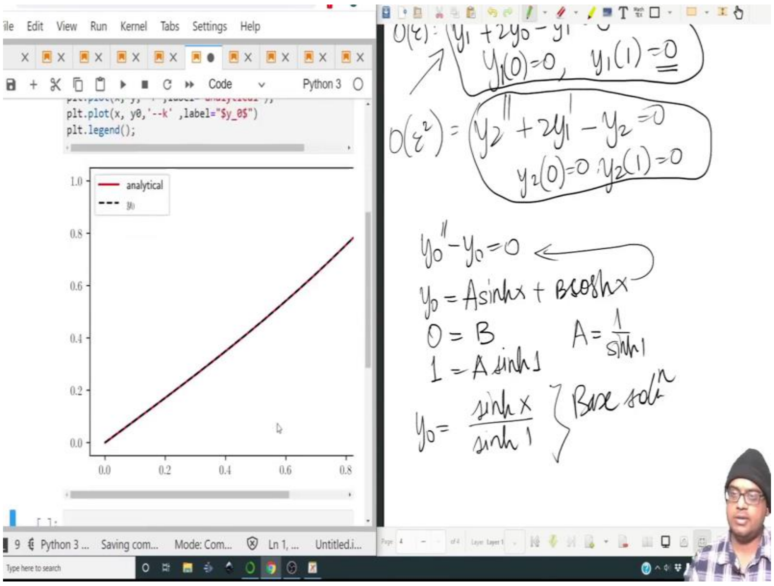

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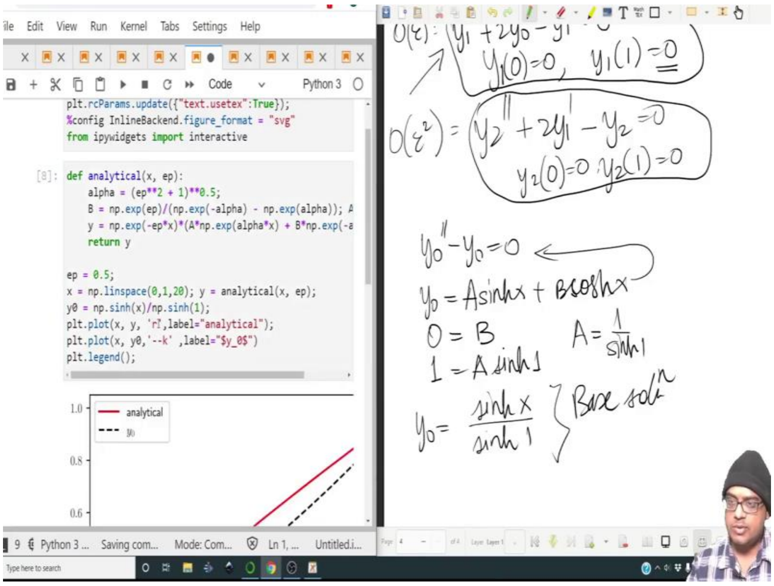

The perturbation parameter becomes 0.5 maybe they do not match.

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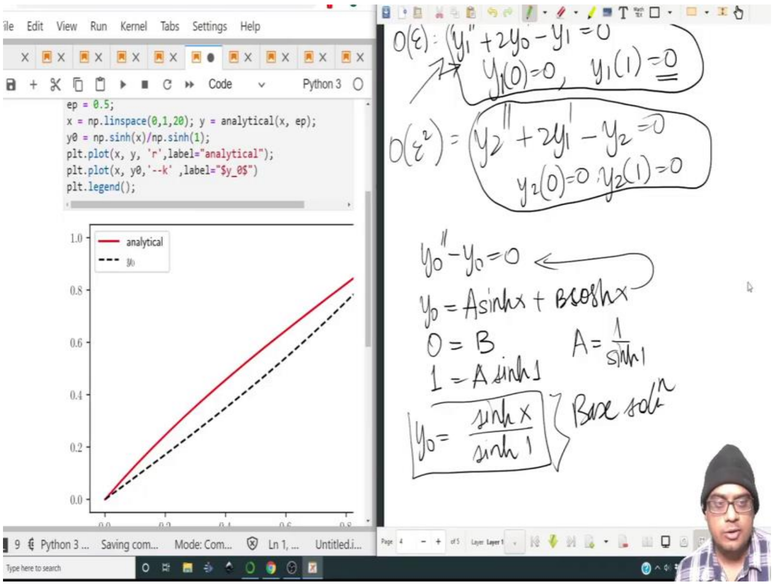

So well they do not match. So, well and that this information is quite critical because, while it does not match well it does encode all the basic information of the monotonous increase and so on

So, now you know that the next correction to y_0 which is going to be through y_1, its, is going to make the broken line approach the analytical curve. So, we have this base solution which was which is arguably very easy to obtain and now what we can do is use this equation to find out what y_1 is going to be.

