Vibrations of Structures
Prof. Anirvan DasGupta
Department of Mechanical Engineering
Indian Institute of Technology – Kharagpur

Lecture - 06
Modal Analysis - I

So in the previous lectures, we had been looking at ways of deriving the equation of motion of continuous systems. So we derived equation of motion, the boundary conditions and we also looked at initial conditions, which close the system that means we will then have unique solutions of our system. Now so this part we derive those equations by two ways. One was the Newtonian approach the other was the variational approach.

Now, in the next few lectures what we are going to do this analyze these equations or solve these equations. So the first thing that we note is that we have the Equation of motion.

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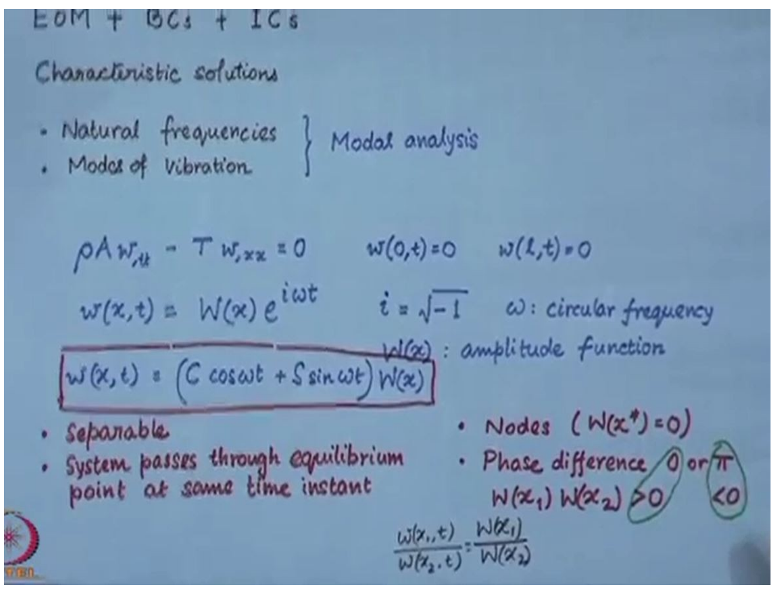

Then we have the boundary conditions and we have the initial conditions. So this is what we have in hand. Now what can we do with these, so in the general situation what we can do is given these 3 things we can solve the system, we can find out the solution, we can find out the behavior of the system. But then with change in initial conditions, for example the solution will change. Now, if the solution changes then we are not able to get a feel of the systems.

So the question that we put is whether we can find out some characteristic solution of the system. So, what we are interested in is finding out certain characteristic solutions of the system. By characteristic solutions, I mean that things that will not change, the solutions will not change, those things will not change with change in initial conditions or certain properties of the system that will not change with change in initial conditions. So, can we find such characteristic solutions?

So the answer to this question is an affirmative and we know this from our study of vibrations of discrete systems, so in there we solve for what are known as the natural frequencies of the system. Natural frequencies, so how fast the system is going to vibrate, so that is, I mean the idea of that is given by the natural frequencies of the system. By natural we mean that the system is in free motion, it is not being forced or disturbed from outside as such.

The other is the mode of vibration. So one thing is the natural frequencies and the corresponding modes of vibration of the system. So these two constitute the characteristic solutions of the system and finding them out is known as Modal analysis. So when we do modal analysis, what we are doing is actually searching for solutions of very special form.

So if you recall the equation of motion of a string. So suppose it is a fixed string of length l, so we have the equation of motion and the boundary conditions. Now our field variable is this w (x, t). Now this is a general function of the special co-ordinate x and the temporal co-ordinate t. Now, when we do modal analysis we are searching for solutions which are of very special form, which look like this, where this i is the square root of - 1.

Omega is known as the circular frequency and W is the amplitude function. So we are searching for solutions of this very special structure in which, in a certain way, the special function and the temporal function are somewhat separated. Now, they look separated but remember that we have introduced, so now this solution is in the complex form. So, this W may in general be a complex function. In that case, the solution is not strictly separated or separated in space and time.

But, we will encounter such solutions later in this course, so for the time being we introduce this complex solution structure and we say that the actual solution because our equation and boundary conditions are all real. So the actual solution is obtained by taking either the real part. So the actual solution will be obtained by taking either the real part of this quantity or the imaginary part of this quantity or a linear combination of the real and imaginary parts.

