Advanced Concepts In Fluid Mechanics
Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture – 18
Interfacial Boundary Conditions and Example on Thin Film Flows

In the previous chapter, we have discussed about some exact solutions of the Navier Stokes equation where we started with the very simple case of flow through a parallel plate channel.

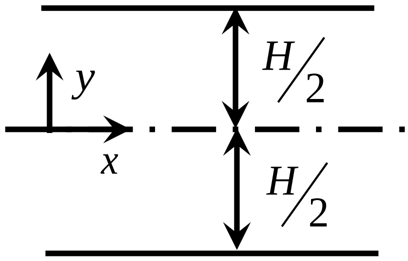

Figure 1. Schematic of the flow through a parallel plate channel of height *H*.

We have considered the fully developed flow through a parallel plate channel of height *H*. The schematic of this flow is depicted in figure 1 where the co-ordinate system (i.e. *x* co-ordinate and *y* co-ordinate) is also shown. In the previous chapter, we derived the expression of the velocity profile $\frac{u}{\bar{u}} = \frac{3}{2}\left[1 - \frac{y^2}{(H/2)^2}\right]$. One can realize that, there is an error in the expression of the velocity profile of the last chapter where it was written as $\frac{u}{\bar{u}} = \frac{3}{2}\left[1 - \frac{y^2}{H^2}\right]$. So, in the earlier expression of velocity, although the parabolic nature of the velocity distribution remains same, the term *H* should be replaced by *H*/2. This is an example of the flow where the entire domain is filled up with one fluid. But we can have a situation, for example, in case of an interfacial flow; we can have two fluids or two phases of the same fluid having distinct physical appearance separated by an interface. So question arises what are the governing equations and the boundary conditions that are required to obtain the velocity distribution and this is the main focus of the present chapter.

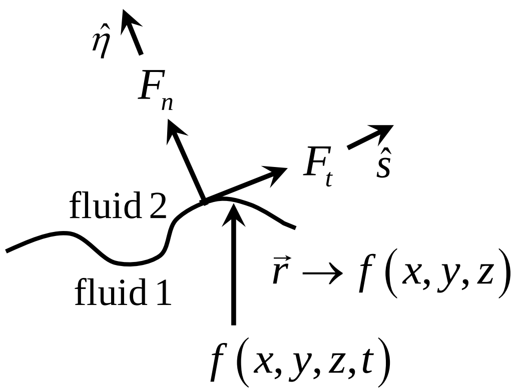

Figure 2. Schematic of the interfacial flow, i.e. flow at the interface between two fluids.