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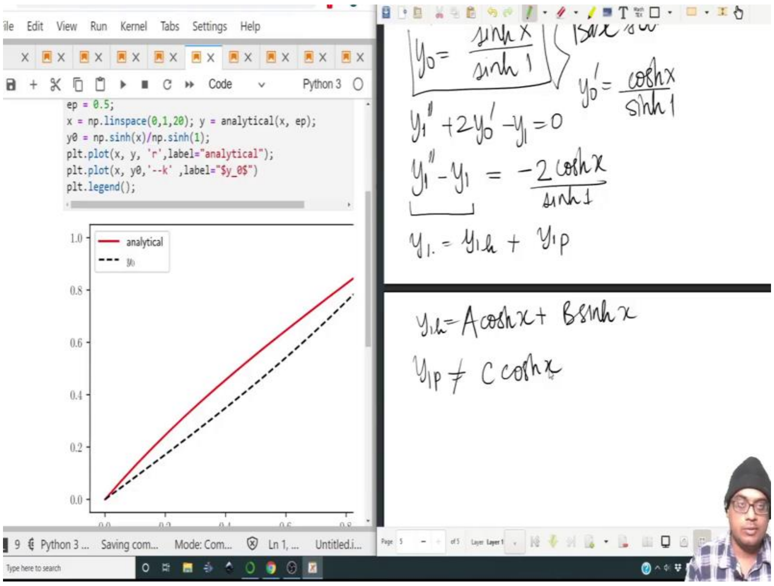

So, let me write down the equation for y_1. So, it is $y_1'' + 2y_0' - y_1 = 0$. So, now what is y_0'? $y_0' = \dfrac{\cosh x}{\sinh 1}$ so, what do we have? $y_1'' - y_1 = \dfrac{-2\cosh x}{\sinh 1}$ right.

So, the solution for this, the homogeneous part. So, $y_1 = y_{1h} + y_{1p}$ and; obviously, the homogeneous solution is going to be again $y_{1h} = A\cosh x + B\sinh x$ this is going to be the homogeneous solution.

What about the particular integral? Look the particular integral has the functional form cosh x so; obviously, the particular integral cannot have a form C cosh x i.e $y_{1p} \neq C\cosh x$, it cannot because cosh x is or already the homogeneous part. So, it cannot have the same form.

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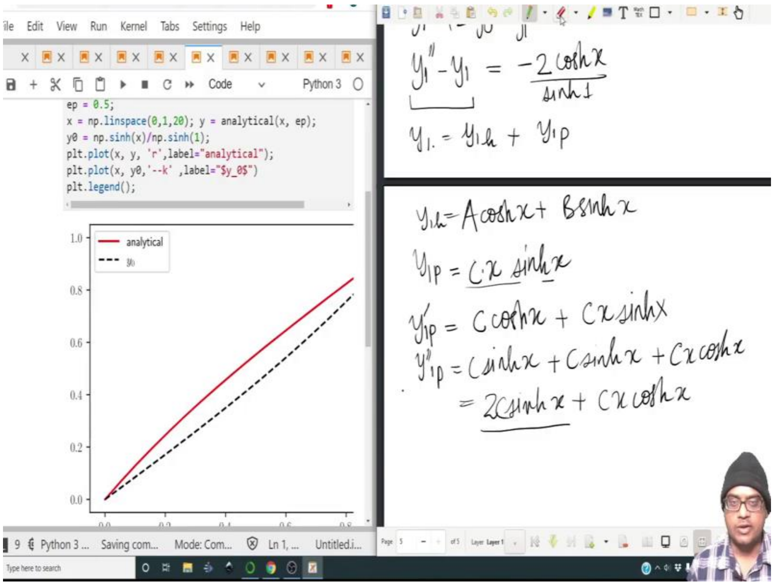

But it can have and you can use various techniques that you have learnt in a differential equations course, it can be of this particular form. And it is because the homogeneous and the particular the homogeneous part has the same form as the homogeneous solution.

