**Introduction to Fluid Mechanics**
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**Lecture – 52**
**Navier - Stokes equation - Part-II**

Now, so far we have considered only linear momentum conservation. So, we will take a little bit of distraction from that and we will now think of angular momentum consideration.

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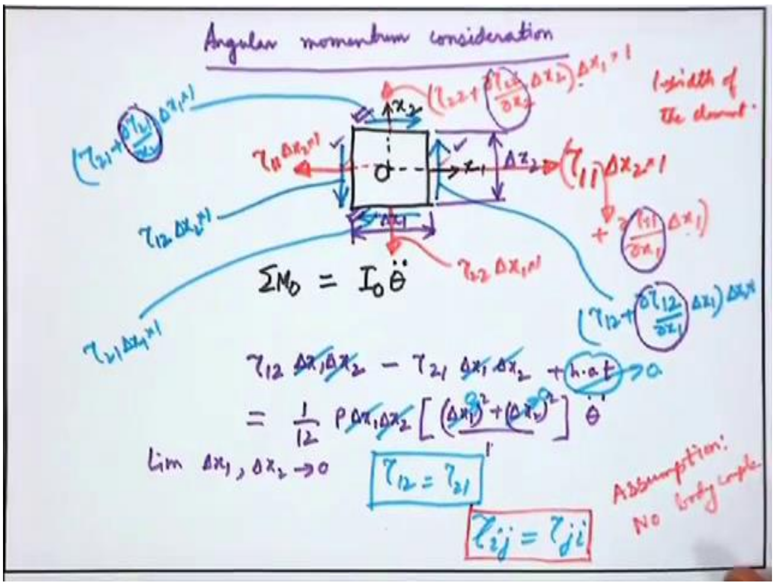


Just like linear momentum refers to translation, angular momentum refers to rotation. Now, let us say that we are interested about rotation with respect to z axis. Now, we can decouple the rotation in terms of rotation with respect to x axis, rotation with respect to y axis and z axis, x means $x_1$ y means $x_2$ and z means $x_3$. So, we can isolate these effects and consider one at a time. So, we can consider for example, rotation with respect to the z axis or $x_3$ axis. Then it is important that we consider only the forces which take place which are there in the xy plane or $x_1x_2$ plane.

So, let us say this is the $x_1$ axis and this is the $x_2$ axis and let us say that we give dimensions to this such, that this is delta $x_1$ and delta $x_2$. Now, we are interested to have rotation with respect to or an equation for rotation with respect to x 3 axis an axis that passes through O. So, for rotational motion we can write resultant moment of all forces with respect to an axis which is normal to this plane and passing through is equal to the moment of inertia with respect to the same axis times the angular acceleration.

$$\sum M_O = I_O \ddot{\theta}$$

So, let us identify the forces here; let us write the normal forces and the tangential forces. let us say 1 is the width of the element. On the left side it would be $\tau_{11} \Delta x_2 \times 1$ . To the right it would be $\left( \tau_{11} + \frac{\partial \tau_{11}}{\partial x_1} \Delta x_1 \right) \Delta x_2 \times 1$ .

On the bottom it would be $\tau_{22} \Delta x_2 \times 1$ .

On the top face it would be $\left( \tau_{22} + \frac{\partial \tau_{22}}{\partial x_2} \Delta x_2 \right) \Delta x_1 \times 1$ .

Because all these forces pass through this point O therefore, these do not contribute to the moment with respect to an axis passing through O. Let us draw the tangential forces. So, first let us show their proper directions and then we will write the force.

