Vibrations of Structures
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Lecture - 08
Properties of the Eigenvalue Problem

In the last two lectures, we started discussions on the model analysis of continuous systems. Now the performance of model analysis was found to be essentially solving an Eigenvalue problem. Now today, we are going to look at some properties of this Eigenvalue problem that comes while we perform model analysis of continuous systems. So let us start by revisiting the model analysis problem.

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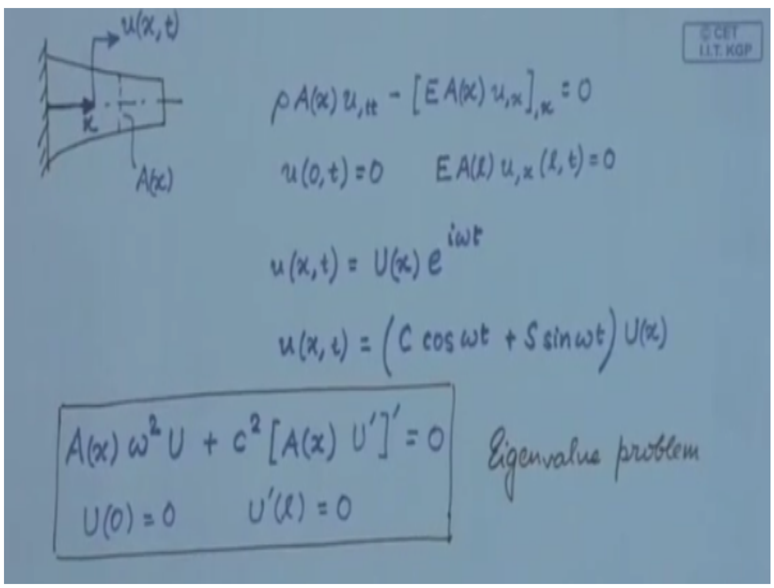

So in our last lecture, we have discussed the problem of bar with varying cross section. The equation of motion of the system was given by this and the relevant boundary conditions for this problem were given by u at 0 is equal to 0 for all time and on the right boundary, there were natural boundary conditions and in order to do the model analysis, we were searching for solutions of the special form or structure.

So the field variable is expressed as a product of an amplitude function, which is a function of x and harmonically varying time function. Now, we discuss the properties of this solution and we found that the solution is actually separable in space and time, so when we write the actual solution in its real form, it appears in this structure. So it was separable in space and time. The other observation is all points, therefore vibrate at the same frequency omega.

The same circular frequency omega. Thirdly, all points of the system pass through the equilibrium point at the same time instant. The time instant when this temporal function is 0, the whole solution is 0, which means is in its equilibrium state. So all points will pass through the equilibrium point at the same time. Then, we observe that phase difference between any two points on the bar is either 0 or pi.

And finally we observe the existence of nodes that means points at which U, the amplitude function U of x is 0. So the properties of the model solution are known to us. So once we substitute a solution of this structure into the equation of motion, we obtained a differential equation in terms of this amplitude function and the corresponding boundary conditions. This forms the Eigenvalue problem for the system.

So the differential equation along with the boundary conditions. Now, we will represent this in a slightly abstract form in this manner.

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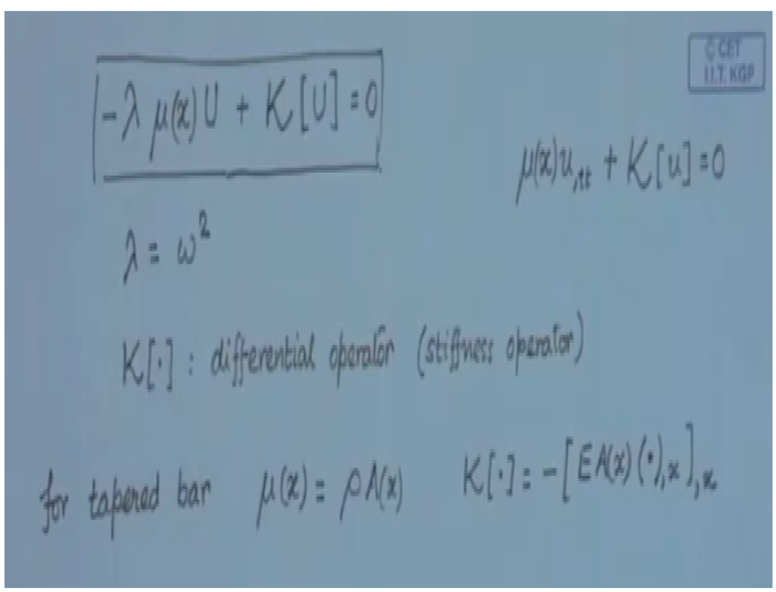

Our equation of motion can be written like this, so if you write a general equation of motion of string or a bar in this form, then the differential equation of the Eigenvalue problem may be represented in this manner, where lambda is omega square, so this actually is plus, so this is the differential equation and the corresponding differential equation for the Eigenvalue problem is given in this form, where lambda is omega square and this K is a differential operator.

