Introduction to Fluid Mechanics
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Lecture – 04
Cauchy's Theorem

So, we will now go to our next objective that is, given these components of the stress tensor how we may utilize this concepts or these components to designate the state of stress on any arbitrary surface which is neither oriented along x, y or z.

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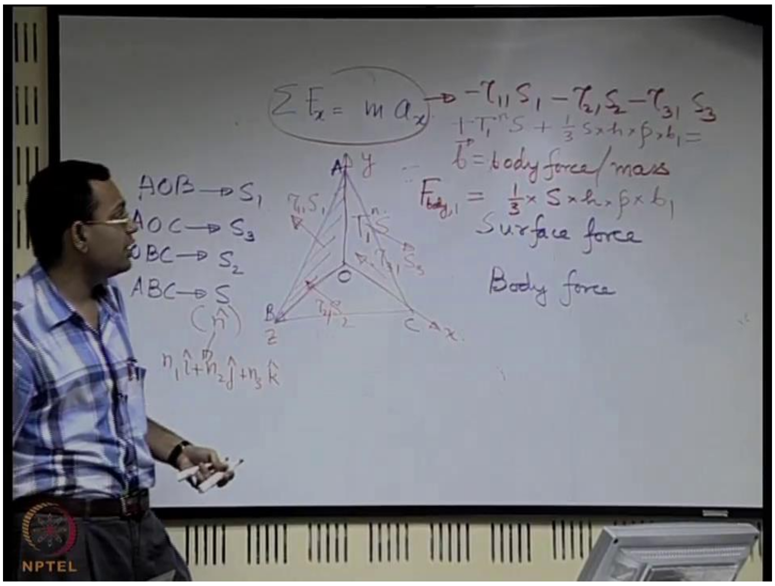

For that what we do is we consider an elemental volume like a tetrahedron. We give the point certain names for convenience. So, there are surfaces like this the surface AOB let us call it as S1, surface AOC, we call it as S3; why such 1 and 3 because this index we are trying to preserve for the direction normal of those surfaces. So, these 1 is for the fact that the direction normal of these AOB is along x.

Student: x.

Similarly, this one is S2 and ABC let us give it a name S. Can you tell: what is the motivation of this taking such a volume?

Student: (Refer Time: 02:07).

See, whenever we are deriving something in the class it is like it will appear to you that yes it has to be done like that. Remember, this is not a ritual; do not accept anything whatever we are learning in the class as a ritual. Always try to ask yourself a question; why are we taken such a volume, what is the motivation behind taking such an element.

So, if you see this element has four faces. Out of these four faces, three are the special ones which I have that direction normal either along x, y or z. The fourth one is not a special one, it is arbitrarily oriented. So now, by considering the equilibrium of these element, by considering the forces which are acting on it we will be able to express what is there on that odd surface in terms of what is there on the special surfaces. So, that is what is the motivation behind taking this one.

Now, whenever we are coming to such an element our objective will be say to write the Newton's second law of motion for this. So just, resultant force equal to mass into acceleration. Question is what forces are acting on this element. So, when we say what forces are acting on the element, we will be classifying the forces in continuum mechanics in two categories; one is a surface force another is a body force.

These names are almost self explanatory. So, when you say surface force it means that these are forces which are acting on the surface or surfaces which are comprising the volume element chosen and body force is a force which is acting over the volume of the body example is body force. One of the examples of body force is the gravity force which acts throughout the body; volume of the body. Surface force pressure is an example force due to pressure is a surface force.

So, whenever we are having forces we will categorize in terms of surface force and body force. So, wherever we have surface force the surface force may be expressed in terms of the traction vector because the traction vector we have defining such a way that on a surface it represent the resultant force per unit area. So, it represents the cumulative effect of all forces which are acting at that point on the surface.