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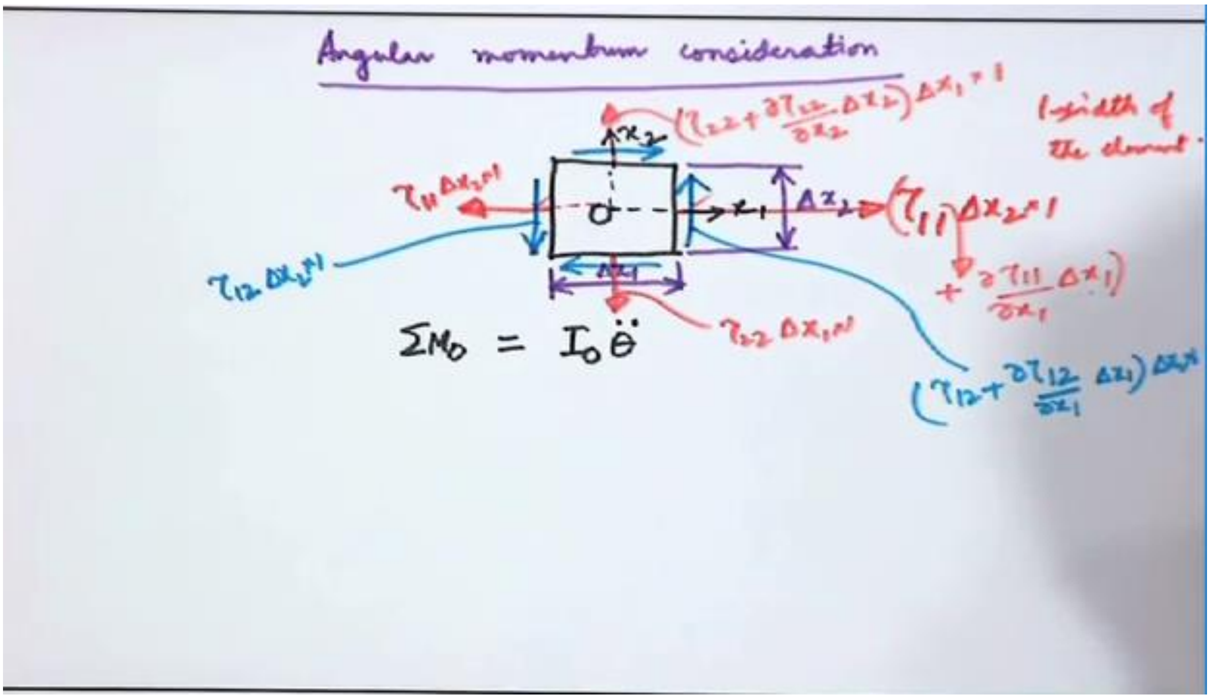