So, if you consider this possibility and we will initially most of the systems that we are going to study will have this capital W, the amplitude function as real. In that case, we can rewrite this solution in the form. So if you consider that W is a real function and as I mentioned that you can take the linear combination of the real and imaginary parts of this complex solution form as the general solution, then your solution looks like this.

Now from this structure, we can deduce immediately a few properties of this solution. So here you can immediately see that the solution is strictly separable in space and time. So, space part and the time part they are separated, then this, the temporal part of the solution can become 0 at a certain time instant. In that event, all points for all x the solution is 0, which means that the motion of the string or the continuous system passes through the equilibrium point, all points pass through the equilibrium point at the same time.

So, the system passes through the equilibrium point, so all points of the system pass through the equilibrium point at the same time instant. Then there are points, there can be points at which W (x) is 0. Such points are called nodes, so there is a possibility of existence of nodes, where W (x) is 0. The fourth property of this solution structure is that the phase difference between any two points of the system is either 0 or Phi.

So, if you take any two points x1 and x2 and if you observe the product of this amplitude function if this is positive then the phase difference is 0, if it is negative the phase difference is Phi. There is another interesting property of the solution that is the ratio of amplitudes at x1 and x2 that is independent of time that is to say, this ratio, so essentially this is independent of time as we can see. So, we will start with this structure of the solution and see or tried to find out the characteristic motion of the system.

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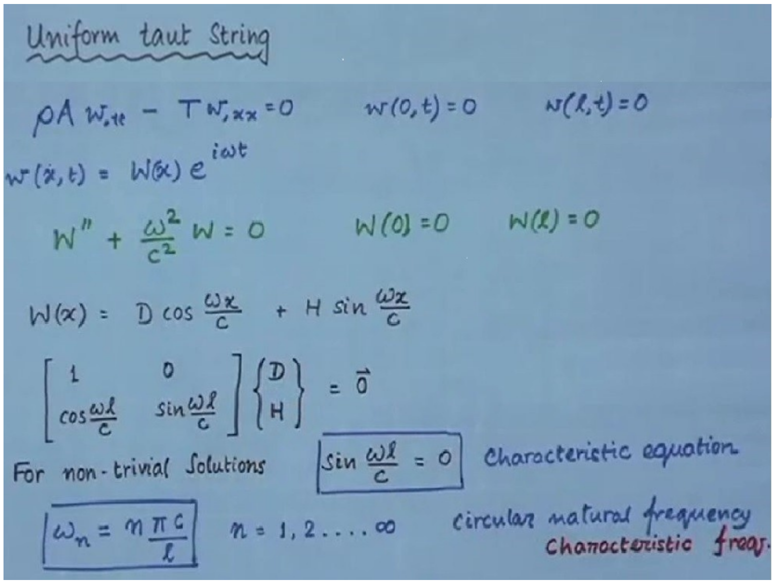

So to do this, we will start with the example of the Uniform taut string. So, the equation of motion and the boundary conditions look like this. Now, we are going to substitute this solution structure and if you do that and with a little rearrangement, you can easily see the equation reduces to this, because Exponential i Omega t will never be 0 for any time. So the remaining part the coefficient of Exponential i Omega t, which is this must be 0.

And along with this, we must have the conditions act x equal to 0 and x equal to l. Now this differential equation is very familiar and the solution can be written directly, now this is the general form of solution of this differential equation along with this, now we have these boundary conditions at x equal to 0 and x equal to l. So, we will substitute this solution form in the boundary conditions and we can rewrite the boundary conditions in this form.

Now if you want to have non - trivial solutions of D and H, which is what we desire, then the determinant of this matrix must vanish. So that implies, so determinant of this is nothing but Sin Omega l over c and that must be 0. This equation is known as the characteristic equation because it yields, this condition yields certain characteristic solutions or properties of the system and they are the, these circular frequencies which are also known as the natural frequencies, circular natural frequencies of the system.

So from this condition we can immediately write that Omega, now there are discrete solutions of these characteristic equations but there are infinitely many of them, so there are countable infinite many points at which this will be satisfied for special values of Omega, so we will put an index n and write this as, where n, this index n goes from 1 to infinity. So there are countable infinitely many natural frequencies of a string, of a fixed string which are given by these values.

They are called the circular natural frequency, also sometimes known as the characteristic frequencies of the system.