So for example, in the case of the tapered bar, mu of x is rho times the area and the differential operator K, which is also known as the stiffness operator is the spatial derivative of this quantity Ea the derivative of the argument. So this is a structure of the differential equation of our Eigenvalue problem. Now here, as mentioned here that this is known as the stiffness operator because this term comes from potential energy in the Lagrangian formulation.

While this term mu of x is the kinetic energy operator because it comes from the kinetic in the Lagrangian formulation.

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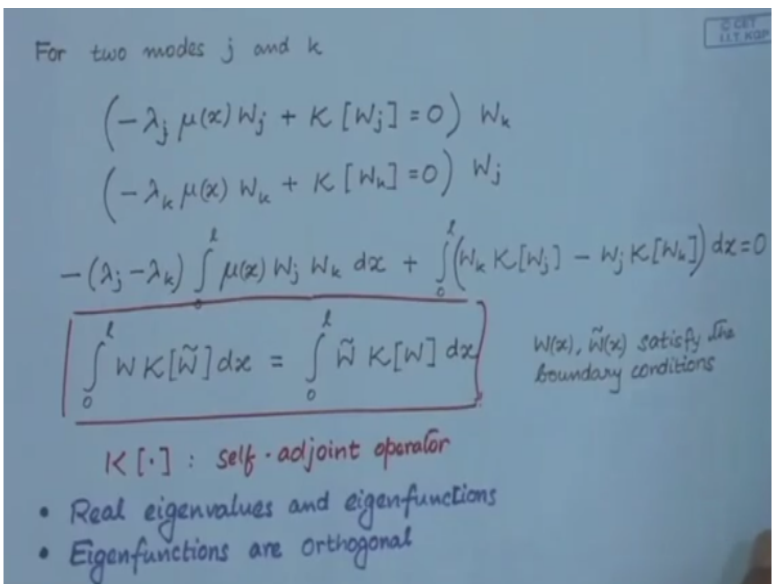

So suppose for two modes j and k, we can write this differential equation. For the j-th mode, let us say, we can write the differential equation of the Eigenvalue problem like this, while for the k-th mode, the differential equation becomes this. Now the objective of this analysis is to determine certain properties of the Eigenvalue problem. So let me multiply the first equation with Wk and the second equation with Wj and subtract one from the other.

Then after some rearrangement, I am also integrating over the domain of the system. So I multiplied the first equation with Wk and the second equation with Wj, subtracted one from the other and integrated over the domain of the problem and this is what I obtained upon rearrangement. Now, suppose that this integral vanishes, so let us consider the situation when this property holds.

Where this W and W till J are functions that satisfy the boundary conditions of the problem. If this property holds, then this operator K is known self adjoint. So if this property is satisfied by the stiffness operator, then it is known as the self adjoint operator. Now, this self adjointness of an operator is connected to symmetry, so as you know that the stiffness operator has the corresponding matrix.

For example, in vibrations of discrete systems, you have come across stiffness matrix. So self adjointness of the stiffness operator is nothing but the symmetry of the stiffness matrix, the corresponding stiffness matrix. So what are the consequences of the symmetry? As we know, that the matrices are symmetric, the Eigenvalues are real and the Eigen functions are also real, and the Eigen vectors are orthogonal.

So in a similar manner, we have this properties, which can be shown very easily that the Eigenvalues and Eigen functions are real whenever the stiffness operator is self adjoint. Secondly, the Eigen functions are orthogonal with respect to an inner product, that we will find out in the course of this lecture.