Here the discussion is mainly on interfacial flows. As shown in figure 2, we have two fluids, fluid 1 and fluid 2. These two fluids are separated by an interface which in a functional form can be written as $f(x, y, z, t)$, i.e. a function of the position vector $\vec{r}$ or the co-ordinates *x*, *y*, *z* and time *t*. Now the interface is expected to dynamically evolve with position and time and that is why the interface is described as a combined function of position and time. Being a combined function of position and time, for any point on the interface; the change in the position will be dictated by the total derivative of the function $f(x, y, z, t)$. So, if there is a point on the interface which is identified, we can say from pure kinematic considerations that the point located on the interface will be located on the interface forever. The interface may dynamically evolve but it will still be located on the interface. This is known as kinematic boundary condition. This kinematic boundary condition talks about the interfacial kinematics. So, this essentially boils down to the total derivative of the function *f* equal to zero, i.e. $\frac{Df}{Dt} = 0$ or $\frac{\partial f}{\partial t} + \vec{V}_S \cdot \nabla f = 0$. Here, V_S is the velocity of a point located on the interface. The boundary condition $\frac{\partial f}{\partial t} + \vec{V}_S \cdot \nabla f = 0$ will be satisfied irrespective of the form of the interface. A special case of this boundary condition is the scenario when one of the two fluids is replaced by a solid boundary. In that case, the kinematic boundary condition will be nothing but the no-penetration boundary condition. So, no-penetration boundary condition becomes a special case of the kinematic boundary condition which tells that on a flat interface, the normal component of the velocity at the wall is zero, i.e. fluid cannot penetrate through a wall. Now, we have to keep in mind that $\vec{V}_S$ is not the velocity of the fluid. When there is no transport of fluid across the interface, then only $\vec{V}_S$ is equal to $\vec{V}$. However, if there is a transport of fluid across the interface, then we need to take into account the relative velocity $(\vec{u}_r)$ normal to the direction of the interface $\vec{u}_r = \vec{V} \cdot \hat{\eta} - \vec{V}_S \cdot \hat{\eta}$. Here, $\vec{u}_r$ is the velocity of the surface relative to the fluid or the velocity of the fluid relative to the surface. $\vec{V} \cdot \hat{\eta} - \vec{V}_S \cdot \hat{\eta}$ is the difference in the normal component of the fluid velocity and the velocity at a point on the surface with $\hat{\eta}$ being the unit vector along the normal direction. $\hat{\eta}$ can be calculated as the gradient of the function *f* divided by the modulus of the gradient of the function *f*, i.e. $\hat{\eta} = \frac{\Delta f}{|\Delta f|}$. So, in case of a flow across the interface, knowing the relative velocity of the fluid with respect to a point on the surface and the normal component of relative velocity one can evaluate $\vec{V}_S$ which can be further substituted in the kinematic boundary condition. This may be a little bit more involved but in case of immiscible phases $\vec{V}_S$ is equal to $\vec{V}$. This kinematic boundary condition does not understand the presence of the acting forces. However, we also have boundary conditions based on these forces. First we will consider the normal force balance. Because the interface is of any arbitrary shape, we can write the tangential force (F_t) and the normal force (F_n) and the resultant force becomes the vector sum of the tangential and the normal component. Normal force balance implies that we are considering the $\hat{\eta}$ direction and the tangential force balance means we are considering the $\hat{s}$ direction. If the fluid is static, then the pressure difference $p_1 - p_2$ can be written as $p_1 - p_2 = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$ which is known as the Young Laplace equation. To explain this, a schematic of a membrane (which is the symbolic interface here) is shown in figure 3(i) in which R_1 is the radius of curvature of one side and R_2 is the radius of curvature of the other side. Here, the two points *a* and *b* are the two centers of curvature and σ is the surface tension coefficient between the two phases which are separated by the interface. If it is a flat interface, R_1 and R_2 are both tending to infinity and then there is no pressure difference. So, the pressure difference at the interface due to surface tension can be attributed only to the curvature of the interface. However, we need to keep in mind that this is only under static condition. The equation $p_1 - p_2 = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$ comes through a

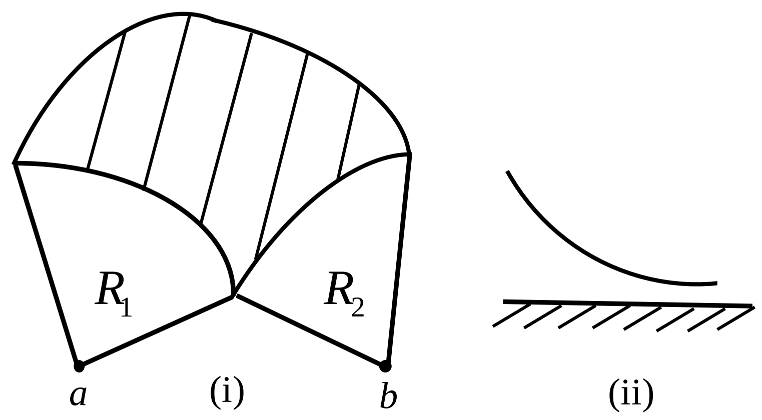

Figure 3. (i) Schematic of a membrane which is a symbolic representation of the interface. (ii) Interface in case of thin film of atomic scale or molecular scale dimension.

force balance under work energy consideration. In the course of fluid mechanics, the derivation of this equation is beyond the scope but normally it is taken up in a basic course of Physics. Otherwise in micro scale flows it is also covered since surface tension is very important. If one is interested to understand the derivation of this equation, one can go through the NPTEL lectures on microfluidics where this expression has been derived.

