Introduction to Fluid Mechanics
Prof. Suman Chakraborty
Department of Mechanical Engineering
National Institute of Technology, Kharagpur

Lecture - 19
Stability of solid bodies in fluid-Part-II

We were discussing earlier about the Stability of floating bodies and we will continue with that.

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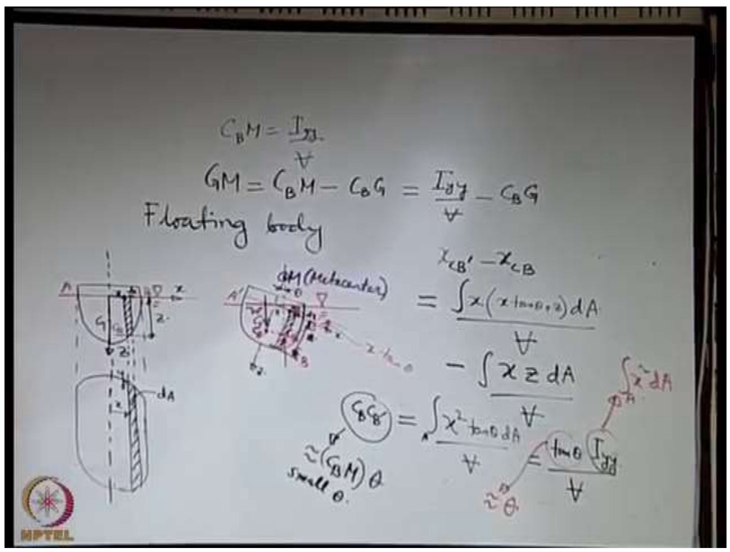

So, if you recall this sketch which we were discussing in the previous class, we identified a point known as metacenter and the motivation behind identifying this point is that when it's a floating body and it gets displaced and its submerged portion within the water changes its configuration. The centre of buoyancy does not remain fixed at its own position. So, referring to a fixed centre of buoyancy, might not work as a stability criterion.

Now we will try to find out as what we have done for the case of totally submerged bodies that how we can find out a stability criterion for floating body bodies. As you can see the presence of or the existence of this metacenter is something which is equivalent to the presence or existence of the centre of buoyancy, for a case when the centre of buoyancy remains at its own position.

So, the metacenter somehow reflects the location of the centre of buoyancy with respect to the axis of symmetry of the body. So, it has some relationship in the centre of buoyancy definitely, but by specifying the metacenter or by specifying the centre of buoyancy just as they are absolute locations, it is not just possible to talk about the stability criterion. So, we have to look into more details. To look into more details we will set up some coordinate axis say we have x axis like this; say we have z axis like this and the y axis is perpendicular to the plane of the figure.

So, the y axis is this dotted line which represents an axis in the other view. Now if we consider that how we calculate the submerged volume, the volume of the solid which is there within the fluid. Let us say that at a distance of x we take a small element of width dx and this element has a depth of z. So, this element if you look into the other view will be like this which is basically located at a distance x from the y axis. Let us say that this shaded area is dA.

So, you can say that z.dA is a representation of the elemental volume of this shaded portion. Now, when it is tilted; even when it is tilted we fix the x and z axis with respect to the body and we keep them as same. So, we still have this as the x axis and maybe this as the z axis, these are fixed relative to the body. So, although the body has tilted, we are using a body fitted set of axis so to say.

Now, let us say that the angle of tilt is theta. If the angle of tilt is θ, then what happens? We can find out that what is now the displaced volume. So, if you consider again say at a distance x from here some strip of width dx. Now, if you consider the displaced volume; the displaced volume is of course, corresponding to this part. This part is like z, same as this one because it is the same axis that has got tilted. So, if this is z this displaced part is also z, but you have the additional displaced part which may be let us mark it with a different colour. So, this is an additional displaced volume.

So, this additional displaced part, we will correspond to a particular length along the z axis. If this angle is θ then the angle of tilt of the x axis with respect to the horizontal is also theta. So, what is this dimension? You know that this is x, so, that is $x \tan\theta$.

