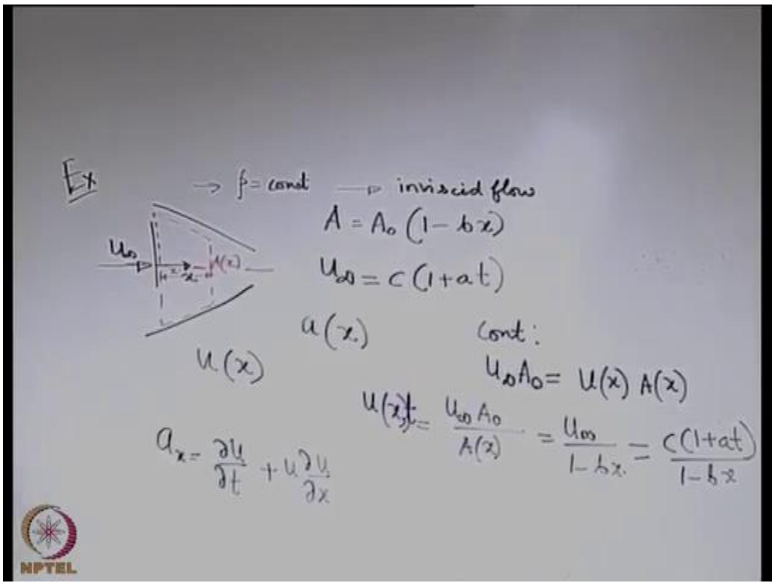
Introduction to Fluid Mechanics

Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 27
Deformation of fluid elements-Part-II

We were discussing about the continuity equation last time and let us work out a problem from your textbook to go ahead. So, we have a nozzle like this.

(Refer Slide Time: 00:29)



And it is carrying a fluid with density equal to constant and some other input data are given for the problem. The area of the cross section of the nozzle, it varies with the area of the cross section of the inlet by $A = A_0(1-bx)$, $u_\infty = c(1+at)$. What we are asked to find out that what is the acceleration as a function of x.

One important assumption is density equal to constant. The other assumption that may not be stated explicitly in the problem, but we will assume to go ahead is that it is an inviscid flow. So, when we assume it as an inviscid flow; that means, we are not bothering about the variation of velocity along the transverse direction, we are assuming that at each section the velocity is uniform.

So, only flow is through the inlet and through the surface at x. So, from the continuity equation, density equal to constant $u_0 A_0 = u(x)A(x)$

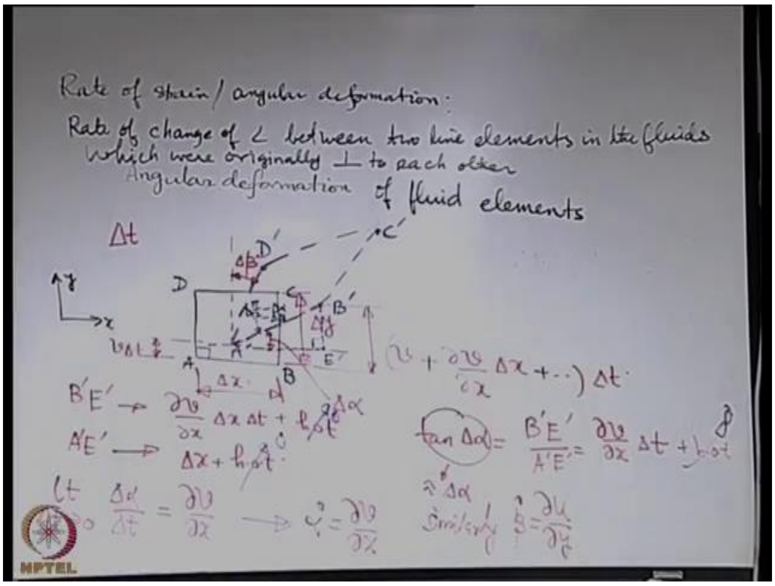
$$u(x,t) = \frac{u_\infty A_0}{A(x)} = \frac{u_0}{1-bx}$$

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}$$

There is no acceleration along other any other direction because it is just a one dimensional flow. So, here you can see an example where you have both the temporal component as well as the spatially varying component of the acceleration. So, given this acceleration, it is also possible to find out that how much time a fluid particle will take to traverse say from one end of the nozzle to the other or given in fact, the velocity components or just here one component of velocity, you may work out that what should be the time necessary for a fluid particle which is injected here to move along the centreline from one end to the other.

