Vibrations of Structures
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Lecture – 03
Axial and Torsional Vibrations of Bars

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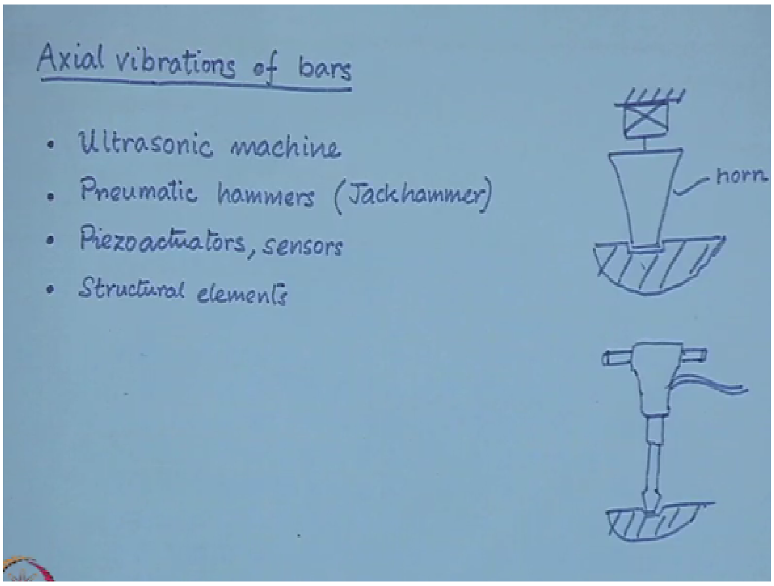

So today we are going to look at 2 examples on vibrations of one dimensional elastic structures that we have started with strengths. So today we are going to look at axial vibrations of bars and torsional vibrations of circular bars. So let us begin with axial vibrations of bars. So where do we finds bars in axial vibrations? So some examples are; in an ultrasonic machine, so in an ultrasonic machine, what you have is a bar which is shape like this, which is connected to an actuator.

So this bar is called a horn of the ultrasonic machine and this in ultrasonic generator which passes ultrasonic waves in this bar and because of this shape, you have large amplitude motions at this work piece, so such a machine is used for machine in brittle materials, for example. Then you find bars in axial vibrations and Pneumatic hammers, sometimes also known as jack hammers.

So in a jack hammer, it looks roughly like this, so these are used for drilling or chipping operations in construction sites. So here you have a bar, which is also in axial vibrations. Then you have piezo actuators, sensors in which you find a bar, which made of piezo electric material which is under axial vibrations. Then in various structural elements, you may find bars in axial vibrations.

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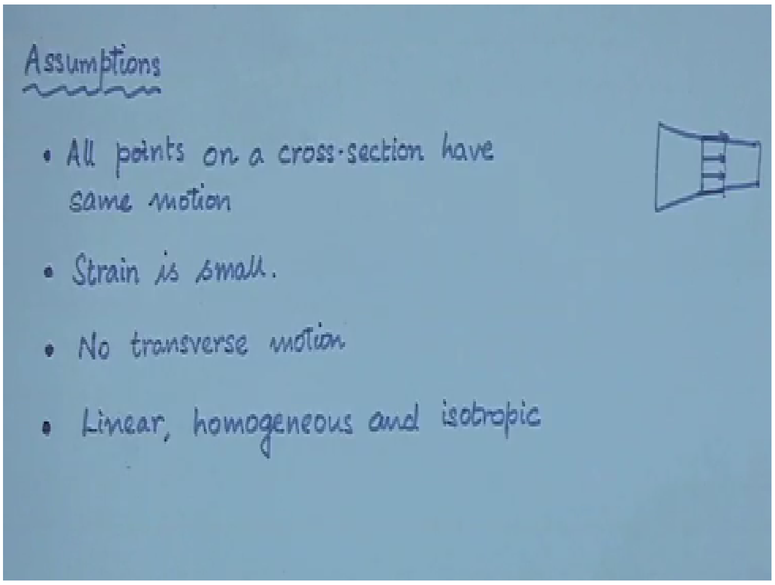

Now in order to model the dynamics of bars in axial vibrations, we begin with some assumptions that we make on this modelling. So the first assumption that we make is that all points of the bar are in all points on a cross section, have same motion. So what I mean by this is suppose you have a bar, at a certain cross section all points will have the same motion. The second assumption is that we make is that the strains, the strain is small.