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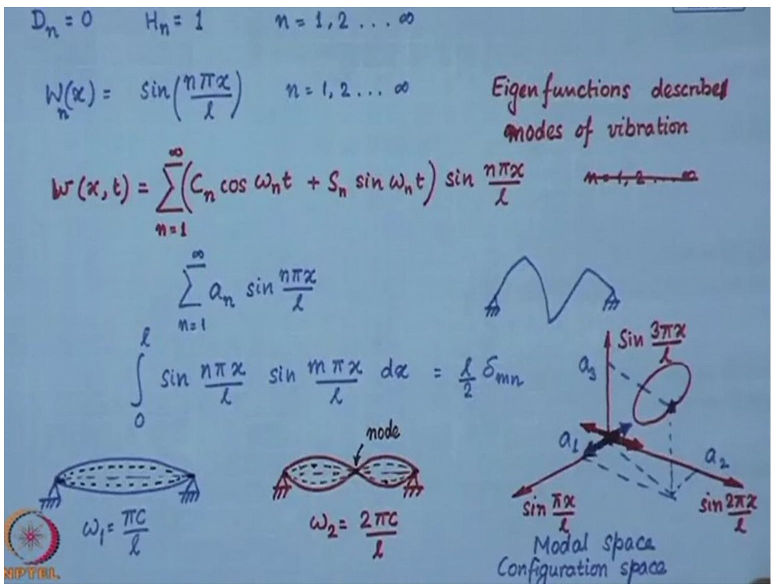

Now once we find these special values of Omega in this matrix for which we have non - trivial solutions, then we can really find those non-trivial solutions by finding the corresponding D and H values, which look like, so if you substitute these values of Omega n into the boundary conditions, now it can be easily seen that the solutions are can be written in this form.

So anything proportional to this, will be a solution, where again n goes from, so corresponding to each circular natural frequency, you have this vector, which gives you the actual solution, which is obtained as, these are called the modes of vibration or they are actually the Eigen functions and again they are, they correspond to the natural frequencies of the system. So for every natural frequency there is an Eigen function, which defines or describes the mode of vibration.

So the total solution now looks like, so everywhere we have this index and so here also I must have this index n. So, this is the solution. For each n, I can have a solution and since our system is linear, a superposition or a Summation of this solution is also a solution. So, I can finally sum all these solutions and construct the most general solution of a vibrating string with fixed support. Now, this structure looks very familiar, if you think of the Fourier Sine series.

So this represents a function, which is periodic and so the periodicity is l, so l is the, where l here refers to the length of the string. So, this structure of the solution finally what we have is like a Fourier Sine series. So, which means that any shape of the string, since we know from Fourier series theory that any shape between two supports. If I have this shape, by continuation I can construct the Fourier or represent this as a Fourier series, Fourier Sine series like this.

So, any shape of the string is now at any time instant if you think about the shape of the string at any time instant then, this is a constant, which is represented here as 'a n' and I am just expanded in terms of this function Sin n Phi x over l. Now we also know from Fourier series theory that these form the basis functions which are Orthogonal. In the sense that, so if you integrate perform this integration.

So if you take any two Eigen functions with different index n and m, multiply them and integrate from 0 to l, the length of the string then it is l over 2 the Kronecker Delta m n. So which means if m is not equal to n this is 0, if m equal to n this is l / 2. So, these are orthogonal, so Sin Phi x over l is orthogonal to Sin 2 Phi x over l etc. So if I make graphical representation of this solution, I can think about it with the slide stretch of imagination.

Suppose I draw axis which are orthogonal to represent this orthogonality conditions and I call it Sin Phi x over l, call this axis as Sin 2 Phi x over l etc. I cannot draw all these infinitely many axis, but I appeal to your imagination that you can consider this to be an infinite dimensional space, where axis are orthogonal to represent the orthogonality of these functions.

And then a solution of this form or a representation in this form for a particular shape of the string is actually a point in this infinite dimensional space, this is a1, this is a2, this is a3 and like this you can have all these coefficients a1, a2, a3, a4 up to ‘a’ infinity. So in this space the configuration of the string at a time instant, at a particular time instant is nothing but a point in this, in final dimensional space.

So as the string moves, so it is nothing but the motion of this point. So if the string executes a periodic motion in this space, then it would be a perfectly closed curve. Now in this space then what is the simplest motion that is possible, so the simplest motion would be for example the string moving only along this axis, only along this axis, this is a modal solution, the motion of the string in the first mood of vibration.