So, now let us use this form i.e. $y_{1_p} = C \cdot x \sinh x$. So, let us substitute in the equation. So, what do we have? What is y_{1_p}'? It is going to be $y_{1_p}' = C \cosh x + Cx \sinh x$ and then we take another derivative. So, $y_{1_p}'' = C \sinh x + C \sinh x + Cx \cosh x$. So, this is what is $2C \sinh x + Cx \cosh x$. So, now we actually have a form of 2 sinh x which means that the solution has to be not of a cosh form or it has to be of a sinh form. Well yeah, it has to be of the form 2 sinh form.

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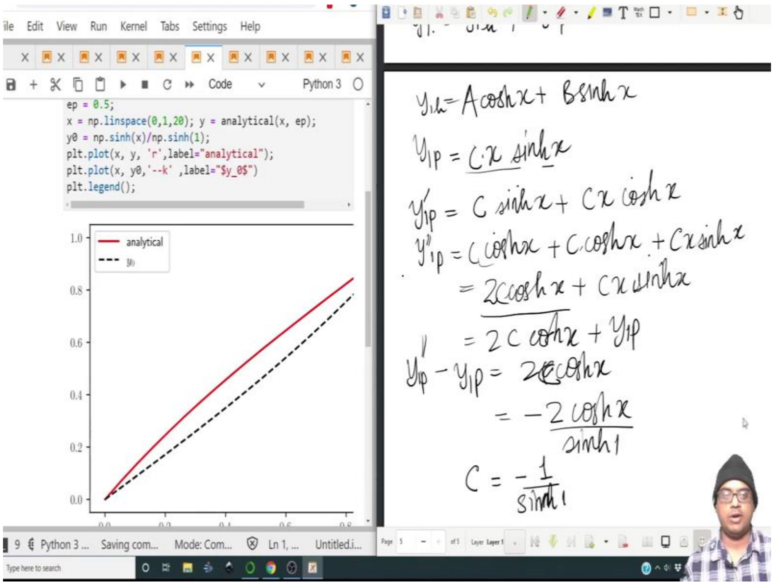

So, once we take the derivative this will become sinh x this will become 2xcosh x. Now, this second derivative will become this will become cosh, this will become again this will become cosh and this will become Cxsinh. So, I request you to do this on your own and have a look. So, this becomes 2Ccosh x and this becomes Cxsinh x. So, this is essentially 2Ccosh x.

So, now Cxsinh x is y_{1p}. So, this is y_{1p}. So, essentially we had $y_{1p}'' - y_1'$ = 2Ccosh x. But we already know that the particular solution has to satisfy this as - 2cosh x / sinh 1, this is equal to - 2 cosh x / sinh 1 equating these two we get an expression for C. So, $C = \frac{-1}{\sinh 1}$ great.

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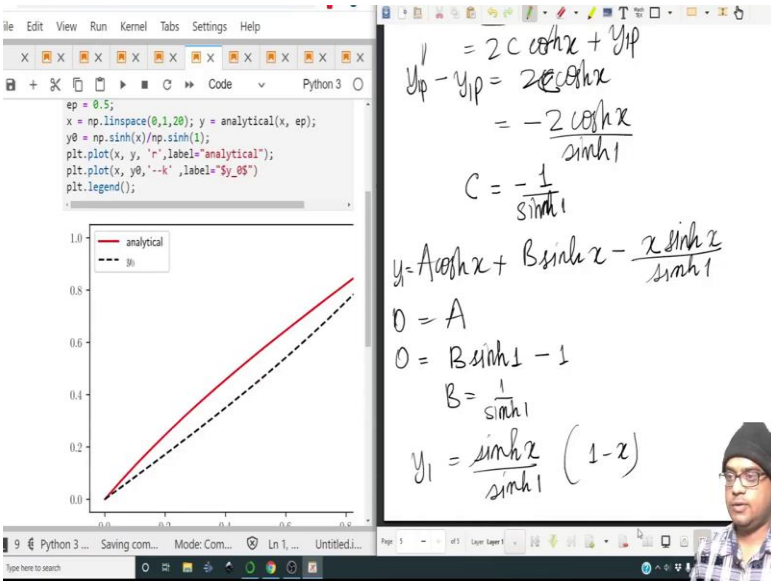

So, now the solution is $y_1 = y_{1h} + y_{1p}$. So, $A\cosh x + B\sinh x - \frac{x\sinh x}{\sinh 1}$, now the boundary conditions. So, y_1 is this $y_1(0) = 0$. So, 0 = A everything else is 0 and at $y_1(1) = 0$. So, this what is $0 = B\sinh 1 - 1$, so $B = \frac{1}{\sinh 1}$ when A is 0. So, y_1 becomes $y_1 = \frac{\sinh x}{\sinh 1}(1 - \text{x})$. A lot of derivation, but yeah.