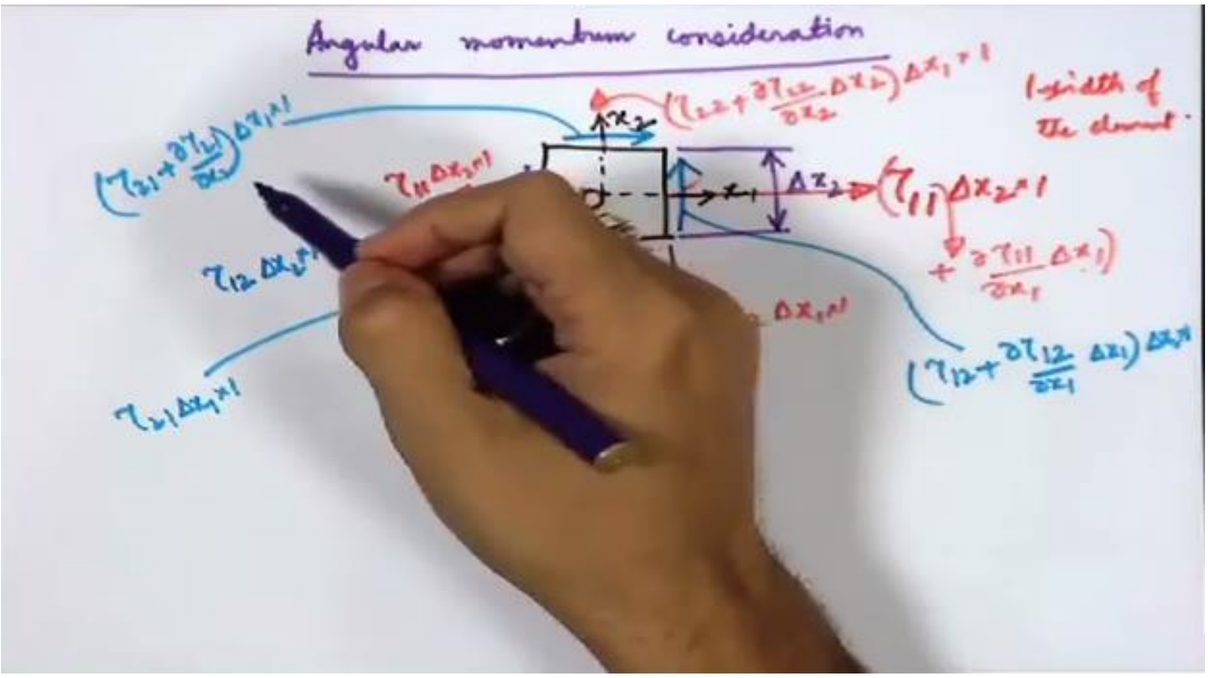


For calculating the moment of the forces these incremental changes will not essentially matter; these changes will not essentially matter.

So, essentially for calculating the moments these two forces are like almost equal and opposite; they are actually not equal there is an incremental change, but in the limit they are almost equal opposite and anti-parallel. So, they will form a couple. This force is upward and this force is downwards. So, this is creating an anti clockwise moment. So, this is a positive moment. So, this is $\tau_{12}\Delta x_1 \Delta x_2$ .

$$\tau_{12}\Delta x_1\Delta x_2 - \tau_{21}\Delta x_1\Delta x_2 + h.o.t. = \frac{1}{12}\rho\Delta x_1\Delta x_2\left[\frac{(\Delta x_1)^2 + (\Delta x_2)^2}{1}\right]\ddot{\theta}$$

Lim $\Delta x_1, \Delta x_2 \to 0$

$$\Rightarrow \tau_{12} = \tau_{21}$$

If there is a body force is still because, its moment with respect to O is 0 until and unless it is asymmetrically distributed. But, if there is a body couple then because of the body couple there could be an additional couple moment. So, the assumption is no body couple.

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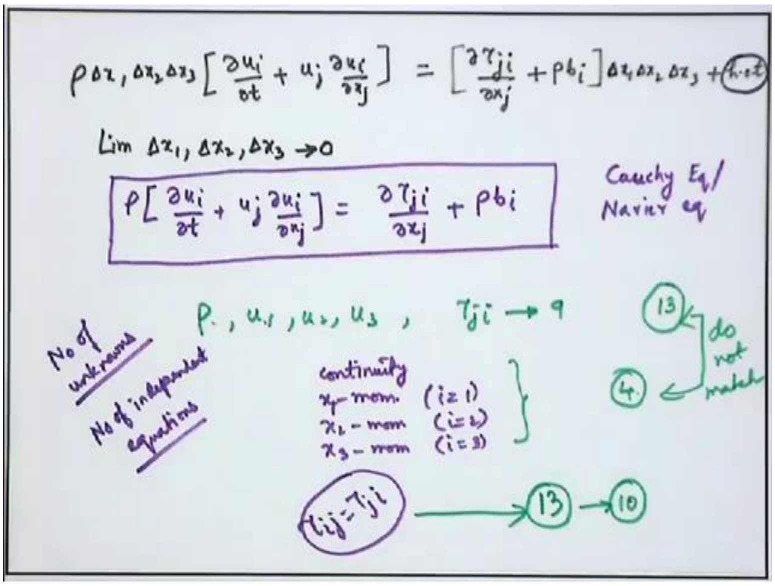


I will bring the previous page. So, what you see here is that we had 13 number of unknowns and 4 number of independent equations. Now, we have proven that $\tau_{ij} = \tau_{ji}$ .