So, this particular element has four surfaces, let us write the four forces which are acting on these four surfaces and write the Newton's second law of motion along say x-direction. So, what we are going to write is resultant force along x is equal to the mass of the fluid element times the acceleration along x. So, when we write the resultant force, we will write it in terms of the surface force and body force.

So, first let us come to the surface say AOB. So, on this surface which component of the stress tensor will give a force along x in term of τ_{ij} ?

Student: τ_{11}.

T_{11}, right. So, what would be the positive sign convention direction of τ_{11} ?

Student: (Refer Time: 05:41).

Along this so, τ_{11}, this is the force per unit area. So, what is the area on which it is acting in to S_1. Similarly, for S_2, what is the force which is acting on along 1? τ_{21}, right. What would be its direction? Just like these, because the it is outward normally is along negative to direction therefore, the positive sense of τ_{21} on the surface is along negative one, ok. So, and these multiplied by S_2 similarly this will be τ_{31} S_3. There is a fourth surface which is really the back surface here which has its normal neither along x, y or z.

So, let us say the normal to these is $\hat{n}$, the normal vector outward normal vector of S. So, this $\hat{n}$ say it has its components like this $n_1\hat{\imath} + n_2\hat{\jmath} + n_3\hat{k}$. where n_1, n_2, n_3 are the components of this along x, y and z we are now going to write the force on S. So, the force on S let us say that it is T^n.

So, now, we cannot use the τ notation for that because it is not a special surface τ notation you can write for a special surface where the normal are along x, y or z. So, we use the T notation this is for the ABC. This component wise it is along one and the area on which it is acting is S and, obviously, by default we are taking it as along positive x.

These are the surface forces, what is the body force? Say we call that $\vec{b}$ is the body force per unit mass. So, we if we find out what is the mass of the fluid element; so, what is the mass of the fluid element. So, let us say that we find out first what is the volume of the fluid element. So, for this type of element we can say that it is one third of one third into the area of this ABC times the perpendicular distance from O to ABC let us say that perpendicular distance is h.

So, we first find what is b_1. So, what is b_1? $\frac{1}{3} \times S \times h$, where h is the perpendicular distance from O to ABC, $\frac{1}{3} \times S \times h$, are volume; mass of this, you multiply it by the density. So, ρ is the density. So, $\frac{1}{3} \times S \times h \times \rho$ is a mass and $\frac{1}{3} \times S \times h \times \rho$ multiplied by the body force per unit mass(b_1) along x we will give the total body force along the direction 1 or x. So, this is the mass, this is the body force per unit mass along x. So, the product is the total body force along x, right.

So, we write the Newton's second law of motion, this particular expression now we write the forces. So, $-\tau_{11}S_1 - \tau_{21}S_2 - \tau_{31}S_3 + T^n \times S$ that is the total surface force plus the total body force, that is the net force which is acting on it and that net force is equal to its mass into acceleration.

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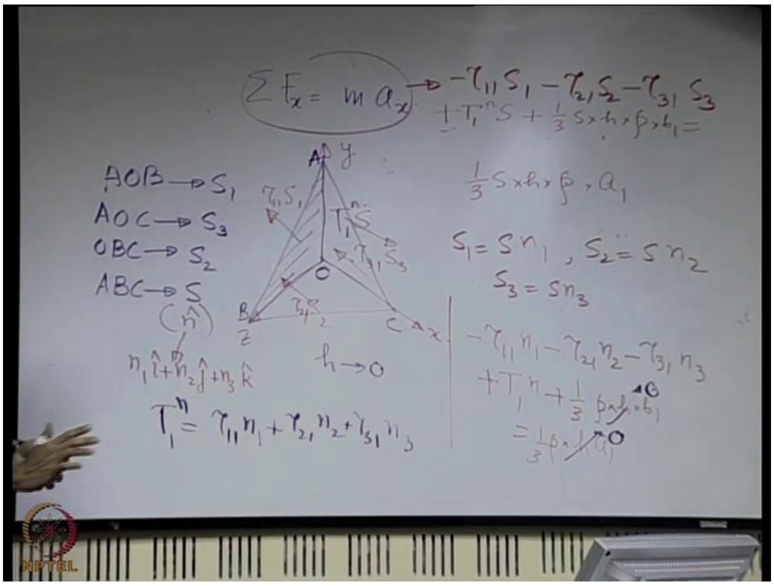

So, that is equal to what is it's mass $\frac{1}{3} \times S \times h \times \rho$ that is the mass. Let us say a is the acceleration, so, a_1 is the acceleration along x_1.