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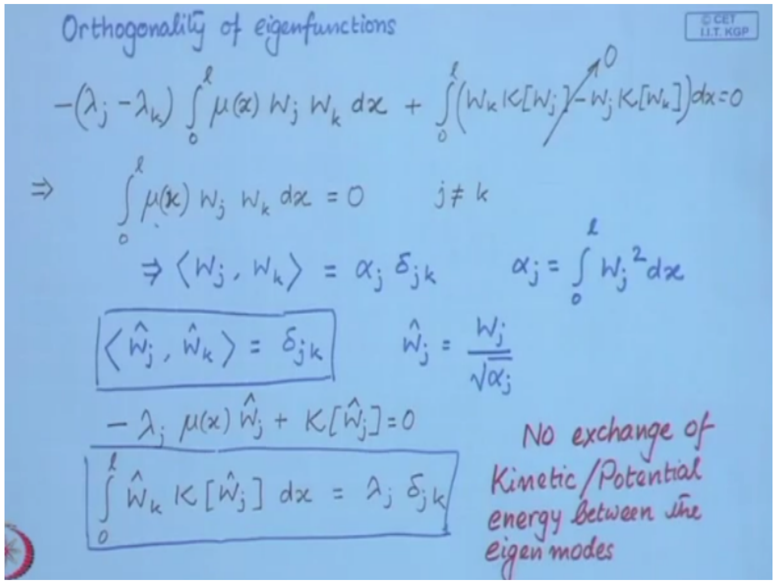

So we will be discussing this orthogonality property. So recall that we had this equation. So if the operator k is self adjoint, then this term vanishes, so this implies, this integral must vanish. Whenever j is not equal to k, which means if I take two distinct amplitude function modes, Eigen functions, Wj and Wk, then this satisfy this property that this integral must vanish and this we define as the inner product of these two Eigen functions.

And in a compact form, we will write this that the inner product of two Eigen functions can be written in this form where alpha j is given by this integral. Now, one may normalize this property by appropriately scaling this Eigen functions because as we know that any scaled form of this Eigen function is also an Eigen function. So we can scale appropriately to have orthonormality of the Eigen functions with the respect to this inner product that we have defined.

So here this Wj hat is Wj over square root of alpha j. So here we have orthogonality with respect to the inertia operator, so if you consider that this mu of x represents inertia operator, then this orthogonality is with respect to the inertia operator and correspondingly we can write. So for the j-th mode, this is the differential equation. So if I multiply this equation, so this can be written also for the hat, the normalized Eigen function and if I multiply this with WK hat and integrate.

So this shows that the Eigen functions are orthogonal also with respect to the stiffness operator k. Now what is the physical implication of this orthogonality with respect to inertia and stiffness operators. So the physical implication is that there is no exchange of kinetic or potential energy between the Eigen modes. And this orthogonality property is also very useful as we will see in due course for solving initial value problems or other problems related to continuous systems.

And this orthogonality already we have come across when we discussed about model analysis of strings.

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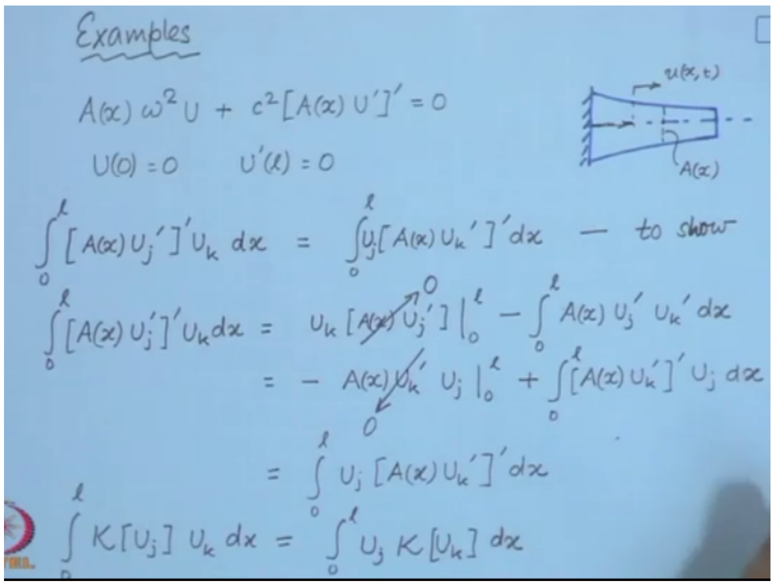

Then we discuss some examples and we will determine the orthogonality relations for these examples. So we once again go back to this bar with varying cross section and follow the steps that we have done in detail. So our Eigen value problem, now let us check that this stiffness operator that we have here is really self adjoint. So what we have to show? So this we have to show. So we have to show that these two are equal where Uk and Uj are two Eigen functions of this Eigenvalue problem.

So we start integrating by parts, let us say from the left hand side. So we take this as the first function and this as the second function, this is what we obtain and here I will integrate by parts, this term once again and here I will use the boundary conditions. So the boundary terms here that I have so this term will be evaluated at L and at 0. Now at L, U prime at L must be 0, so this term must vanish at L and U 0 is 0, so Uk at 0 must be 0.