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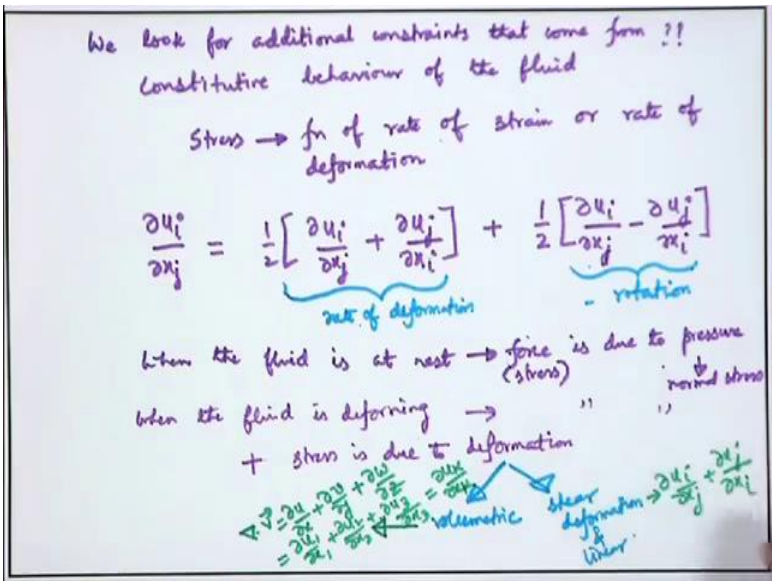


For fluids the stress is a function of rate of strain or rate of deformation ok; function of rate of strain or rate of deformation.

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] + \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right]$$

$$\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] \rightarrow rate \text{ of deformation}$$

$$\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right] \rightarrow rotation$$

It is deformation that gives rise to stress. But it is also true that when the fluid is not under motion or under deformation then also there is a stress.

So, when the fluid is at rest when the fluid is at rest, the force is due to pressure. When the fluid is deforming still then the pressure is present; stress is due to pressure + stress is due to deformation. The deformation can be linear or angular, linear will give rise to volumetric deformation and shear deformation, angular deformation will give rise to shear deformation.

The expression for the volumetric deformation is divergence of the velocity vector

$$\nabla.\vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

Shear $deformation \rightarrow \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$

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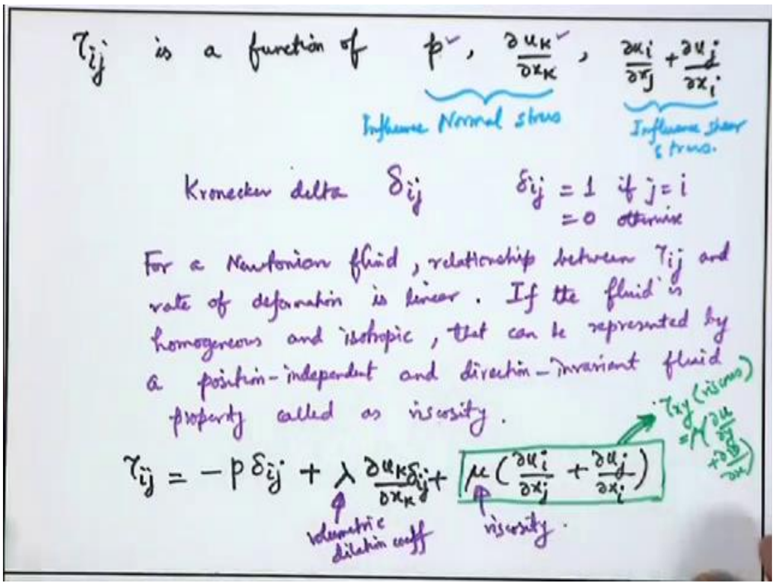


$\tau_{ij}$ is a function of pressure(P), $\frac{\partial u_k}{\partial x_k}, \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$

$\frac{\partial u_k}{\partial x_k}$ influence normal stress

$\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$ influence shear stress

Kronecker delta $\delta_{ij}$ $\delta_{ij} = 1$ if j=1

=0 otherwise

So, for a Newtonian fluid relationship between $\tau_{ij}$ and rate of deformation is linear. If the fluid is homogeneous and isotropic, isotropic that can be represented by a position independent and direction invariant fluid property called as viscosity.

$$\tau_{ij} = -p\delta_{ij} + \lambda \frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial uj}{\partial x_i}\right)$$

$\lambda \rightarrow$ volumetric dilation coefficient, $\mu$ is viscosity.