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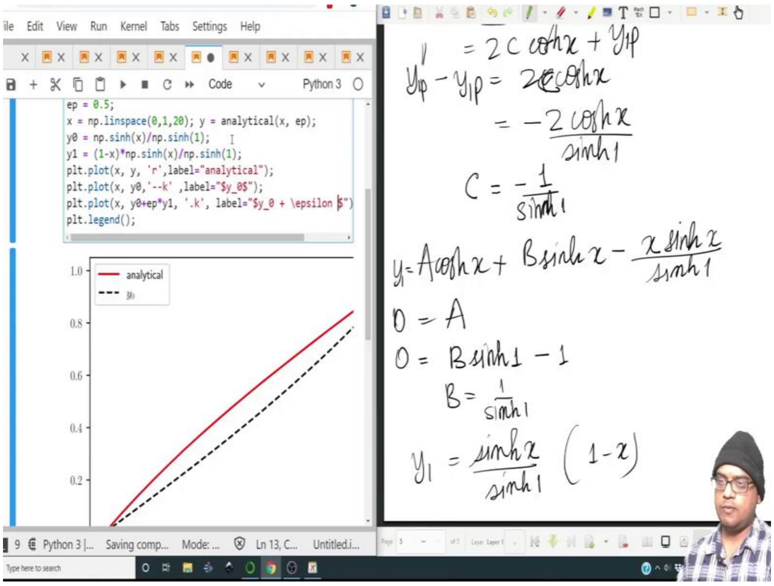

So, y1 = (1-x)*np.sinh(x)/np.sinh(1). So, now with this let me do plt.plot(x, y0+ep*y1, '-.k', label="$y_0 + \epsilon y_1$"). Well let me yeah.

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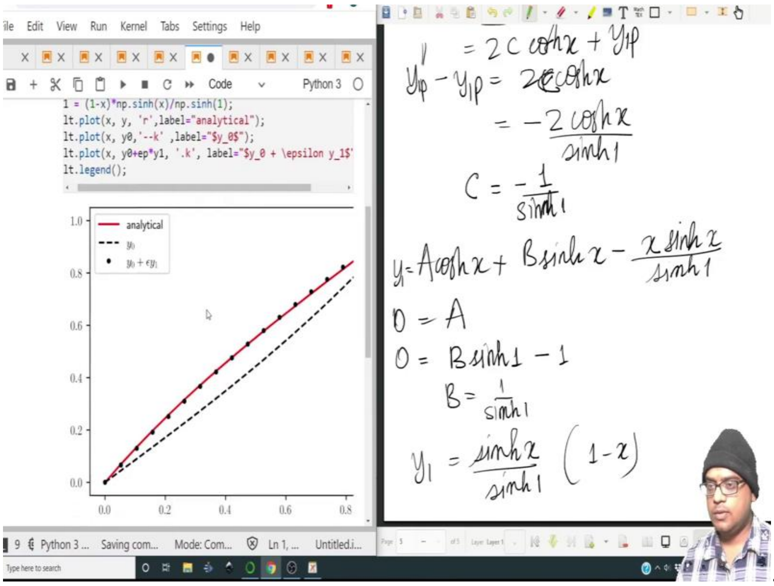

