Advanced Concepts In Fluid Mechanics
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Lecture - 16
Navier Stokes Equation - Part II

Navier-Stokes equation (continued)

We recapitulate what was obtained in the last lecture. The governing equations for flow are the continuity equation and the three momentum conservation equations:

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_j}(\rho v_j)=0 \qquad (1)$$

$$\rho\left[\frac{\partial v_i}{\partial t}+\rho v_j\frac{\partial v_i}{\partial x_j}\right]=\frac{\partial \tau_{ij}}{\partial x_j}+b_i \qquad (2)$$

For a Newtonian fluid, the stress tensor τ_{ij} is related to the strain rate tensor e_{ij} as

$$\tau_{ij}=-p\delta_{ij}+\lambda e_{kk}\delta_{ij}+2\mu e_{ij} \qquad (3)$$

The strain rate tensor e_{ij} is simply

$$e_{ij}=\frac{1}{2}\left[\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right] \qquad (4)$$

Substituting e_{ij} from equation (4) into equation (3), we have

$$\tau_{ij}=-p\delta_{ij}+\lambda\frac{\partial v_k}{\partial x_k}\delta_{ij}+\mu\left[\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right] \qquad (5)$$

Putting *i* and *j* both as say 1 in equation (5) gives us the expression for τ_{11}, i.e. the normal stress along x_1, as

$$\tau_{11}=-p+\lambda\frac{\partial v_k}{\partial x_k}+2\mu\frac{\partial v_1}{\partial x_1} \qquad (6)$$

The presence of $2\mu\frac{\partial v_1}{\partial x_1}$ in RHS of equation (6) implies that viscous effects can contribute to normal stress, and therefore, pressure is not the only contributor to normal stress, a common misconception. Similar conclusion can be arrived at for the normal stresses along x_2 and x_3.

Now, substituting τ_{ij} from equation (5) into equation (2), we have

$$\rho\left[\frac{\partial v_i}{\partial t}+\rho v_j\frac{\partial v_i}{\partial x_j}\right]=\frac{\partial}{\partial x_j}\left\{-p\delta_{ij}+\lambda\frac{\partial v_k}{\partial x_k}\delta_{ij}+\mu\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\right\}+b_i$$

$$=-\frac{\partial p}{\partial x_j}\delta_{ij}+\lambda\frac{\partial}{\partial x_j}\left(\frac{\partial v_k}{\partial x_k}\right)\delta_{ij}+\mu\frac{\partial}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)+b_i$$

$$=-\frac{\partial p}{\partial x_i}+\lambda\frac{\partial}{\partial x_i}\left(\frac{\partial v_k}{\partial x_k}\right)+\mu\frac{\partial^2 v_i}{\partial x_j^2}+\mu\frac{\partial}{\partial x_j}\left(\frac{\partial v_j}{\partial x_i}\right)+b_i \qquad (7)$$

$$=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 v_i}{\partial x_j^2}+\lambda\frac{\partial}{\partial x_i}\left(\frac{\partial v_k}{\partial x_k}\right)+\mu\frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j}\right)+b_i$$

$$=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 v_i}{\partial x_j^2}+\lambda\frac{\partial}{\partial x_i}\left(\frac{\partial v_k}{\partial x_k}\right)+\mu\frac{\partial}{\partial x_i}\left(\frac{\partial v_k}{\partial x_k}\right)+b_i$$

$$\Rightarrow\rho\left[\frac{\partial v_i}{\partial t}+\rho v_j\frac{\partial v_i}{\partial x_j}\right]=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 v_i}{\partial x_j^2}+(\lambda+\mu)\frac{\partial}{\partial x_i}\left(\frac{\partial v_k}{\partial x_k}\right)+b_i \qquad (8)$$

Stokes' Hypothesis:

The pressure *p* appearing in equation (8) is called thermodynamic pressure, and it is the outcome of all possible degrees of freedom, rotational, translational, vibrational, etc. In contrast, we have mechanical pressure p_m, which is the pressure purely out of translational degrees of freedom, and is defined as the negative of the mean of the normal stresses:

$$p_m=-\frac{\tau_{kk}}{3}=-\frac{\tau_{11}+\tau_{22}+\tau_{33}}{3}$$

$$=-\frac{1}{3}\left[\left(-p+\lambda\frac{\partial v_k}{\partial x_k}+2\mu\frac{\partial v_1}{\partial x_1}\right)+\left(-p+\lambda\frac{\partial v_k}{\partial x_k}+2\mu\frac{\partial v_2}{\partial x_2}\right)+\left(-p+\lambda\frac{\partial v_k}{\partial x_k}+2\mu\frac{\partial v_3}{\partial x_3}\right)\right]$$

$$=-\frac{1}{3}\left[-3p+3\lambda\frac{\partial v_k}{\partial x_k}+2\mu\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\right)\right] \qquad (9)$$

$$=-\frac{1}{3}\left[-3p+3\lambda\frac{\partial v_k}{\partial x_k}+2\mu\frac{\partial v_k}{\partial x_k}\right]$$

$$\Rightarrow p_m = p - \left(\lambda + \frac{2\mu}{3}\right)\frac{\partial v_k}{\partial x_k} \qquad (10)$$