So this motion is nothing but only Sin Phi x over l, is represented by Sin Phi x over l. All the other coefficients are 0 a2, a3 etc. are all 0 except a1. So this is the first mode of vibration of the string which looks like. Similarly, if you consider motion only along the second axis, Sin 2 Phi x over l, then the motion of the string looks like this.

So, essentially the string is vibrating between these two extreme configurations and the corresponding frequencies, corresponding circular natural frequencies are given by these values Omega 1 as Phi c over l and Omega 2 as 2 Phi c over l. This infinite dimensional space is sometimes called the modal space or sometimes also can be called the configuration space of the string.

So in this space, as the configuration of a string at any time instant is represented by a point and it is nothing but when it moves from one configuration to the other is nothing but the motion of this point, in this space and we have seen that the elemental motion or the motion along these axis they represent the Modal solution. Now, I am going to demonstrate with a small experiment these solutions.

Before that let me just state, point out here that as you can see in this solution there is no node within the domain of the string, whereas here there is one node, where the string remains fixed. This point is not moving out from the equilibrium position, whereas other points are vibrating. Here all points are vibrating in the same phase, show the string essentially moves through intermediate configurations like this and then again comes back and passes through the equilibrium points.

So at the equilibrium point the string again become straight and then starts moving down and reaches this extreme again, repeats, starts fresh. Here the intermediate configurations look like this. So you see a point here and a point here, they are simply out of phase, so when this is moving up, this is moving down and vice versa. So, but two points in this loop they are moving in phase.

So the phase is either 0, phase difference between two points of the string is either 0 or it is Phi. So, now I will demonstrate using a small experiment, these modes.

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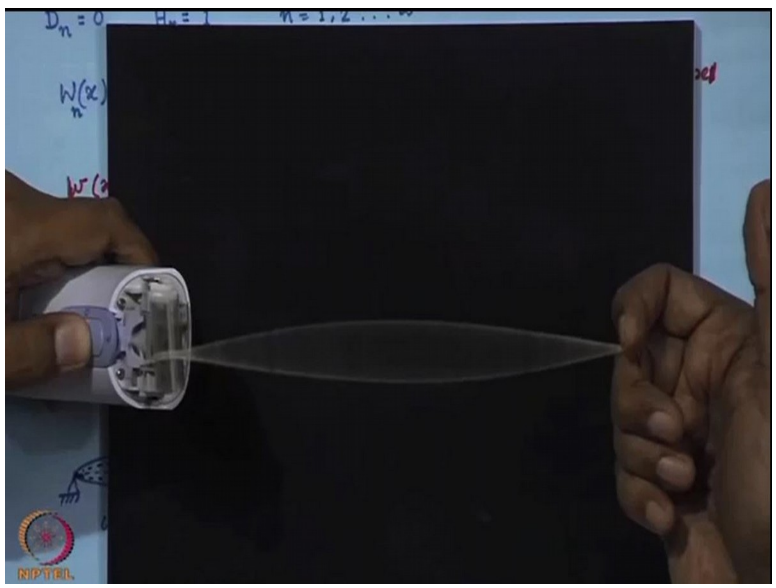

So, here I have a string, which I have made taut by pulling and here there is an exciter, which is nothing but a simple electric shaver. Now as I pluck the string you see the motion ceases after sometime. This is expected since all real systems they have internal damping but in our model that we have considered till now we cannot have damping. So to cancel the effect of damping, I must have an exciter, which must pump the same amount of energy that is being dissipated due to the internal damping in the string.

Now let us see, so this is the first mode of vibration of the string. This is how it will look like when it is vibrating in the first mode. Now let me try the second mode.

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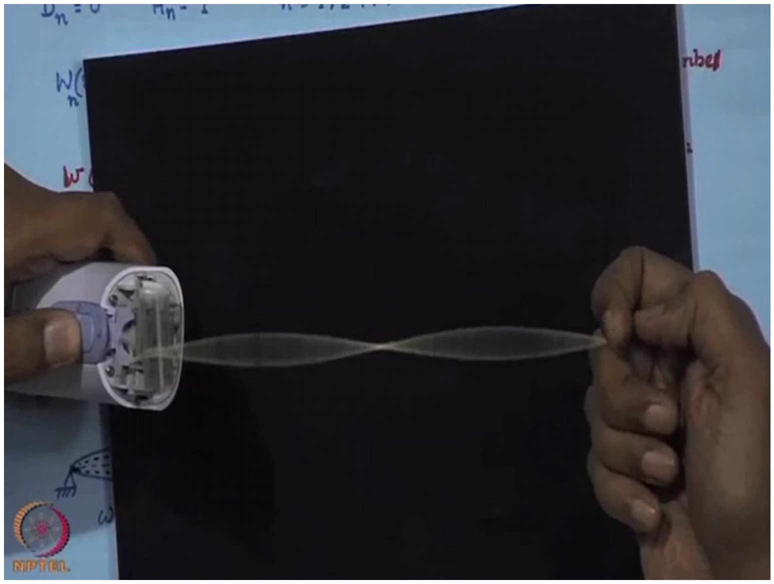