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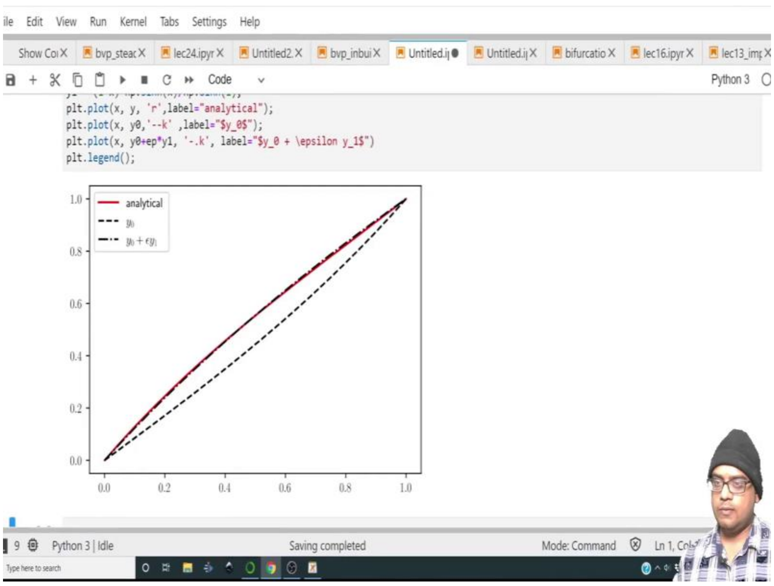

So, what do we see? The red is the analytical curve the broken lines are simply y_0 and the broken dot lines are the correction. So, now look because of the corrections the solution matches quite well with the analytical solution.

So, if you are like me if you are not you are not looking forward to analytical solutions every time and you look forward to an approximate solution, regular perturbation is a great way of getting around things. You will obtain very easy hierarchical equations which will entail a very easy solution most of the times. It will help you get rid of nonlinearities when the small term multiplies non-linear terms right.

So, not always, but in many cases. It is quite beautiful how with just 1 correction no of course, you can look at higher corrections you have the solution for y_1 you cannot substitute this solution into the approximate equation for y_2 and obtain the solution for y_2 as well and you can make a further tweaking of this ok.

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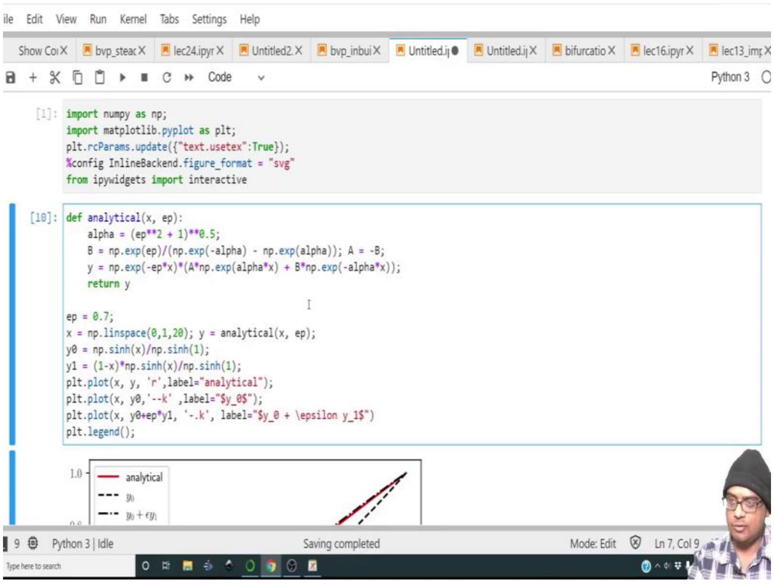

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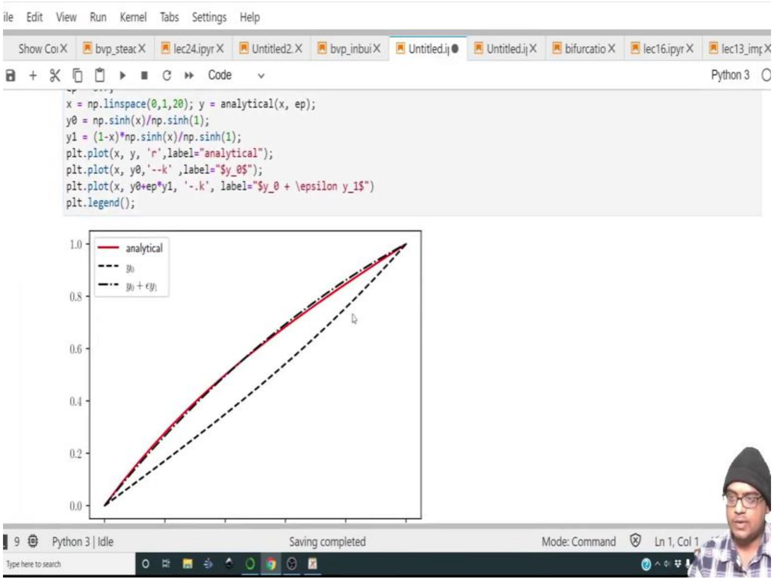

So, let me increase the value of ϵ to 0.7 even at 0.7 it matches quite well, ok.

