Vibration of Structures
Prof. Anirvan DasGupta
Department of Mechanical Engineering
Indian Institute of Technology – Kharagpur

Lecture - 02
Transverse Vibrations of Strings - II

In the second part of our discussions on transverse vibrations of strings, we are going to look at strings with interaction.

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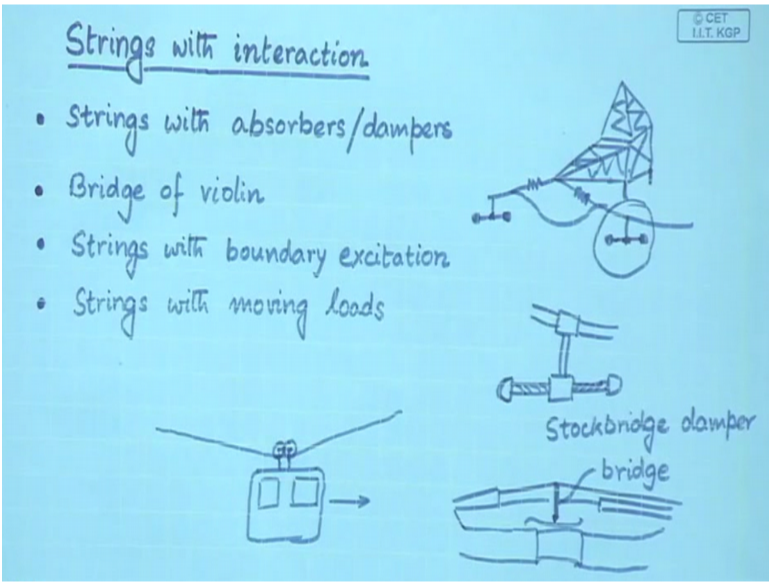

So this term interaction, by the term interaction I would mean that the string is in contact or is connected to another system or an actuator or it could be a passive system like an absorber or a damper, so let us look at some examples of strengths with interaction. So as I mentioned just now we may have strengths, interacting with absorbers or dampers. So such strings are found in for example in high tension cables.

So this roughly looks like this. So here is the pylon. So you must have seen such structures carrying high tension cables. So if you look carefully at around near this support point, you may be able to see something hanging like this. So this is a string, this is a span of a string and near the support point, you have this kind of attachments. So if you look at these devices closely they are mounted on this cable.

And they look like this and here are some masses and this is the cable and here there is multi strand cable which acts like a spring and a damper. This is called a Stockbridge damper. So this is an example of a string interacting with a damper. Such strings are also found in for example the piano. In the piano you have dampers which stop or damp out the vibrations of the strings.

So in order to understand the efficacy of this Stockbridge damper which is essentially put to damp out the vibrations, wind induced vibrations of high tension cables. So to understand the effect of such external devices, so you will have to analyze the string along with these absorbers or dampers. Then interaction of another system is also found in for example the bridge of a violin.

So the use of the bridge is to transmit the vibrations of the string to the sound board or the main body of the violin so that it can be amplified. So if you see a violin, this is the bridge which transmits it to the body of the violin. Then you can have strings which are connected to an actuator which make excite within the span of the string or maybe at the boundary or it could be another system which is vibrating and it is exciting the string at the boundary.

Then you can have strings with moving loads. To typical example would be a cable car. Something like this. So here this cable car will be moving on a taut string. So we see that there are various examples in which a string may have to interact with an external discrete or even sometimes continuous system, another system. So we need to model and analyze such strengths.