Now, under dynamic condition, there will be a normal stress beyond the contribution of the pressure which is the viscous normal stress $\vec{\mathrm{T}}_{dev}^{\hat{\eta}} \cdot \hat{\eta}$. For this, we consider the deviatoric component of the traction vector $\vec{\mathrm{T}}$ and $\vec{\mathrm{T}}_{dev}^{\hat{\eta}} \cdot \hat{\eta}$ is the normal component of it. So, the left hand side of the normal force balance becomes $p_1 - p_2 - \vec{\mathrm{T}}_{dev}^{\hat{\eta}} \cdot \hat{\eta}$. There is a minus sign before the viscous normal stress because pressure by its definition is compressive positive and $\vec{\mathrm{T}}_{dev}^{\hat{\eta}} \cdot \hat{\eta}$ is tensile positive. Interestingly, one can add complicated physics to this boundary condition. For example, if the interface is like what is drawn in figure 3(ii) and there is a solid boundary and a thin film of atomic scale or molecular scale dimension, inter molecular forces become important. In that case, in addition to the present forces, there will be an additional contribution of Van der Waals force of interaction and it is called as disjoining pressure (p_d). So, the normal force balance equation becomes $p_1 - p_2 - \vec{\mathrm{T}}_{dev}^{\hat{\eta}} \cdot \hat{\eta} = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + p_d$. In this particular course, one can simply interpret p_d as molecular interaction contribution of pressure. From this boundary condition, one thing we can clearly assess that if there is no curvature of the interface, i.e. if the interface is flat, then this boundary condition does not have any significance unless the molecular effects themselves play a very critical role. However, even if we have a flat interface, the tangential force balance may play a very critical role. So, the next thing is to demonstrate the tangential force balance.

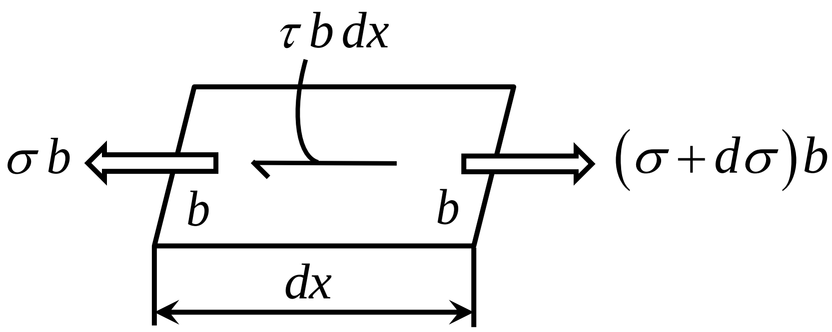

Figure 4. Schematic of the tangential force balance.

Let us assume that there is a flat interface of dimension '*dx*' as shown in figure 4. The dimension perpendicular to the interface is *b*. Let us consider σ to be the surface tension coefficient. So, on the left side, there is a force σb and on the right side there is a force $(\sigma + d\sigma)b$. If there is a fluid at the top and a fluid at the bottom, then the interface is present in between them; there exists a shear stress difference between the fluids at the top and at the bottom. This shear stress difference is denoted by the symbol τ. Sometimes in proper technical jargon it is called as the jump in the shear stress of difference in the shear stress. But in practical purposes, if there is a liquid at one side and air on the other side, then the shear stress on the air side is negligible. So this is effectively the shear stress from the other side if it is liquid air or liquid vapor interface. If there are two liquids on the two sides, we need to calculate the difference in the shear stress very judiciously. The force due to this shear stress difference is equal to $\tau\, b\, dx$. For the equilibrium of the interface, all these forces are balanced, i.e. $-\sigma b + (\sigma + d\sigma)b - \tau\, b\, dx = 0$ from which we get $\tau = \frac{d\sigma}{dx}$. This is true if the interface is flat as we have chosen an example of a flat interface. Now, if the interface is not flat, we replace this by $\vec{T}^{\hat{\eta}} \cdot \hat{s}$ which is the tangential component of the traction vector. In a vector form, the tangential force balance can be written as $\left(\vec{T}^{\hat{\eta}} \cdot \hat{s}\right)\hat{s} - \nabla_s \sigma = 0$ where ∇_s is the surface gradient operator. In order to explain the surface gradient operator ∇_s, let us think of acceleration as a vector $\vec{a}$ which is the sum of the tangential component $\vec{a}_s$ and the normal component $\vec{a}_n$, i.e. $\vec{a} = \vec{a}_s + \vec{a}_n$. The normal component of $\vec{a}_n$ is $\hat{\eta} \cdot \vec{a}$ which can be written in a vector form as $\vec{a}_n = \left(\hat{\eta} \cdot \vec{a}\right)\hat{\eta}$ where $\hat{\eta}$ is the unit vector in the normal direction. So, $\vec{a}_s = \vec{a} - \left(\hat{\eta} \cdot \vec{a}\right)\hat{\eta}$. Thus, the vector $\vec{a}_s$ is helping us to resolve the vector $\vec{a}$ in the *s* direction. Similarly, ∇_s is allowing us to resolve the grad operator ∇ in the *s* direction. So, $\nabla_s = \nabla - \left(\hat{\eta} \cdot \nabla\right)\hat{\eta}$. So, now we have understood the three important boundary conditions to summarize the interfacial flow, namely, kinematic boundary condition, tangential force balance and normal force balance. So, with this little bit of introduction of the boundary conditions for interfacial flows, for the remaining portion of the present chapter we will discuss a simple classical problem. This is a very simple classical problem where all these boundary conditions will not be required at all, i.e. it is a thin film flow with a flat interface.