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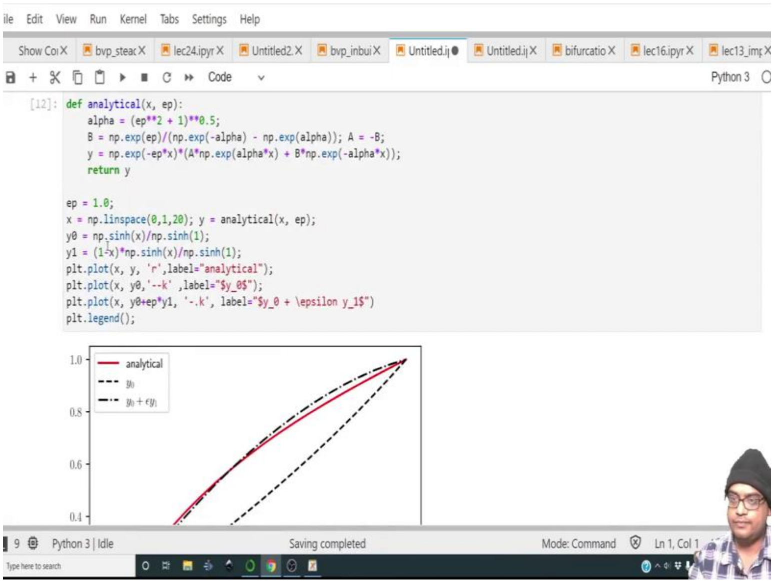

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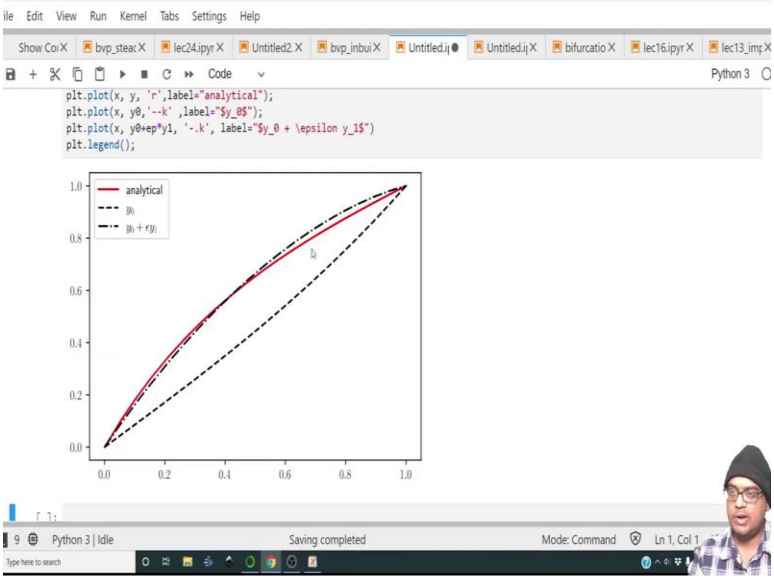

And 1 and 1 the significant deviation. Well this whole thing is not supposed to work for large values of epsilon it is supposed to work for epsilon the magnitudes of ϵ being much less than 1.

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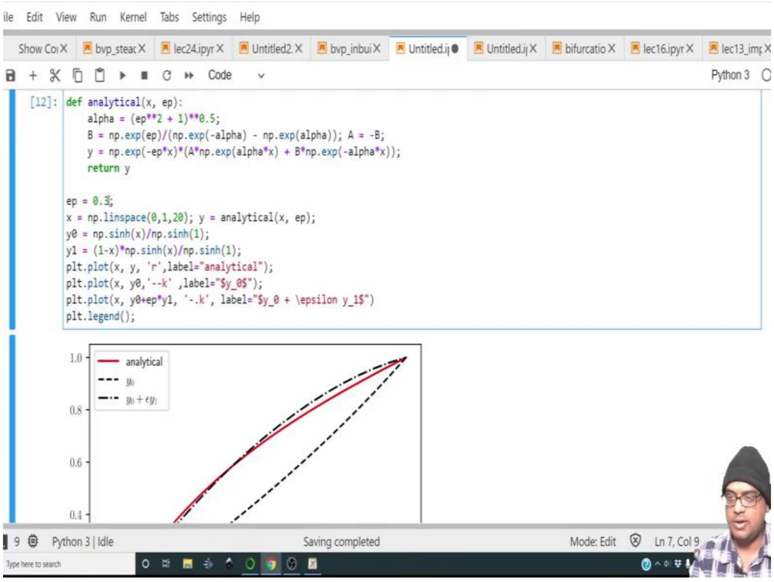