So, if we calculate what is that displacement in centroid of the displaced volume; that is what is the difference between the x coordinate of say the original centre of buoyancy; let us give it a name that there was an original centre of buoyancy say C_B and now the new centre of buoyancy is $C_{B'}$. So, the difference in x coordinate of $C_{B'}$ and C_B that is what we are interested to find out. What is that? Centre of buoyancy you can calculate by utilizing the formula for centroid of a volume which is here the displaced volume.

$$x_{CB'} - x_{CB} = \int \frac{x(x\tan\theta + z)dA}{V} - \int \frac{xzdA}{V}$$

It may be easier if we mark the original C_B and the displaced C_B here. So, there has been a displacement from C_B to $C_{B'}$, the location of the centre by centre of buoyancy has got shifted.

$$C_B C_{B'} = \int \frac{x^2 \tan\theta dA}{V} = \frac{\tan\theta I_{yy}}{V}$$

$x^2 dA$ is the second moment of area with respect to the y axis. $I_{yy} = x^2 dA$. Now, referring to this figure, you can say that for a small angle theta $C_BC_{B'}$, this particular small distance is a small arc of maybe a circle.

So, $C_BC_{B'} = C_BM\theta$ for small θ it is just like $s = r\theta$ for a small part of an arc of a circle because this is very small. We are assuming a small displacement; we testability not with a large displacement, but with a small displacement.

Now for the same smallness $\tan\theta = \theta$. Of course, you are expressing θ in radian that is understood implicitly. So, then if you equate the two parts, what you will get? $C_B M = \frac{I_{yy}}{V}$.
Now, look into this figure. In this figure, you see that M is above G; when M is above G, the couple moment created by the forces is trying to restore the body to its original configuration.

However, if G was above M, if it would have been just the opposite case that you can clearly visualize. So, what is important here is the location of M relative to G; just like for a totally submerged body, it was location of B relative to or the centre of buoyancy relative to G.

$$GM = C_B M - C_B G = \frac{I_{yy}}{V} - C_B G$$

C_B is the location of the original centre of buoyancy relative to the body, G is the location of the centre of gravity relative to the body and the distance between those two that does not change with deflection.

So, we are able to express something which should be a function of deflection in terms of certain things which are not functions of deflection. Therefore, this is also not a function of deflection and M remains sort of fixed; but you have to keep in mind that there are certain assumptions that go behind this. What are the important assumptions?

θ is small. So, this analysis is valid only when you are having a small θ ; not only that we have implicitly assumed that the z axis is our axis of symmetry for calculating or for coming up with this expression. So, we are essentially dealing with a simplified case with see symmetric bodies that we have to also keep in mind. Let us look and look into an example, where we illustrate the use of this expression for finding out the stability criterion. So, what is the stability criterion here?

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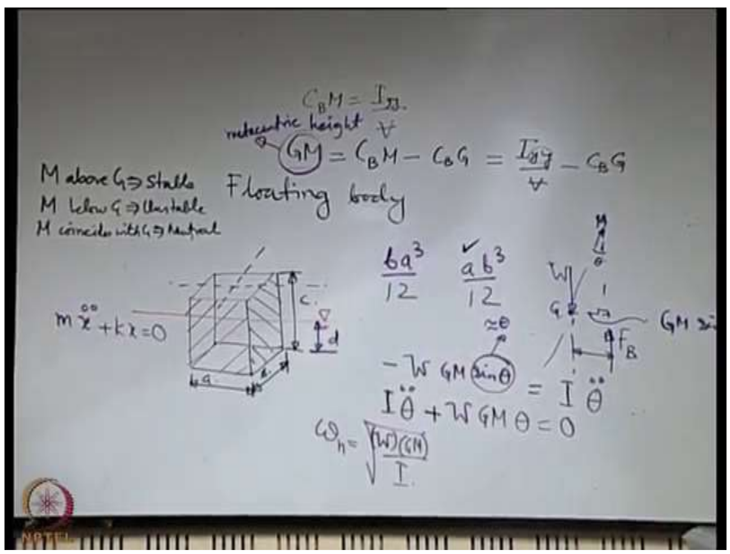

Let us summarize if M is above G, it is stable. If M is below G, it is unstable and obviously, if M coincides with G it is neutral. So, you can clearly understand that M plays the role of equivalent role of centre of buoyancy in this case, but it is not exactly the centre of buoyancy. We consider a very simple example and we will illustrate the use of this through that example. Let us say that you have a body of this shape rectangular parallelepiped shape and you can give dimensions say let us say A, maybe this is B and let us say this is C. So, this is partially immersed in a fluid, so, its like a floating body.