So, it is just like tracing the path line and finding it out when along that path it moves and comes to the end of the channel. So, that is a straightforward extension of this one ok. So, we will move on to our next concept and that concept is related to the angular deformation of the fluid elements. So, till now, we have discussed about the linear deformation of the fluid elements. And from the linear deformation of the fluid elements we found out that we got as a consequence, a very important equation known as the continuity equation.

(Refer Slide Time: 08:07)



For the angular deformation, we will try to first sketch that how a fluid element when deformed angularly will look and then try to quantify it in terms of the velocity components.

Let us say that we start with a rectangular fluid element like this. When we are think about angular deformation, the deformation is in terms of change of an angle and that angle change may take place in a plane. So, although the general fluid element is a 3 dimensional element, but you can always take a 2 dimensional element and consider the deformation in the plane because no matter how complicated deformation is, it may be resolved in different planes. So, let us say that this is one such plane in which the deformation is taking place

This is an x, y plane. So, for example, if there is a rotation with respect to the z axis, then that occurs in this plane. So, it does not mean that if we are focusing our attention on a plane, we are actually restricted to 2D, we are actually restricted to one component of the deformation and other components will be very similar. Let us say that name of this fluid element is ABCD and it has it is dimensions say Δ x and Δ y along x and y.

Now, let us stretch our imagination and assume that this fluid element has got deformed with time. When it has got deformed, it is possible that it has got deformed in two ways; one is its volume might have got changed which is like extension of the linear deformation, but it is shape might have also got distorted which is more common if it is under shear. So, if it is shape is distorted, let us say that maybe it has come to this shape. This may not be a very regular shape. This is just a schematic.

So, now if you see that what are the important parameters that are characterizing this deformation. If you draw two lines through this new location of a say one along x and another along y, the first important parameter that will come very much apparent is this angle say $\Delta\alpha$. Here, we are considering a small interval of time Δ t within which this deformation has occurred because if we allow large time, we will not be able to keep track of the deformation fluid is under continuous deformation. So, we take a small time interval. In that small time interval, the element A, B which was originally oriented along x now is oriented at an angle $\Delta\alpha$ with x.

Similarly, let us say that this angle is $\Delta\beta$. Our objective will be to quantify the time rate of change of these angles say alpha or beta in terms of the velocity components, here u and v because it is in the x, y plane. To do that, we may make some simple geometrical construction it is not actually a construction, but just to figure out what is happening, so, if you think of a right angle triangle like this maybe A', B', E'. This right angle triangle is important because in that right angle triangle, if you know that what is B', E', then you may possibly be able to express $\Delta\alpha$ or $\tan(\Delta\alpha)$ in terms of that and A' E'. So, let us try to figure out what is B', E', that is the next objective. To do that, we first understand or we first try to figure out that what is the vertical displacement of the point A.

Our consideration is the vertical component of the displacement. So, first we find out what is the vertical component of the displacement at A, also we find out what is the vertical component of the displacement at B. The net difference between these two is this length B’, E’ that we are talking about. So, what is this displacement? Let us say that the velocity at the point A is given by the two components, u and v (u , v) are the components of the velocity vector. So, at A, if you allow a time of Δt, the vertical displacement is vΔ t.

Next, we try to find out that what is the corresponding vertical displacement for the point B?

So, v at A is v, v at B is $\left(v+\frac{\partial v}{\partial x}\Delta x+...\right)\Delta t$

$$B'E' \rightarrow \frac{\partial v}{\partial x}\Delta x\Delta t + h.o.t.$$

$$A'E' \rightarrow \Delta x + h.o.t.$$

$$\tan(\Delta\alpha) = \frac{B'E'}{A'E'} = \frac{\partial v}{\partial x}\Delta t + h.o.t.$$

So, when you are taking the limit as, Δ t $\rightarrow$ 0, this higher order terms are tending to 0. So, effectively in the denominator, you are left with only Δ x. In the numerator, you are left with only this term; other terms are vanishing is small in comparison to this dominant term.