Because these are Eigen functions and they satisfy the boundary conditions of the Eigenvalue problem. So this term is actually 0. So we are left with only this term and this I will integrate by parts once again as we had here, these boundary terms must also vanish, so we are left with, which is nothing but the right hand side of this equation. So we have shown that this operator acting on. So we have shown the self adjointness of the stiffness operator of a tapered bar.

So we can write this orthogonality in terms of the inner product as we were defining for the tapered bar.

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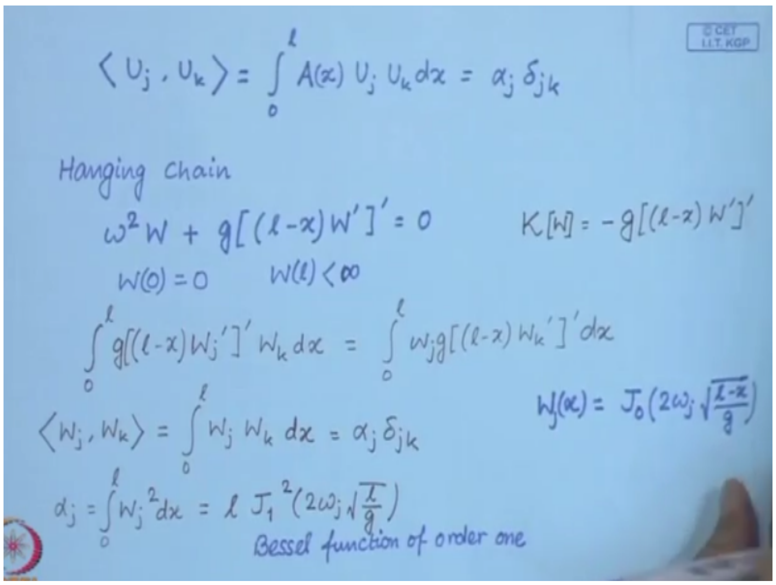

Next we are going to look at the hanging string or hanging chain. So for the hanging chain, the Eigenvalue problem read this. So in this case, the stiffness operator is given by this term and in a similar manner you can check that this self adjointness property holds for the stiffness operator of the hanging chain and once you use this property, you can derive the inner product of the Eigen functions of the hanging chain with respect to which the Eigen functions are orthogonal.

So let me just write down this Eigen function that we have already derived in a previous lecture. So this is the structure of the Eigen function of a hanging chain, and this satisfy the orthogonality relation in this form where so this alpha j is the square of the Eigen, say the j-th Eigen function, and integrated over zero to L. And this turns out to be, where this j1 is the Bessel function of order one.

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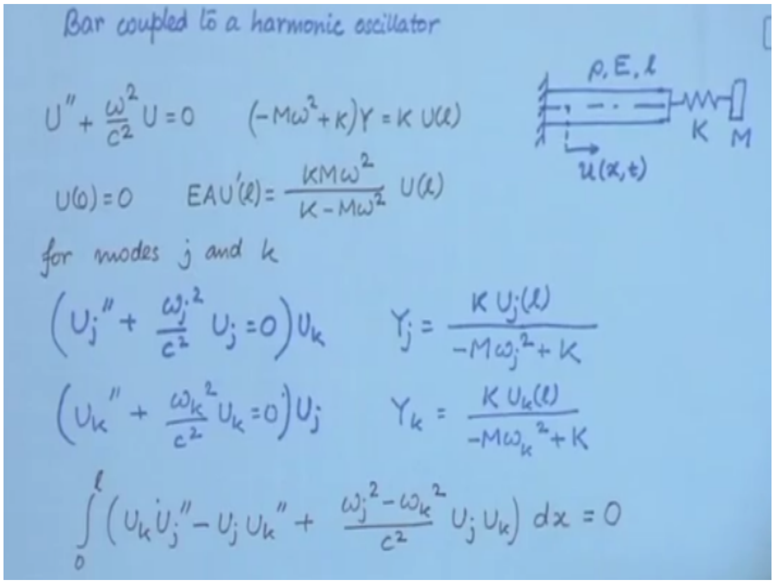

Now let us consider the example of a uniform bar, which is coupled to a harmonic oscillator, which we have discussed in our previous lecture. So the Eigen value problem for this system was written as, as obtained in our previous lecture. So once again for mode j and mode k we can, so these are the two differential equations, and these are the boundary conditions for the bar. So for the modes j and k, we can write.

Now we will once again multiply this with, the first equation for the bar with Uk and the second with Uj, subtract and integrate over the domain of the bar, and upon rearrangement you can very easily obtain. So there are few standard steps so to obtain from here to here, that can very easily perform and come to this condition. Now if you integrate by parts, let us say this, first term, so integrate by parts this first term two times, and use the boundary conditions for the boundary terms that you generate, then you can check that.