Now let us try to see that how we can apply this stability criterion here and what are the important issues. First important issue is that when you apply this criterion, what should be the y axis with respect to which you are considering the second moment of area. If you look into this problem very critically, you will see that there are certain nontrivial issues. Like you can consider maybe this as one of the frontal surfaces; you might consider this as one of the frontal surfaces and even the third one. So, accordingly it is not very straight forward to say.

So, when we drew the picture of the boat. See it could be this front part, it could be the side part; both are exposed to the water, I mean both are within the water. So, if you could say that what should be the corresponding axis that you need to consider or does the choice of the axis change if you shift your attention from the frontal area to the side area. At the end what you are bothered with? You were bothered with the plan view. So, when you have the plan view which is like the top view of this one; what is the plan view it is the intersection of the body with the free water surface or free fluid surface.

So, that particular view, it boils down to the same when you consider the these surface or this surface, but again when you look from the side and when you look from this one maybe in one case, you are considering this as the y axis; in another case you are considering this as the y axis; both axes are relative to the top view.

So, question is with respect to which axis you should evaluate this? See this is what this is like evaluation of a safety criteria that it will be stable. So, when you want that it should be stable; what should be the guideline? The guideline should be that like you want these to be positive right; that means, this height is greater than this height. Then, like M is above G.

So, this the way in which this is written the right hand side has to be positive. So, more positive means what to say it is like the it is more stable. So, to say, so, if we take the least of this one; least possible value of this one and still find that the right hand side is positive, then for all other cases it should be positive. So, it is like a safety design. So, you look into the most adverse condition make sure that it is positive even for that.

So, for better conditions it would always be better. So, out of these two axes; for one axis, $I = \frac{ba^3}{12}$; another will be $I = \frac{ab^3}{12}$. So, you take the smaller one; that means, if b is smaller than a as an example, you take this one. So, then when you substitute that in this expression and still you calculate this as positive, you are assured that it is safe because with respect to other axis when you calculate the metacentric height, it will definitely be greater.

This is known as metacentric height and really when you calculate different metacentric heights based on different views; these represent different types of angular motions like rolling, pitching, these are different technical names based on with respect to what type of axis it is tilting. So, but just for design simple design you can consider the smaller one and you divide it by the volume not the total volume, but the volume of the submerged part.

So, that you can easily calculate based on what part is submerged. Let us say that d is the submerged depth. It is easy to calculate this because you can use the equilibrium that for equilibrium, the buoyancy force must be equal to the weight. So, from that if you know the density of the fluid and density of the solid body, you can come up with what should be the equilibrium d. Just by very simple equating of the two forces. So, you can calculate what is the this; what is the volume which is immersed in the fluid, centre of buoyancy location you can get from the centroid of the displaced volume; centre of gravity also has a fixed position with respect to the body.

So, you can clearly substitute these dimensions and find out what is the metacentric height. So, this effectively requires only the calculation of the immersed volume, the calculation of the second moment of area with respect to these axis and specification of the locations of the centre of buoyancy and centre of gravity with respect to the axis fixed on the body.

So, that is a sufficient information to calculate the metacentric height. So, what we can clearly see is that if M is above G that makes it stable; that means, lower the location of the centre of gravity it is having a greater chance that it will be stable because then it is a greater chance that M is above G lower the location of G. So, location of G if it goes higher and higher, it might make previously stable system or converted into an unstable one. Let us look into an animated example to consider this case.

So, in this animated example what we will see? We will see that how the stability may be disturbed because of the shifting of the distribution of weight of the body. So, let just look into it carefully. So, there is a body which is given a slight displacement and you will see that is for small displacement it oscillates like a pendulum. We will we may easily derive what is the time period of that. Now, you see that the distribution of the weight is being altered. So, the centre of gravity is being shifted higher and higher by putting the load more and more towards the top and you see that it topples.