So that we do not have nonlinear effects. We are going to discuss on the linear vibrations of bars. The third assumption is that, there is no transverse motion of the bar. So this bar, the material points are vibrating only along the axis of the bar. There is no transverse motions and the fourth assumption are, that we make is that the material of this bar is linear, homogenous and isotropic.

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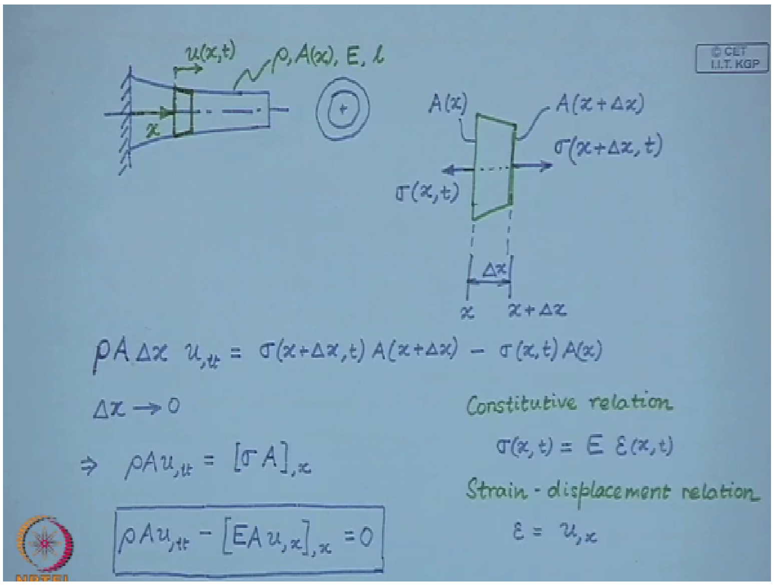

So with these 4 assumptions, we are going to now look into the equation of motion of a bar in axial vibration. So let me draw a bar, so this bar is made of material of density, say rho as an area of cross section A; which may be a function of the spatial coordinate x, has young's modulus E and has the length l. Now at any location x of the bar, the displacement of cross section in the axial direction is measured by this field variable u, as a function of x and time t.

Now you are going to derive the equation of motion of this bar using the Newtonian approach. So what we will do is, we will consider, an infinite decimal, a small section of this bar as I have shown here and draw its free body diagram. So we will consider the stress, so this element is of a line delta x, lies between x and x + delta x, so the stress on the right phase, I will write as sigma, x + delta x, t and on the left phase, this sigma x, t.

Let this area be A(x + delta x) and this is A(x). Now we are going to write the equation of motion using Newton's second law, for this infinite this small element. So the mass of this little piece, may be written as rho times A is mass per unit length and the length of this little element is delta x, so this is the mass of this element. This mass times the acceleration in the longitudinal direction; that is the double derivative of the field variable u with respect to time.

That must be equal to the forces in the longitudinal direction. So the force on the right phase is given by sigma times A on the right phase - the force on the left phase this has given by sigma times A on the left phase. Now if I divide this whole equation by delta x and take the limit, delta x time is 0, then that would imply, rho A times the acceleration equal to the partial derivative of sigma * A.

Now we require to represent this stress in terms of the displacement, the field variable. So in order to do that, we will need 2 things. The first is the material constitutive relation, which will relate the stress with the strain so we know that the axial stress is proportional to the strain and the proportionality constant is the Young's modulus, so this is the Hooks law in one dimension. Along with this, we need the strain displacement relation.