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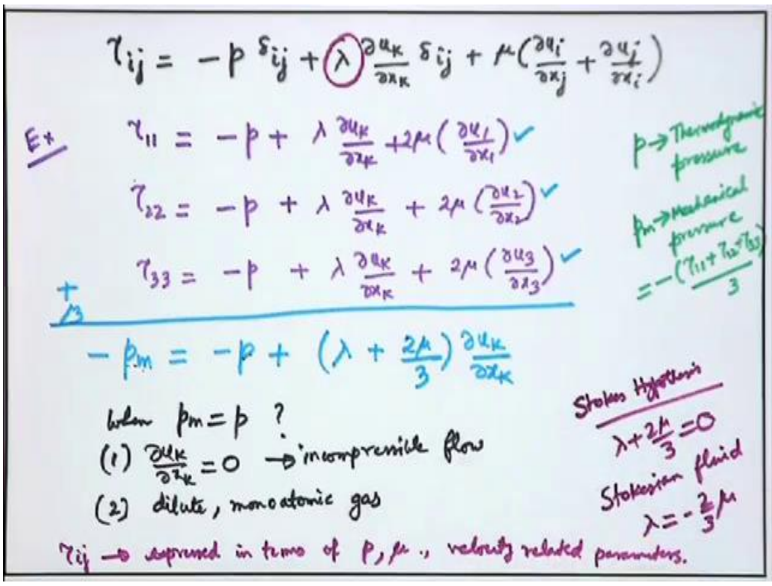


$$\tau_{ij} = -p\delta_{ij} + \lambda \frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial uj}{\partial x_i}\right)$$

**EX**

$$\tau_{11} = -p + \lambda \frac{\partial u_k}{\partial x_k} + 2\mu\left(\frac{\partial u_1}{\partial x_1}\right)$$

$$\tau_{22} = -p + \lambda \frac{\partial u_k}{\partial x_k} + 2\mu\left(\frac{\partial u_2}{\partial x_2}\right)$$

$$\tau_{33} = -p + \lambda \frac{\partial u_k}{\partial x_k} + 2\mu\left(\frac{\partial u_3}{\partial x_3}\right)$$

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P $\rightarrow$ thermodynamic pressure

Mechanical Pressure, $P_m = -\frac{(\tau_{11} + \tau_{22} + \tau_{33})}{3}$

$$-P_m = -P + \left(\lambda + \frac{2\mu}{3}\right)\frac{\partial u_k}{\partial x_k}$$

So, mechanical pressure will usually reflect the translational degrees of freedom of molecules whereas, that thermodynamic pressure will reflect all sorts of degrees of freedom.

When $P_m$=P

(1) $\frac{\partial u_k}{\partial x_k} = 0 \rightarrow$ incompressible flow

(2) Dilute, monoatomic gas

Stokes hypothesis $\rightarrow \lambda + \frac{2\mu}{3} = 0$

Stokesian fluid $\lambda = -\frac{2}{3}\mu$

Every fluid when subjected to a change will take some time to adapt to the change and this time is known as relaxation time. But, there can be a very interesting situation when the time over which the change on the fluid is imposed is shorter than the relaxation time; then the fluid cannot adapt to the change and then a new change by that time has come.

So, then we can say that it is a very rapid process. Let us take an example. So, if you have a bubble which is rapidly expanding and contracting then what is happening is that the time scale of imposition of the change is very rapid. If its relaxation time is not that fast then the fluid cannot relax to a new thermodynamic state before a new change has taken place. And then it cannot equilibrate its properties so, that it can convert all the degrees of freedom to the translational degrees of freedom instantaneously.