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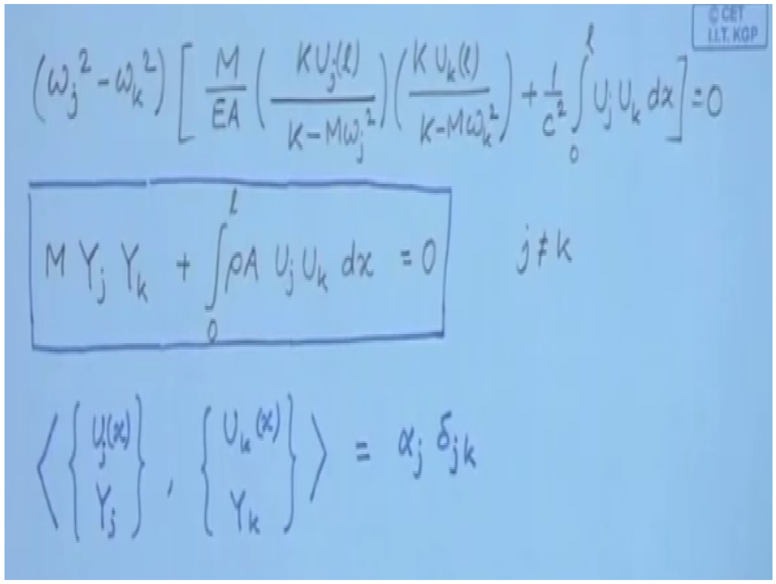

That expression reduces to. So we are integrating this term by parts two times, so at the end of that integration by parts, this term will be same as this. So that will cancel of, but we will generate two boundary terms with single prime and that we have to, to replace that we have to use this boundary condition. So this boundary condition, once you use that you will ultimately come to this expression.

Now when j is not equal to k, and considering that omega j is not equal to omega k, there are no repeated Eigen frequencies, then this bracketed quantity must vanish. And this if you check this can be written as, here I have replaced these quantities by Yj and Yk, which we have obtained these expressions before. So this is our inner product, remember that in the case of discrete of a hybrid system in which we have a continuous and discrete system.

We had this Eigen function vector, which we have discussed in the previous lecture. So here we would say, will write the inner product in this form, where the inner product is now defined in this form. So you see this is not a trivial or a simple inner product that we obtain for the other systems. So this procedure you have to follow in order to determine this structure of this inner product, how this inner product is calculated based on the Eigen functions.

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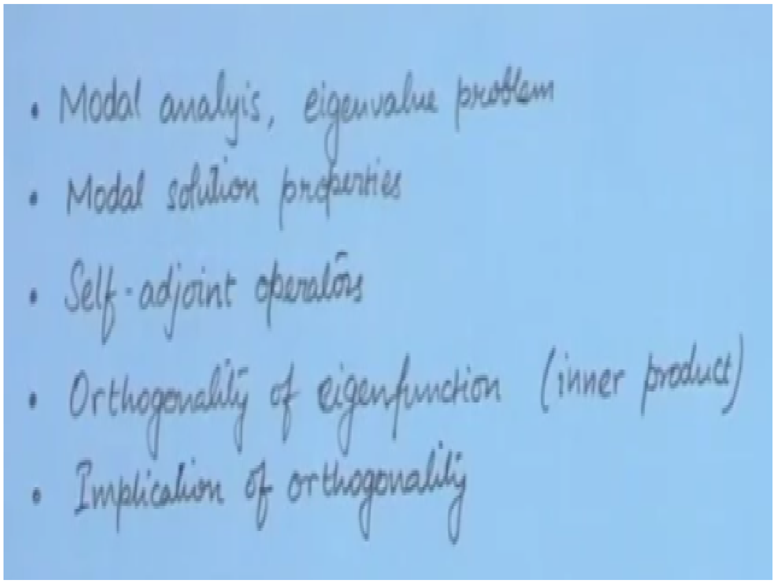

So let us summarize what we have studied today. So we had, so we have revisited this model analysis problem and the Eigenvalue problem. Then we looked at the properties of the model solution. Then we discussed about self-adjoint operators, and the consequence of the stiffness operators being self-adjoint, these are real Eigenvalues and real Eigen functions. Then, we have discussed about the Orthogonality property of Eigen functions.

And we have determined the inner product. The outline steps to determine the inner product with respect to which this orthogonality property holds. And we have looked at the implications of the orthogonality property of Eigen functions. So if the Eigen functions are orthogonal that implies, that there is no exchange of energy, kinetic or potential between the Eigen modes or Eigen functions. So with that we conclude this lecture.