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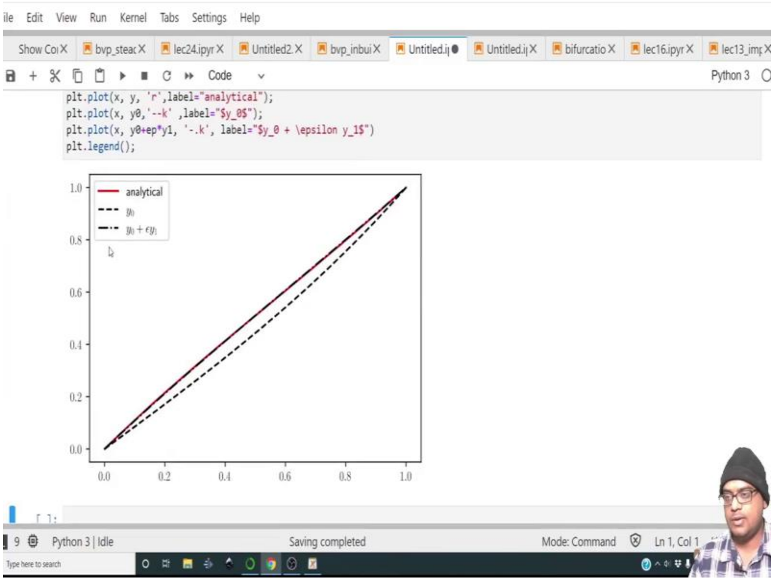

So, let me just put it back to 0.3 measures quite well. So, this is all about regular perturbation. Keep in mind that the order of the equation, that we are starting with that is this it is a second order equation yes y'' and this case the small parameter that is ϵ is multiplying something which is not the highest order term.

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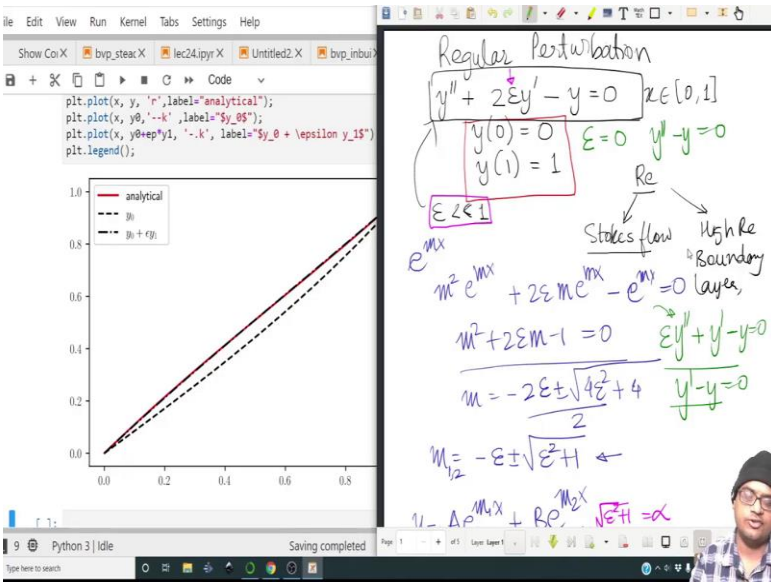

Meaning in the base case we have $\epsilon = 0$. So, this is the base case. So, base case is still something like this which is still second order. So, you are not sacrificing the order of the equation when you let epsilon go to 0. Now, all such things fall beneath the purview of regular perturbation, but if you had something like this $\epsilon y'' + y' - y = 0$.

If we were to now so, this equation would still be governed by two boundary conditions, but now if we let ϵ go to 0 the basic equation becomes this. So, it is a first order equation whereas, the actual equation was a second order equation such kinds of things where you are causing a dropping of the order they will not fall under the purview of regular perturbation, but they will fall in the purview of singular perturbation and we will consider singular perturbation in our next lecture.

Until then it is goodbye, have a nice day, bye.