So this is the second mode of vibration of the string and I may also try the third mode which may not be visible very clearly, because the amplitudes are so small. But the second mode is quite clear and the first mode as well. So, this is what we have observed we have seen the first two modes quite clearly, third mode because the amplitudes are so small has been less visible and so on for the higher modes. But then, whenever, I pluck a string I actually excite a number of these modes.

So the motion of the string if I pluck a string or at an arbitrary point, then I am actually exciting a number of these modes, so the solution is more complicated. But I can by special means excite only these individual modes and we have seen that in a demonstration.

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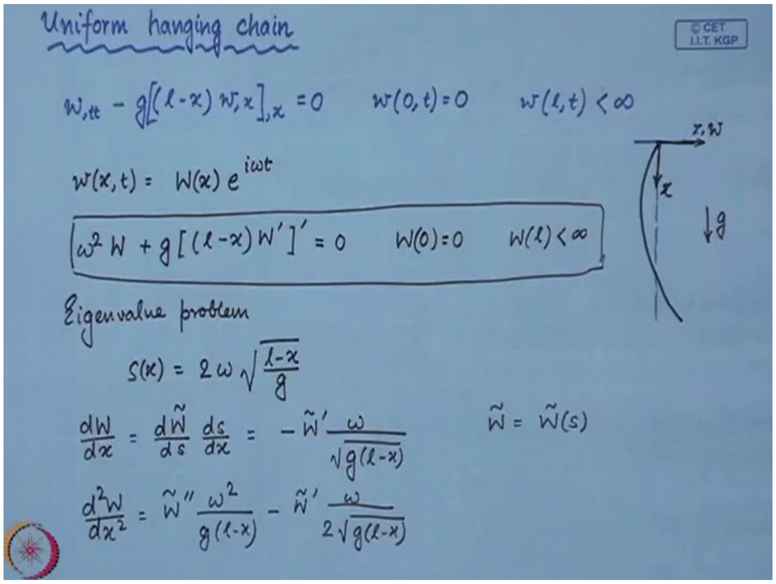

The next example that we are going to look at is that of a Uniform hanging chain. We have derived the equation of motion of uniform hanging chain previously and it looks like. So this is the equation of motion, this is the boundary condition at the fixed end. Now, once again we are going to attempt a solution with this structure and we obtain with the boundary conditions. Now in the case of a string of a uniform taut string.

We obtained the equation previously after substituting the solution structure; we obtained a differential equation and boundary conditions. Here again we have obtained a differential equation and boundary conditions. This problem is known as the Eigen value problem. So in the case of a string, this was our Eigen value problem. Now this structure of equation looks tantalizingly familiar.

Because we have studied equations of this structure when we discussed Sturm-Liouville problems in mathematics. Now to convert this because this is a particular, I mean this structure is very special, we can convert this into a more familiar form by using transformation of the independent variable x, so let us consider a variable s which is a function of x. If you use this, new variable ‘s’, then you can replace which you can check very easily, where W Tilde is a function of this new variable ‘s’ and similarly you can work out the higher derivative.

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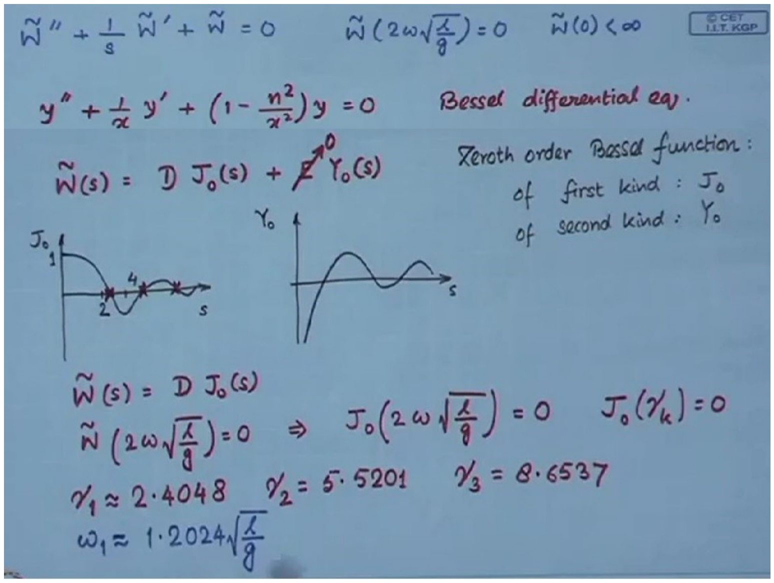