Student: Sir (Refer Time: 11:20).

Yes.

Student: Can you, sir, will you interpret physically what is surface force and body force?

The physical definition and the mathematical definition is identical here. Surface force is a force which is distributed over the surface which is the envelope of the volume, that is being considered and body force is something which is acting within the volume of the body.

Student: Within the body?

Yes.

Student: So, like surface force is shear force.

Shear force is a surface force as an example.

Student: Only shear force (Refer Time: 11:51).

Also normal force just like pressure is a normal component, it is not a shear component.

Student: Actually one term (Refer Time: 12:00) when just $\tau_{11}S_1$

Yes.

Student: It is also acting normally to a body.

But, it is a force which is which is acting on the surface I mean whether normally to the body or not it is a matter of direction. But the force may act on the surface or the force may be acting throughout the volume of the body that is how this is classified. So, obviously, when we are considering these this is something which is acting on the surface. Obviously, it will have a direction right I mean there is no contradiction with that with a surface force and a body force. Surface force will have a direction body force will also have a direction.

So, when you have this expression next is you can write S_1, S_2 and S_3 in terms of S. How can you write it? See S_1 is like the projection of S on the y-z plane, right. So, how do you find out the projection? You find out basically the component of this so called vectoral representation of ABC on the y-z plane. So, that means, when you want to find out the component you basically find the dot product of the corresponding unit vectors. So, this has a unit vector in the direction of ABC has unit vector in the direction of n, AOB has unit vector in the direction of $(-\hat{\imath})$. So, but obviously, here the plus minus you are already taking care of through these sign convention. So, you are not duplicating it once more. So, the dot product of those two directions will be n 1 is this vector another is i so, the dot product will be n 1. So, in terms of these are all magnitudes. Their senses have already been taken care of with plus minus. So, S_1 is nothing, but a Sn_1 by taking the component of so called S in the direction of the so called S_1. Similarly S_2 is Sn_2 and S_3 is Sn_3.

So, in that way if you substitute in this equation you will see that S gets cancelled out. So, what you will get $-\tau_{11}n_1 - \tau_{21}n_2 - \tau_{31}n_3 + T_1^n + \frac{1}{3} \times \rho \times h \times b_1 = \frac{1}{3} \times \rho \times h \times a_1$. When you have these equations, the next consideration that we have to make is something which is subtle, but important to understand. We will shrink this volume to a point such that this entire volume as if converges to the point O, because our end objective is to find out the state of stress at a point in terms of an area chosen around that point.

So, we will be considering a vanishing area or a vanishing volume so to say, not a vanishing area so that everything converges to O; that means, we will taking the limit as h tends to 0. So, when you take h tends to 0, the entire volume will converge to the point O then whatever we describe basically is the description of state of stress at a point O. So, when you take that limit as h tends to 0 you will see very beautifully these terms will tend to 0. So, in that case you are left with a very simple expression for $T_1^n = \tau_{11}n_1 + \tau_{21}n_2 + \tau_{31}n_3$. You can see that this is a very excellent expression because it relates the traction vector on an arbitrary surface with the components of stress tensor.

What are the inputs? The inputs are that you must know the state of stress on those specified orient planes with specified orientations and the components of the unit vector direction of the arbitrary plane that you are considering, everything at a particular location. The other thing is that there is a way of writing this symbolically in a more compact manner.