So, if it cannot convert all the degrees of freedom to the translational degrees of freedom instantaneously then the equality of mechanical pressure and thermodynamic pressure will not take place. Therefore, for very rapid changes where the timescale of imposition of the change is faster than the relaxation time of the fluid; then the Stokes hypothesis will no more be valid. So, it is very important to understand that Stokes hypothesis is not a law it is a hypothesis.

So, it does not have a proof, but there are physical arguments that for most of the practical scenarios the time scale of imposition of the disturbance is significantly larger as compared to the relaxation time scale. So, that fluid relaxes almost instantaneously so, that it attains an equilibration of mechanical and thermodynamic properties.

$\tau_{ij} \rightarrow$ expressed in terms of $P, \mu$ and velocity related parameters

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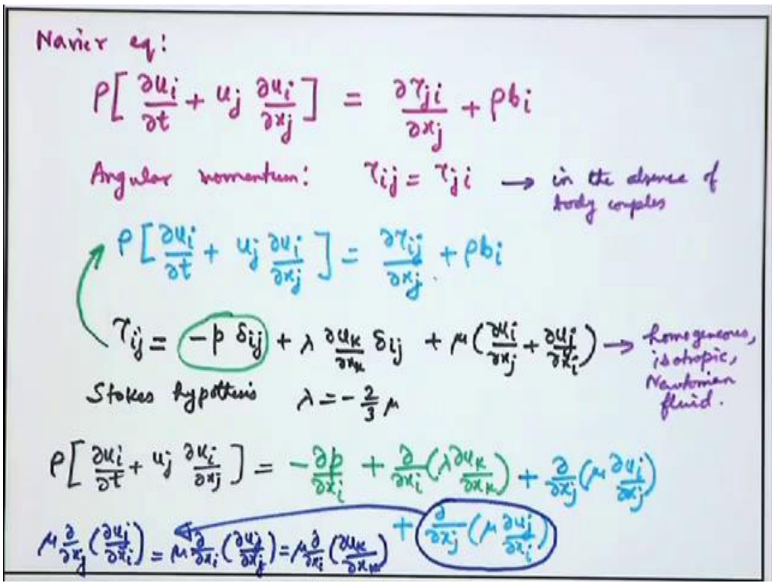


Navier's Equation:

$$P\left[\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right]=\frac{\partial \tau_{ji}}{\partial x_j}+\rho b_i$$

Angular momentum: $\tau_{ij}=\tau_{ji}$

$$P\left[\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right]=\frac{\partial \tau_{ij}}{\partial x_j}+\rho b_i$$

$$\tau_{ij}=-p\delta_{ij}+\lambda\frac{\partial u_k}{\partial x_k}\delta_{ij}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\rightarrow \text{homogenous, isotropic, Newtonian fluid}$$

Stokes hypothesis $\lambda=-\frac{2}{3}\mu$

$$\rho\left[\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right]=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_i}\left(\lambda\frac{\partial u_k}{\partial x_k}\right)+\frac{\partial}{\partial x_j}\left(\lambda\frac{\partial u_i}{\partial x_j}\right)+\frac{\partial}{\partial x_j}\left(\lambda\frac{\partial u_j}{\partial x_i}\right)$$

$$\mu \frac{\partial}{\partial x_j}\left( \lambda \frac{\partial u_j}{\partial x_i} \right) = \mu \frac{\partial}{\partial x_i}\left( \lambda \frac{\partial u_j}{\partial x_j} \right) = \mu \frac{\partial}{\partial x_i}\left( \lambda \frac{\partial u_k}{\partial x_k} \right)$$