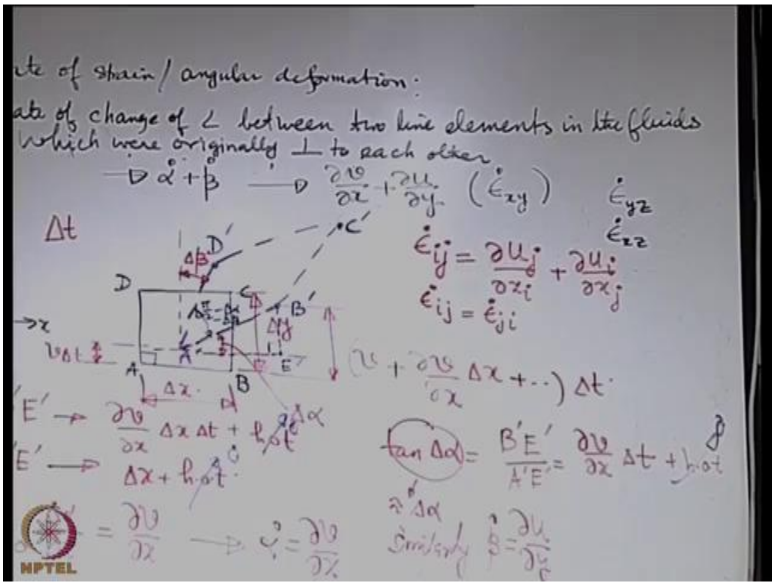
$$\underset{\Delta t\rightarrow 0}{Lt}\frac{\Delta\alpha}{\Delta t} = \frac{\partial v}{\partial x} \rightarrow \alpha = \frac{\partial v}{\partial x}$$

It will be the partial derivative of u with respect to y. So, whatever geometrical consideration you had on this side, if you have it on the other side, it will exactly lead you to the same thing. So, we have got a quantification of the rate of change of these angles and these are like rates of deformation.

Now, this rate of deformation we have to quantify in terms of certain parameters. So, when we quantify the rate of deformation, we have to keep certain thing in mind that we have to give a definition to what is rate of deformation. These are just changes in angles now how we translate that into a more formal definition. So, to have a more formal definition, we may say that we are defining the rate of strain or the rate of deformation in the following way; rate of strain or angular deformation. There is a very important quantification of the distortion in the angle. What is that if you consider two line elements say AB and AD, they were originally at an angle 90^0 with each other. Now, these line elements are no more at an angle 90^0 with each other.

They are now at an angle $90^0 - \Delta\alpha - \Delta\beta$. So, this difference between 90^0 and this one gives an indication of the angular deformation because if there was no angular deformation, this angle would have remained as 90^0. We are trying to identify what is the change in angle between two line elements in the fluid which were originally perpendicular to each other. So, these are representatives of two line elements which were originally perpendicular to each other, but with deformation, they are no more perpendicular to each other. So, what we are interested to find out that what is the rate of change of angle between two line elements in the fluid which were originally perpendicular to each other. So, if we want to quantify that, we now know that we can quantify that in terms of Δα and Δβ and the rate in terms of $\dot{\alpha}, \dot{\beta}$

(Refer Slide Time: 23:17)



So, $\dot{\alpha} + \dot{\beta} \rightarrow \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\left(\dot{\varepsilon}_{xy}\right)$. The indices are the coordinate directions along which your original line elements were oriented.

And now, because of deformation, those line elements are not oriented anymore along those directions.

$$\dot{\varepsilon}_{ij} = \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}$$

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ji}$$

So, if you write it in the form of a matrix where you can see there are two indices each index varies from 1, 2 3; that means, you can write a 3 by 3 matrix using these components. Again out of that, you will have six independent components because of the symmetry.

So, this is written in a, this may be written in a matrix form, that is the components of this not only that what is the other thing that you get from this. See this is also a second order tensor because it may be shown that it maps a vector onto a vector and not only that it requires two indices for it is specification in the Cartesian notation

So, we have seen the rate of angular deformation. But one important thing is we have seen examples earlier that there are cases when the fluid element is not having angular deformation as such like this, but it may be rotating like a rigid body. So, if the fluid element is rotating like a rigid body without any angular deformation, then there also will be a change in some angle that will not be an angular deformation, but the change in angle because of rigid body rotation.

So, that is also an aspect of angular deformation. And in place of deformation, we can just call it rotation because it is not actually a deformation. It is the change in some angle because of a rigid body type of motion. So, we will now see that what is the way in which we may parameterize or we may describe the angular velocity of a fluid element. And that angular velocity definition should be such that whenever there is a rigid body motion, the angular velocity will only be there, but the rate of deformation will be 0.

So, keeping that generality in mind, we have to define the angular velocity. These are all definitions. We know that what we want to qualitatively represent and we are now trying to put into some mathematical definitions to quantify those physical features. That is what is the exercise we are undergoing at the moment. So, the next objective therefore, is finding the angular velocity of the fluid element.