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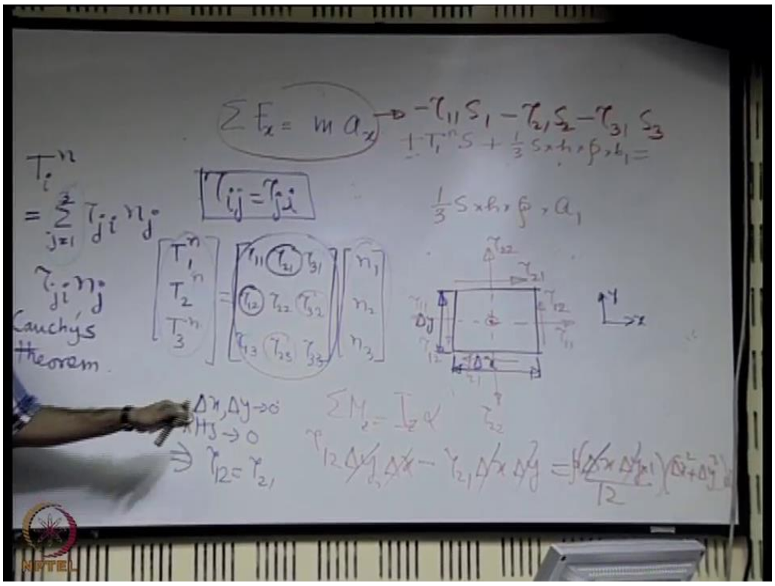

You can see that here you are having two indices. So, the first index is what is varying, the second index is something which corresponds to this one. So, you can just generalize $T_i^n = \sum_{j=1}^{3} T_{ji} n_j$. So, you have one index i which is fixed, the other index j which is a variable which varies from 1 to 3. Now, because this type of notation is very common, the general rule is again the general notation is that this summation is omitted. So, this becomes invisible. So, this is also written as just $\tau_{ji}n_j$. How you will know that there is a summation whenever there is a repeating index you have to keep in mind that there is an invisible summation in it.

So, from now onwards many times you will be using this notation without using the summation symbol, but have to keep in mind that whenever there is a repeating index there should be an invisible summation over that this is known as Einstein index notation this was first introduced by Einstein. So, this type of notation gives a very compact way of writing these terms of writing this traction vector in terms of the components of the stress tensor. So, $\tau_{ji}n_j$ is known as Cauchy's theory.

So, what it does is, it expresses the traction vector on any arbitrary plane in terms of the corresponding stress tensor components. It is also possible to write it in a matrix form. So, what you can do you are having components of the traction vector on a given plane say with orientation n, three components you have. So, this if you just follow you will see that

$$\begin{bmatrix} T_n^1 \\ T_n^2 \\ T_n^3 \end{bmatrix} = \begin{bmatrix} \tau_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

So, you can see that this is nothing, but a matrix way of writing the three components of the traction vector. So, whatever equations that we are written this is not something new, just put i equal to 1, it will correspond to the first row of these i equal to 2 and i equal to 3. So, what you can see is that here you get those. So, called nine components of the stress tensor and these all these nine are not independent we will see, but this is something what it is mathematically doing you can see here.

Look into these quantities, what is this? This is a vector, it has its three components n_1, n_2, n_3, these also a vector. So, this is acting like a transformation which maps a vector on to a vector. So, it is a very important characteristic of a second order tensor that a second order tensor maps a vector onto a vector. Similarly like a fourth order tensor maps a second order tensor on to a second order tensor like that as an example. So, tensor is also like a transformation tool or a transformation which tries to transform one vector into another vector. If it is a second order tensor that next thing that follows from this is that are all these independent or all these τ_{ij}'s are independent or we should be in a position to express these τ_{ij}'s some of these in terms of the other.