Now having done this, if you substitute back in the equation and convert this equation in terms of 's' then what you obtain is with the boundary conditions, this can be checked very easily. Now this is a very familiar equation, which is the Bessel differential equation. So this equation is the Bessel differential equation and in this equation, if you put n as 0 then it reduces to this differential equation.

And the solution of the Bessel differential equation with n equal to 0, which is the solution for our W Tilde, where J 0 and Y 0, they are known as 0th order Bessel function of the first and second kinds respectively. Now, if you look at these Bessel functions of 0th order, then you will find that this second kind has a logarithmic singularity at 's' equal to 0. So, if I make an approximate plot of J 0, so at 's' equal to 0, y is - infinity.

So, this will violate this condition, this boundary condition, which is the boundary condition at the free end. So finiteness of the solution at the free end forces us to select E as 0, so E must be 0. So therefore which can be written, so I can write this down in terms of my original variable x, but before I do that, let me look at the boundary conditions, so we know that, so this implies this must be zero.

Now you can see from this figure there are discrete points but infinitely many at which this function J 0 is 0. So these countably infinitely many solutions of this, will give us the natural frequency of the system and if I represent this by something like Gamma k, then this Gamma 1 is approximately 2.4048, Gamma 2 is 5.5201, Gamma 3 is 8.6537 and like this you can find out the values and from here, from these values of Gamma k therefore you can find out say for example we can easily write Omega 1 as 1.2024 under root l over g.

So, you can see that compared to a mathematical pendulum, this is 1.2 times higher.

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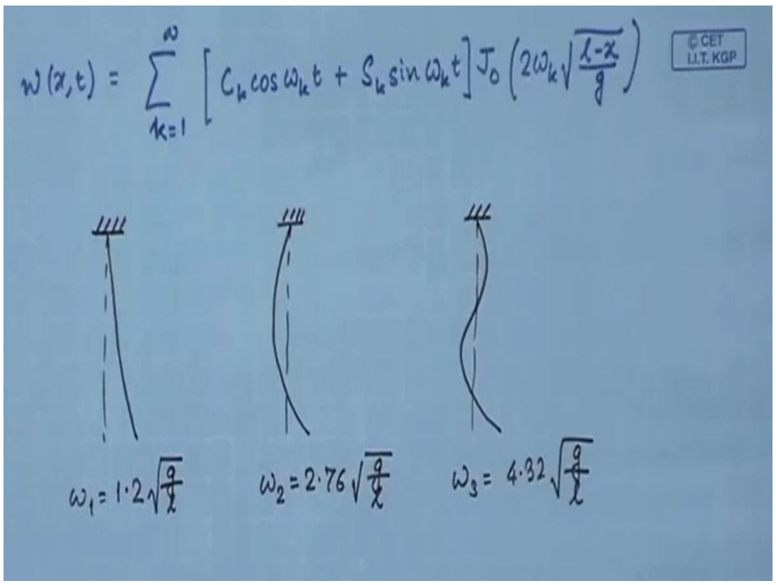

Now finally once you have this, you can write the general solution once again, using superposition. So these are the, these represent the modes, these are the Eigen functions and the modes of vibration of a uniform hanging string are defined by these Eigen functions. So the modes of vibration of a hanging string are defined by these Eigen functions and they look. So this is the first mode, this is the second mode and this represents the third mode.

Now once again I will demonstrate to you the modes of vibration of a hanging chain. So if you see this chain, so this is the first mode of vibration, now we will try the second mode, so this is the second mode of vibration of the chain. Exciting the higher modes is little more difficult. So in this lecture, what we have studied is the modal analysis of continuous systems, so we started off with the modal analysis of a taut string.

And then we looked at the properties of these model solutions and finally we also looked at the example of a uniform hanging chain. So with this we come to an end of this lecture.