So which means, epsilon the strain is del u del x. So if I substitute this expression in the constitutive relation and put this back in the equation of motion, what I obtained on rearrangement of terms, so is therefore the equation of motion for axial vibrations of a bar. Now remember that this equation has been derived by considering a small element of the bar that in no way tells us or describes to us the full physical picture of the bar.

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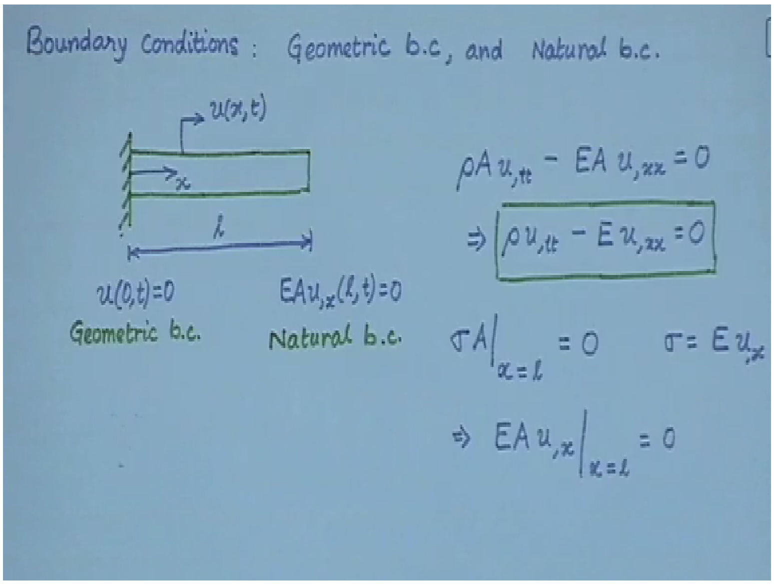

So what I am trying to get at is, we need to complete this description of this bar in an axial vibrations, we need the boundary conditions. Now as we have discussed before, this boundary conditions are of 2 types, geometric and natural. Now let us look at some examples and identify the boundary conditions. So we begin with the uniform bar. So the equation of motion in this case, simplifies to this equation, because the area is now constant, it is no longer function of x.

So it can come out of the partial derivative and this can be simplified further to obtain this equation of motion for a uniform bar. Now the boundary conditions of this bar, as you can see, this end of the bar, is completely fixed at the wall, so at x=0, there cannot be any actual motion of the bar. Now this boundary condition is fixed by the geometry of the problem, so this is the geometric boundary condition.

Now this end is free, free would mean that there is no axial force at this, on this phase of the bar. Now force as you know, force here on any cross section is given by sigma times A, so at x=l, the axial force must be 0. Now we have used the Hook's law and the strain displacement relation previously to obtain this relation between the displacement and the stress, so therefore, these 2 will imply, E times A, del u del x, this computed or evaluated at x=l, must be 0.

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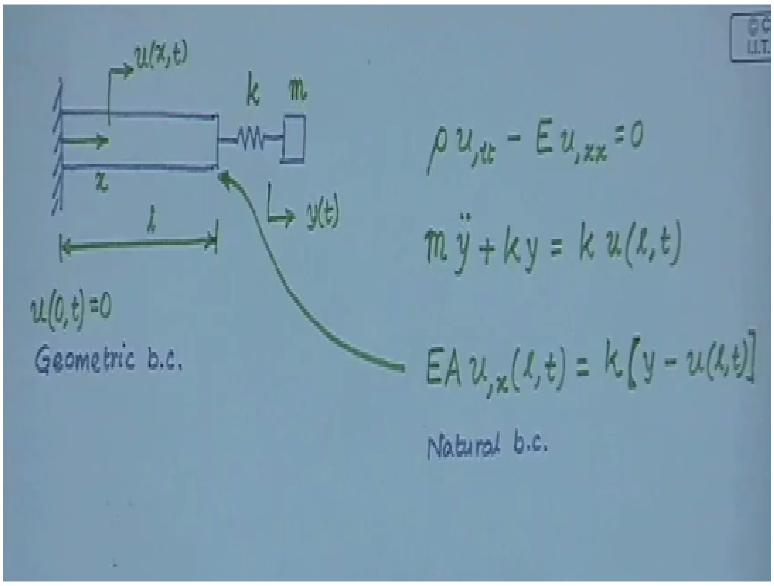