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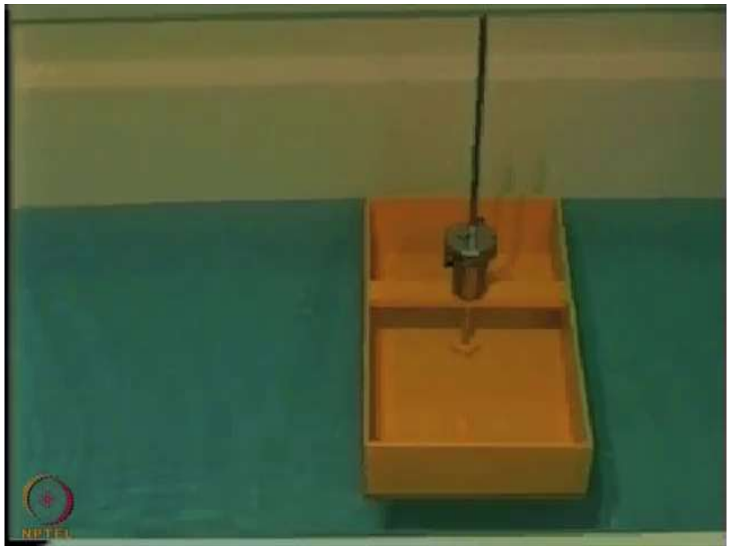

So; obviously, this is a clear illustration of this concept that we have learnt in this example that if you have metacentre above the centre of gravity, it makes it more stable and otherwise it is not. So, now for small oscillations you could see that it or for small deflection, you could see that it oscillates like a pendulum. So, when it oscillates like a pendulum it has a time period and that may be calculated by calculating the moment of the resultant force with respect to the axis. So, if you have a tilted axis like this and if you have force F_B which is the buoyancy force, this force has a moment with respect to the axis of the body.

So, what is the moment of this force with respect to the axis of the body? You can just find out the perpendicular distance of this force from the axis of the body. So, it will be just what will be the perpendicular distance of this buoyancy force line of action of the buoyancy force? I mean you can see that it eventually passes through the axis of the body right. So, it is not this force individually which is important, it is the couple moment that you are having the so called the restoring couple. So, you have also the W and it is basically the couple moment of these forces that you need to consider.

So, the perpendicular distance between these two; in terms of the metacentric height could you express this? So, if you have this as the metacentre M. So, we are interested about this distance. So, it is possible to express it in terms of the metacentric height. So, this angle being theta, perpendicular distance is $\text{GM}\sin\theta$.

So, the moment of this force is WGM $\sin\theta$ and the resultant moment of all the forces is nothing but if it is like rotation of a rigid body with respect to a fixed axis, it's a restoring moment. So, you should have a minus sign associated, that means, whatever is the chosen positive direction of θ ; $\dot{\theta},\ddot{\theta}$ this has a direction opposite to that to bring it to its original configuration.

$$I\ddot{\theta} + WGM\theta = 0$$

I is a real mass moment of inertia with respect to the axis of the body relative to which it is tilting. So, this plus; for small θ again this is approximately equal to θ = 0. So, it is just like equation of a spring mass system mx double dot plus kx equal to 0. So, what is the natural frequency of oscillation of the system?

$w_n = \sqrt{\frac{W(GM)}{I}}$ is the natural frequency of oscillation of the system. It is an angular oscillation not a linear one. So, you can clearly see that greater the metacentric height, greater will be the frequency of oscillation. So, if it is a ship it will be more uncomfortable to the passenger. So, these are two conflicting designs. See greater the metacentric height, you expect it to be a safe in terms of stability. But it will have more oscillation within that stability regime; that means, for a passenger it may be quite uncomfortable.

At the same time if it is for a used for a particular say critical purpose like warfare and so on. So, their stability of the ship is more important than the comfort of the crew and there obviously, it is the stability that should be the driving factor for design. So, when a ship is designed you have two conflicting things; one is the comfort, another is a stability and comfort factor camp comes from this high frequency of oscillation and the stability comes from the metacentric height or location of the metacenter relative to the centre of gravity.