While the second term of RHS of equation (10) becomes identically zero for incompressible flows (i.e. flows with no variation in density over time and space, and thus $\frac{\partial v_k}{\partial x_k} = 0$), we need to resort to Stokes' hypothesis for the thermodynamic pressure and mechanical pressure to be equal for compressible flows. Thus, Stokes hypothesis' states that the second coefficient of viscosity λ and the dynamic viscosity μ are related as

$$\lambda + \frac{2\mu}{3} = 0 \Rightarrow \lambda = -\frac{2\mu}{3} \qquad (11)$$

Any fluid that satisfies the Stokes' hypothesis is called Stokesian fluid. Physically, Stokes' hypothesis implies that the fluid relaxation time is sufficiently small in comparison to the time scale of the dynamics of flow, so that all the other degrees of freedom in the fluid due to an imposed disturbance relax to the translation degree of freedom practically instantly. Furthermore, the second coefficient of viscosity λ is negative (since dynamic viscosity μ is positive). This implies that the $\lambda \frac{\partial v_k}{\partial x_k}$ term in the expression for stress acts to oppose the volumetric change that is occurring – it is negative (i.e. compressive) for an expanding fluid element and is positive (i.e. tensional) for a contracting fluid element.

Substituting the expression for λ from equation (11) into the Navier equation, equation (8), we arrive at the Navier equation for a Newtonian-Stokesian fluid, i.e. the Navier-Stokes equation:

$$\rho\left[\frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j}\right] = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j^2} + \frac{\mu}{3}\frac{\partial}{\partial x_i}\left(\frac{\partial v_k}{\partial x_k}\right) + b_i \qquad (12)$$

In vectorial form, this is written as:

$$\rho \frac{D\vec{v}}{Dt} = \rho\left[\frac{\partial \vec{v}}{\partial t} + \rho \vec{v}\cdot\nabla\vec{v}\right] = -\nabla p + \mu\nabla^2\vec{v} + \frac{\mu}{3}\nabla\left(\nabla\cdot\vec{v}\right) + \vec{b} \qquad (13)$$

Therefore, we have the four governing equations for a Newtonian-Stokesian fluid as the three components of equation (13) and the continuity equation. We are required to solve for the five unknowns, the density ρ, the three components of velocity v_i, and the pressure p, using these. Therefore, we are one equation short for closure. This additional equation emerges out of the prescription for density ρ, which is typically taken as one out of these three options:

1. ρ = constant
2. An equation of state for a simple compressible substrate that is of the form $\rho = \rho(p)$

3. An equation of state for a simple compressible substrate that is of the form $\rho = \rho(p,T)$ - for this equation of state, T appears as another unknown, which brings in the energy equation (taught as part of convective heat transfer) as the required additional equation. An example of this equation of state is the one for an ideal gas, $p = \rho RT$

As a conclusion to this lecture, some salient features of the Navier-Stokes equation are listed below:

1. In its most general form, the Navier-Stokes equation is a non-linear equation (due to the $\rho\vec{v}\cdot\nabla\vec{v}$ term. Therefore, an analytical solution of the general Navier-Stokes equation does not yet exist and analytical solutions for only special cases (that simplify the Navier-Stokes equation) exist.
2. We can see that the pressure in general varies spatially. While the system of equations is closed, pressure does not have a dedicated equation for it. This is remedied either by eliminating the pressure term (in some analytical solutions, by applying the identity 'curl of the gradient of any continuously twice-differentiable scalar field is zero' with the scalar field as pressure) or by substituting it into the continuity equation using some algebraic manipulations (as is done in some computational methods).
3. Ending on a pedagogical note, it is brought to the reader's attention that upon setting the velocity as identically zero and the body force as gravity $\vec{g}$ in equation (13), we recover

   $$0 = -\nabla p + \vec{g}$$

   which is the governing equation for fluid statics. Therefore, we recover fluid statics as a special case of fluid dynamics. This suggests that for advanced courses in fluid mechanics, instructors can start straight-away with fluid dynamics and teach fluid statics as a special case. This helps in salvaging the time that has classically been spent in teaching fluid statics at the start of the course, and utilizing this time in teaching new and emerging advanced concepts.