Since A is uniform, the boundary condition turns out to be this. Now this comes from a force condition, such a boundary condition is a natural boundary condition, so we finally have the equation of motion of a uniform bar and the boundary conditions, which completes the description of this fixed free bar in actual vibrations. Next we look at another example, so this is a bar, at the end of which, this is a fixed free, this is a fixed bar at the end of which, an oscillator is attached in this manner.

So this is a point mass attached with the spring of stiffness k. So the displacement of this mass from the equilibrium position is measured by this coordinate y. So let us first write down the equations of motion for this system, now here you have as you can see a bar and this mass, so here are the field variable which measures the displacement of the bar, the material points of the bar.

And we have this coordinate y, which measures the displacement of this mass, discrete mass M. So the equation of motion of this bar remains the same as before, so you can write, now for this oscillator which is connected at this end, we can easily write the equation of motion as m, y.. + k times y is equals to k times the displacement of the bar at this end. This equation can be easily derived.

If you write down, if you take the oscillator separately and write down its equation on motion. Now these are the equations of motion for this system so as you can see they are coupled. Now let us look at the conditions of the boundary. So at this end of the bar, this completely fixed, therefore, we can easily write the displacement is 0 at this end. On the other hand, at this end of the bar, where this oscillator is attached.

We can expect an interaction force with the oscillator, from the oscillator. So we have already written the force on any cross section as E times area, del u del x, at x=l, a time t, so that is the force, at there been no oscillator, it would have been 0. Now since there is an oscillator you can write the force that this absorber or this oscillator puts on this end, which is easily obtained as k times y, which is the motion of this point – the displacement of the bar at this end. So this is the boundary condition at this end of the bar.

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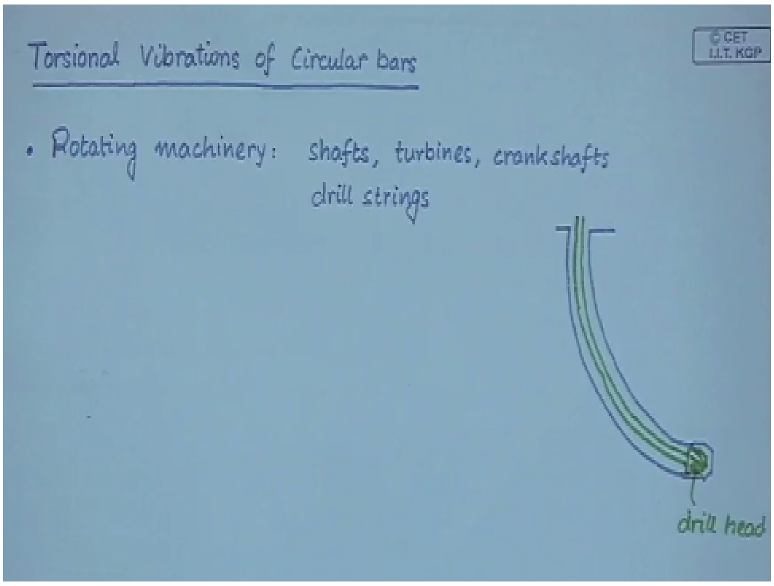

So here we have a geometric boundary condition whereas on the right end, we have a natural boundary condition. So with these 2 examples, we will move on to the next example or the next case that we are going to consider, which is torsional vibrations. So we are going to consider the case of torsional vibrations of circular bars only. So, where do we find torsional vibrations of bars?