Example: A thin film flow with a flat interface

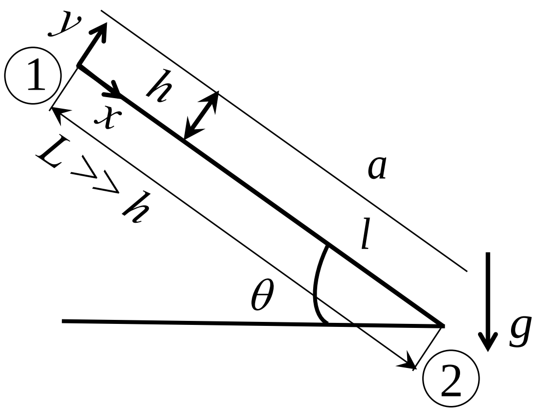

Figure 5. Schematic of the thin film flow with a flat interface.

Let us consider an inclined plane as shown in figure 5. In this inclined plane, the angle of inclination is θ. There is a thin liquid film with a flat interface. The film thickness is *h* which is constant. It does not have any derivative with respect to *x*, *y*, *z* and *t*. So, the kinematic boundary condition becomes redundant here. On one side there is air (denoted by symbol '*a*') and on the other side there is liquid (denoted by symbol '*l*'). The length of the inclined plane *L* is much greater than h, i.e. $L >> h$. This means for all practical purposes, the *y* length scale becomes much less as compared to the *x* length scale. If the y length scale is much less compared to the x length scale we can make certain conclusions and one important conclusion is that (which is left as a homework) the pressure gradient in the *y* direction is much less than the pressure gradient in the *x* direction. That is true only if the condition $L >> h$ holds. If *h* is of the order of *L* or even greater, then it does not hold. From common sense we can understand that in the present case, there will be a pressure difference due to the gravitational effect. So, the fact $L >> h$ basically leads to $\frac{\partial p}{\partial y} << \frac{\partial p}{\partial x}$. So, thin film flow is a situation where the transverse length scale is much less than the axial length scale. Typically it is a low Reynolds number problem but the low Reynolds number is not always demanded although the classical thin film flows are normally low Reynolds number flows.