So, for that we will quickly do one exercise we will consider now a 2-dimensional element. 2-dimensional element is something where everything is occurring in a plane just for simplicity. So, we are assuming that the third direction is like unity or whatever. So, it has its length like say this is delta x, this is δ_y just imagine that it has phases perpendicular to whatever has been drawn in the figure having all width as 1.

So, let us write the components of the stress tensor on these surfaces. So, very quickly we will write it because we have learnt by this time how to write it. So, this is τ_{11}, this is what is this? τ_{21}, here this is τ_{11}, this is τ_{21}, here this is what will be here τ_{12} it is along the positive 1 because the normal is outward normal is along positive 2.

Student: (Refer Time: 23:37).

Yes?

Student: (Refer Time: 23:40).

So, these two will be reversed?

Student: Yes, sir.

So, I would expect that you always correct it. So, the first index is what? Direction normal. So, direction normal is what? 1. Second index is the direction on which it is acting. So, this is τ_{12} and this is τ_{21} here same.

Now, we are interested about the equilibrium of this element. So, when we are interested about the equilibrium of this element we will consider the rotational equilibrium as an example. So, rotational equilibrium let us consider that as if it is an element where we will be writing an equivalent form of Newton's second law for rotation as if like we are writing rotation of a rigid body with respect to a fixed axis something like that where the axis is this, centre.

So, as if we are writing a rotational equilibrium equation with respect to an axis which passes through this centre O and it is perpendicular to this plane of the board. So, that is why it is a 2-dimensional thing we are considering a rotation in this plane basically. So, what we can write resultant moment of all forces we which are acting on this is what say we are writing the moment with respect to which axis; z-axis here is equal to what tells that it is 0? It might be having an angular acceleration. So, it is I with respect to the same axis which is passing through this one and perpendicular to the plane of the board say z times the angular acceleration say alpha.

So, that resultant moment of all these forces what will be that? So, you will see that τ_{11} and $-\tau_{11}$, τ_{22} and $-\tau_{22}$ they cancel. So, moment contribution contributors will be τ_{12} and τ_{21} . So, τ_{12} and this tau 1 2 they form like a couple. So, it is tau 1 2 what is the area on which it is acting into delta y into 1 which is the width times the arm of the couple moment delta x, for the other one it is clockwise. So, $-\tau_{21}\Delta x\Delta y$; we are assuming there is no body couple which is acting on it just like body force there could be body couple fluids usually do not sustain body couple. So, there is no body couple which is acting on it is equal to the moment of inertia is like it is having a dimension of m in to length square. So, m is like $\rho\Delta x\Delta y\times 1$ that is the m.

$$\tau_{12}\Delta y\Delta x-\tau_{21}\Delta x\Delta y=\left(\frac{\rho\Delta x\Delta y\times 1}{12}\right)\left(\Delta x^2+\Delta y^2\right)\alpha$$

If you write it properly it is delta x square plus delta y square by 12 with respect to this axis times the alpha keep in mind that $\Delta x\Delta y\to 0$; so, if you cancel by considering that these are tending to 0, but not equal to 0. So, you are left with what? In a right hand side you have terms because delta x and delta y are tending to 0 you have the right hand side tending to 0 and that will give you a very interesting result tau 1 2 is equal to tau 2 1.

So, in general tau i j is equal to tau j i that means, τ_{21} and τ_{12} are same, τ_{31} and τ_{13} are same and τ_{32} and τ_{23} are same. So, you are left with six independent components in this stress tensor and you see that $\tau_{ij}=\tau_{ji}$ what are the assumptions on under which it is valid?

Student: (Refer Time: 28:17).

There is no body couple that is the only assumption. It does not depend on whether it is at rest or in motion this is a very common misconception that people have that is valid only for static systems, no. I it might be accelerating, but it does not matter because in the limit as $\Delta x\Delta y\to 0$

the acceleration angular acceleration term becomes significant and that is how you get $\tau_{ij} = \tau_{ji}$. Ok, we stop our discussion today, we will continue in the next lecture.

Thank you.