So mostly in rotating machinery, so you have shafts in rotating machinery which transmit torque such as subject to; subjected to torsional vibrations. For examples in turbines, in rotors, in crankshafts; turbines, crankshafts of engines in the dentist drill, so in a dentist drill, you have a wire which is under torsion. These are called drills strings, they are also found in petroleum excavation and mining industries.

So these shafts, for example in a mine like this, so this is the drilling head and here you have a shaft which transmits the torque to this drill head, this shaft is under torsional vibrations. It also has some amount of transverse vibrations but the torsional vibrations are quiet dominant in such situation, these are called drill strings. Si in such situations, you find shafts or circular bars in torsional vibrations.

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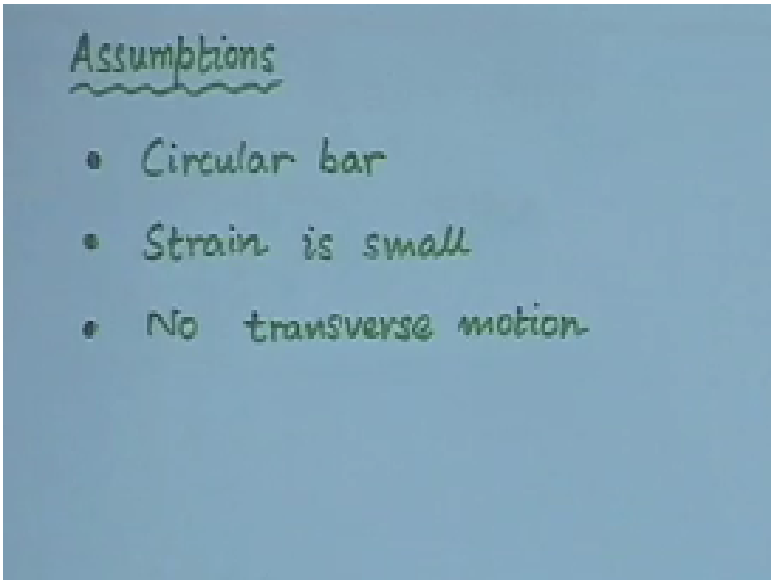

So once again for the mathematical model of such a bar, you make some assumptions, so the first assumptions that we make is that the; we have already said that the bar is circular, so we are going to study only torsional vibrations of circular bars that is to ensure that there is no wrapping of the cross section. We will assume that the strains are small, so that the dynamics can be adequately described by a linear model.

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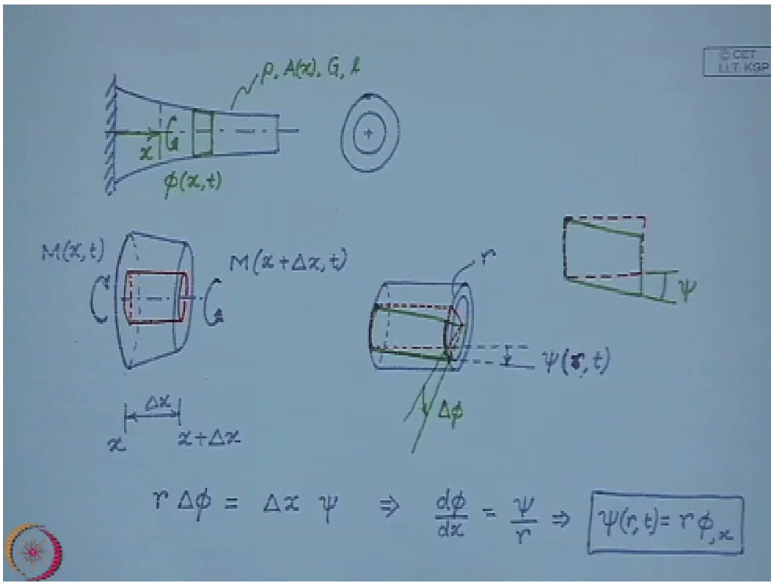

And we will also assume that there is no transverse motion. There is no transverse motion of the bar. So with these assumptions, let us look at a circular bar, so here I have a circular bar, made of material of density say rho as an area of cross section, which may be a function of the special coordinate x, the shear modulus or the modulus of rigidity is represented by G and this has a length l.