So, this is a normal situation but there can be cases of thin film flow where the convective component of acceleration becomes important. So, here we assume that it is a low Reynolds number flow or to be on a safe side a fully developed flow. If we assume a fully developed flow, then the Reynolds number is not important because for fully developed flow the acceleration is anyway zero. So, it is an inertia free flow and therefore, it does not matter whether the Reynolds number is low or high. Now let us write the governing equations. It is having a translational invariance along the *x* direction. Regarding the flow, it may look like the flow between two parallel plates from the figure, but the situation is different here where we have gravity in the downward direction. Here we have the component $g\sin\theta$ in the axial direction which drives the flow. But it is not a major difference because the term $g\sin\theta$ can always be absorbed in the pressure gradient term. So it will effectively give rise to a pressure gradient where the pressure gradient is triggered by the gravity. The major difference with the flow between two parallel plates is that instead of two rigid plates, here we have one rigid boundary and another interface. So, the boundary condition here is the only thing that predominantly changes while all other factors remain the same. Similar to the fully developed flow between two parallel plates, we can write here $\mu\frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} + F_b$ where F_b is the additional body force term $F_b = \rho g \sin\theta$. Now, we will use the consideration of the fully developed flow in which u is a function of the transverse co-ordinate y only; $\mu \frac{\partial^2 u}{\partial y^2}$ can be replaced by $\mu \frac{d^2 u}{dy^2}$. The second consideration is the thin film flow for which the pressure gradient in the y-direction $\frac{\partial p}{\partial y}$ becomes very less as compared to the pressure gradient $\frac{\partial p}{\partial x}$ in the x-direction. So, $\frac{\partial p}{\partial x}$ can be approximated as $\frac{dp}{dx}$. So, $\mu \frac{d^2 u}{dy^2} = \frac{dp}{dx} + \rho g \sin\theta$ where $\mu \frac{d^2 u}{dy^2}$ is a function of y only and $\frac{dp}{dx} + \rho g \sin\theta$ is a function of x only since $\rho g \sin\theta$ is constant, therefore each becomes equal to a constant c. We can also take $\rho g \sin\theta$ in the left hand side and in that case $\mu \frac{d^2 u}{dy^2} - \rho g \sin\theta$ can be treated to a constant k which is equal to the pressure gradient $\frac{dp}{dx}$. The pressure gradient between section 1 and section 2 is the pressure difference $p_2 - p_1$ divided by the length of the inclined plane L. Both p_2 and p_1 are equal to atmospheric pressure because it is a thin film moving in an atmosphere. So, the pressure gradient term $\frac{dp}{dx}$ is equal to zero. In that case the film is purely driven by the gravity. Therefore, we can write $\frac{d^2 u}{dy^2} = \frac{1}{\mu} \rho g \sin\theta$. We can integrate it twice in the transverse direction 'y' and we get the expressions of the velocity gradient $\frac{du}{dy}$ and velocity field u as $\frac{du}{dy} = \left(\frac{1}{\mu} \rho g \sin\theta \right) y + c_1$ and $u = \left(\frac{1}{\mu} \rho g \sin\theta \right) \frac{y^2}{2} + c_1 y + c_2$ respectively. In order to find out the two integration constants c_1 and c_2, we need to apply the boundary conditions. The first boundary condition is very straight forward. At $y = 0$, i.e. at the inclined plane the velocity u is zero which is the well-known no-slip boundary condition. The second boundary condition is the interfacial boundary condition. Since the interface is flat, we do not require any additional explicit treatment of the kinematic boundary condition. Also we do not require the normal force balance because the normal force balance is a special equation if the interface has a curvature. But we require the tangential force balance.

If we recall the tangential force balance, it was $\tau = \frac{d\sigma}{dx}$ for a flat interface. In order to make a gradient of surface tension $\frac{d\sigma}{dx}$ existing, let us apply a temperature gradient here. We can write $\frac{d\sigma}{dx}$ as a product of two terms $\frac{d\sigma}{dT}$ and $\frac{dT}{dx}$, i.e. $\frac{d\sigma}{dx} = \frac{d\sigma}{dT} \cdot \frac{dT}{dx}$. So, if surface tension is a function of temperature (for most of the fluids, surface tension is a function of temperature), by creating a temperature gradient thus can create a surface tension gradient at the free surface. This gives rise to a fluid flow known as Marangoni convection. In the case when there is no temperature gradient in the *x*-direction, which is the description of the present problem, the value of $\frac{d\sigma}{dx}$ is equal to zero. Therefore, we can write τ is equal to zero. This τ is the difference in the shear stresses between the two sides. So, at $y = h$, $\mu_l \left.\frac{du}{dy}\right|_l = \mu_a \left.\frac{du}{dy}\right|_a$ where the subscripts '*l*' and '*a*' correspond to liquid and air respectively. Since the viscosity of air (μ_a) is much less than the viscosity of liquid (μ_l), and $\left.\frac{du}{dy}\right|_a$ the velocity gradient in air is also much less than the velocity gradient in the liquid $\left.\frac{du}{dy}\right|_l$. Therefore, $\mu_a \left.\frac{du}{dy}\right|_a$ can be approximated as zero. Hence, we can approximately write at $y = h$, $\frac{du}{dy} = 0$ in the liquid which is our domain. Here, one important thing to note that, in most of the books this is given as straight forward boundary condition without allowing the students to go through these exercise. Therefore in this case we do not learn the correct boundary condition at the interface; it is essentially continuity in the shear stress. Now, when we replace this problem with liquid 1 and liquid 2 instead of liquid and air the right hand side $\mu_a \left.\frac{du}{dy}\right|_a$ will not be negligible and cannot be approximated as zero. So, with the two aforementioned boundary conditions, one can straight away work out the integration constants c_1 and c_2 by simple straight forward algebra and thus is not shown here.