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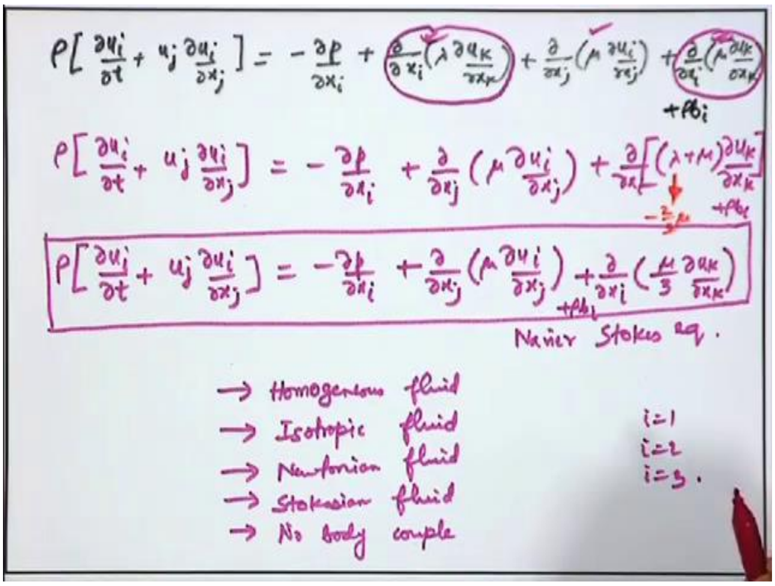


$$\rho\left[ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right] = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}\left( \lambda \frac{\partial u_k}{\partial x_k} \right) + \frac{\partial}{\partial x_j}\left( \lambda \frac{\partial u_i}{\partial x_j} \right) + \frac{\partial}{\partial x_i}\left( \lambda \frac{\partial u_k}{\partial x_k} \right) + \rho b_i$$

$$\rho\left[ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right] = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left( \mu \frac{\partial u_i}{\partial x_j} \right) + \frac{\partial}{\partial x_i}\left( (\lambda + \mu) \frac{\partial u_k}{\partial x_k} \right) + \rho b_i$$

$$\rho\left[ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right] = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left( \mu \frac{\partial u_i}{\partial x_j} \right) + \frac{\partial}{\partial x_i}\left( \frac{\mu}{3} \frac{\partial u_k}{\partial x_k} \right) + \rho b_i$$

This equation is known as Navier-Stokes equation. So, the assumptions that we have taken for this equation are

1) Homogeneous fluid
2) isotropic fluid
3) Newtonian fluid
4) Stokesian fluid
5) No body couple.

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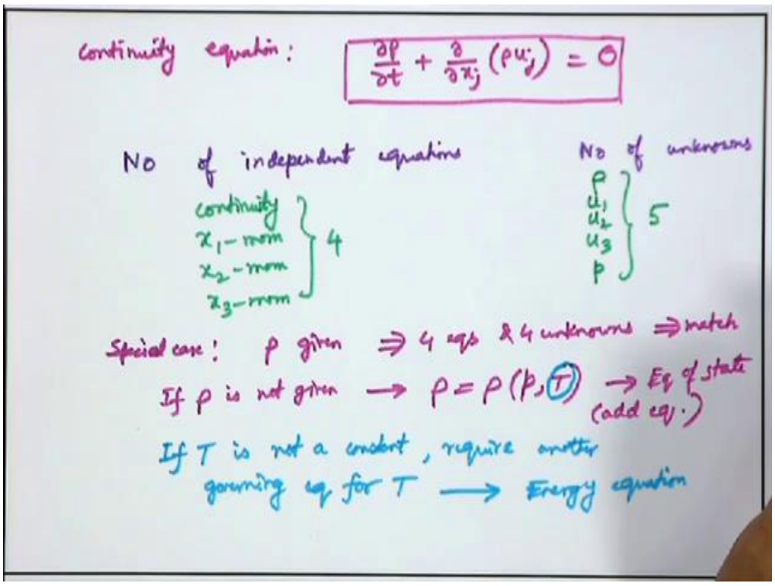


Continuity Equation: $\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho u_j\right) = 0$

Special Case: $\rho$ is given $\Rightarrow 4$ eqns and 4 unknowns

If ρ is not given $\rightarrow \rho = \rho(p, T) \rightarrow$ Eqn of state

If T is not a constant, you require another governing equation for T $\rightarrow$ *Energy* equation

The Navier-Stokes equation is a second order partial differential equation and it is a non-linear partial differential equation and it is a coupled system of partial differential equations.

Stokes law is the law that describes the drag force of a slowly moving sphere in a viscous liquid.