Now at any location x, the field variable that measures the torsion in the bar at the local torsion displacement of the bar is represented by this phi x, t so at any location x at any time t. This is the torsional displacement of any cross section. Once again we are going to derive the equation of motion of such a bar in torsional vibrations using the Newtonian approach, so we are going to consider a small portion of this bar between the special coordinates x and x+ delta x.

Now on the right end of this bar, let us consider the movement represented by M, x + delta x at time t. On the left end, you have the movement on the cross section as M (x, t). Now inside this little element, we are going to consider a ring of certain radius r. So I will draw this portion, this ring separately. Now consider in the undeformed ring, 2 axial lines like this and the corresponding radial lines here.

Now this little element when the bar is under torsion, there will be a differential rotation between the left phase and the right phase, so because of this differential rotation, this red element which is somewhat like a rectangular element is going to take up this configuration.

So if you look at this angle that is the small rotation, the differential rotation between this phase; the left phase and the right phase.

On the other hand, if you look at this angle, this is nothing but the shear strain experienced by this initially undeformed red element. Now this shear strain is a function of the radial position of this element. So if I draw these 2 elements once again, so this was the red element, this was before the deformation, this is green element after deformation, so this is the shear strain in this element.

So with this kinematics, we can write or the radius of; so this is the radial location, so we can write r times delta phi, which is this small length, r times delta phi must be = this length which is delta x times the shear strain, so that would imply upon taking delta x tends to 0, which I can also write as; so this equation that we obtained from the kinematics of this torsional deformation that of the bar.

We can now relate the shear strength at any radius r, at any time t, in terms of our field variable which is phi.

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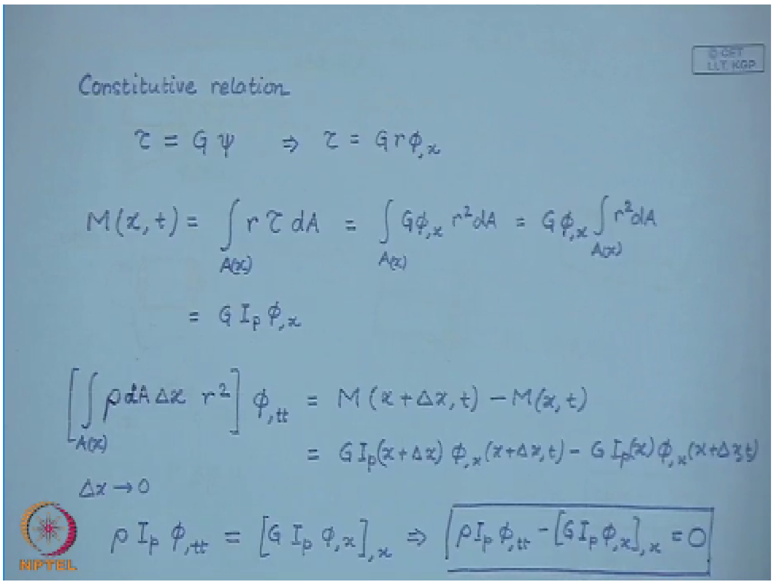

Of course, this is also a function of x, because phi is a function of x. Now we are going to use this kinematic relation further. So let us look at the constitutive relation of the material. So we know from Hooks law, that the shear stress is proportional to the shear strain, so the shear stress is G times the shear strain. So if I use the kinematic relation in this expression, I have the expression of the shear stress, at any radius r, at any location x, at any time t.

So once I have the shear stress, I can integrate this, multiply this with the area and integrate; multiply this shear stress with this area and integrate over the whole phase to get the movement, the torque in the bar. So M at any location x at any time t, I can write from the shear stress, times the small area of this ring and the arm, so the arm times, this shear force, so that will be as the movement, the torque.

Now if you integrate this over the full phase, then you get the total torque on this phase of the bar. So if I substitute this expression, I have this and since these 2 terms, the G and phi, they have nothing to do with this integral of the area, so I can safe them by bring them out and this therefore is the torque and we know we can easily identify this integral as a polar movement of the area.

So now I have related the torque at any cross section in terms of my displacement, the torsional displacement. Now I can write down the equation of motion. So first I will write the movement of inertia of this ring that we have consider. So this movement of inertia of the ring is the mass of the ring times the radius square, so mass of this ring can be written as rho times dA, is a mass per unit length * delta x that is the mass of this ring times r square.

So this if I integrate over the full area, then I get the movement of inertia of this element that we have considering. So this is the movement of inertia times the angular acceleration, which is the double time derivative of our field variable phi and that must be equal to the balance of torques and we have already have this expression of movement or the torque at any phase here, so I can write this as;

Now if I divide this whole equation by delta x and take the limit, delta x times to 0, and if I identify this r square da integral, rho can come out of this integral. So r square dA integral over the phase is once again the polar movement of the area, so I can write rho Ip, the angular acceleration must be equal and that gives us upon rearrangement, the equation of motion of torsional dynamics of a bar; a circular bar.

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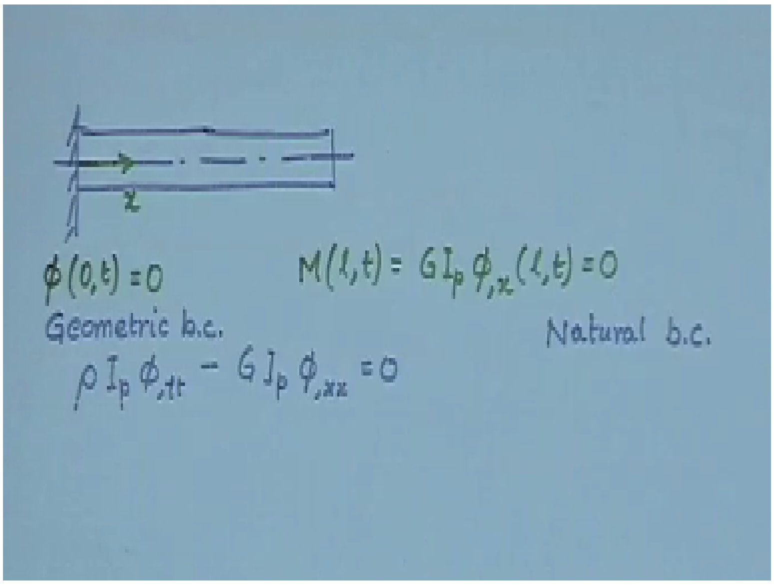

Now let us look at some examples, so if you have say a uniform bar, then the equation of motion once again simplifies to rho Ip phi tt – G Ip phi xx = 0. Now the boundary conditions for this problem, which will complete the description of the physical situation, so here this bar is connected to the wall and therefore this end of the bar cannot have any rotation. So this again is a geometric boundary condition.

On this end of the bar, the torque is 0, which is therefore given by; so this is the geometric boundary condition, whereas this is the on the right end of the bar, we have a natural boundary condition. So with this, we complete our discussions on axial and torsional vibrations of bars. Now to summarise, we have considered axial vibrations of bars, we have derived the equations of motion.

And we have seen the boundary conditions of 2 types; namely the geometric boundary condition and the natural boundary condition, then we have also derived the equations for a bar interacting with the discrete system and then finally we have looked at the dynamics of torsional vibrations of circular bars and we have derived the equation of motion and the boundary condition. So with that we end this lecture.