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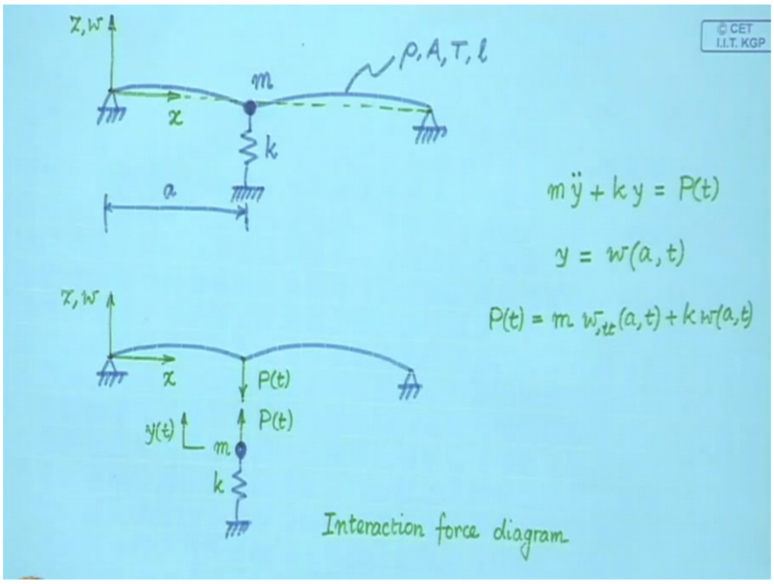

So let us start with a simple example. So let us consider here a string made of a material density rho area of cross section A, under uniform tension and length l. We consider an oscillator connected to the string at a distance a from the origin. Let the mass of this oscillator be m and the stiffness be k. So then I have to analyze this system. What I will do is I will draw what is known as an interaction force diagram.

So this looks like, this in which I have separated the oscillator from the string. So in place of this I will introduce the interaction force. That is a P which will be a function of the time. So this we will call the interaction force diagram. Now let us look at the oscillator. Let us first look at the oscillator. So I know that the equation of motion, so if I consider the coordinate of the mass point as y, then I can very easily write m times y double dot plus k y must be equal to the force.

Now this is the same force that acts on the string at a location a. So here since the mass has to be coupled to the string, we have this y given by w at a at anytime t. So therefore if I make this substitution then I can write, so this is the force, the negative of this force actually acts on the string.

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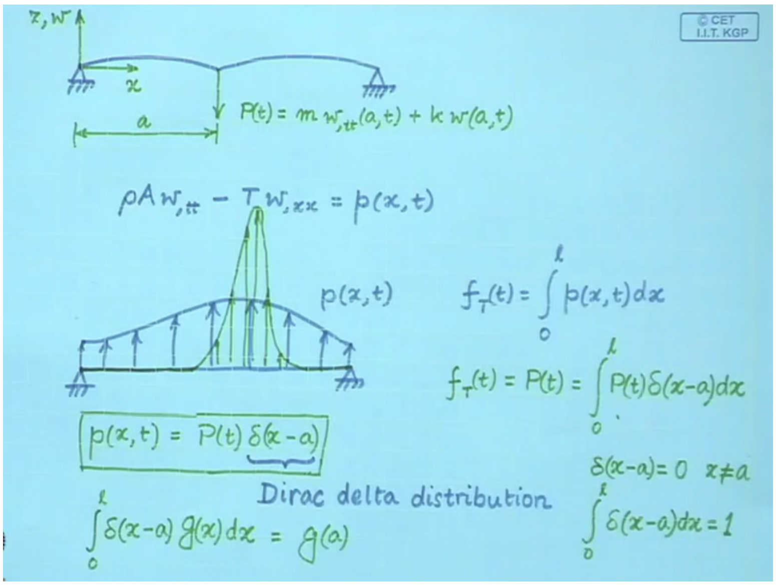

So this is our picture. So this is concentrated force acting, at location a. Now remember that when we wrote out the equation of motion, the transverse motion of a string, so this equation was, so this is uniform string with no actual force. So this was our equation of motion where this small p x comma t is a forced distribution or the force per unit length. Now here what we have shown is a concentrated force acting at x equals a.

So somehow we have to now represent this concentrated force as a distribution. So how do we represent a concentrated force as a distribution, this is what the question is. So let us look at distributed forces. So suppose this is a distributed force on a string, let us say. Now what is the total force, so this total force is obtained as by this integral over the domain of the string of the distributed force because this force per unit length.

So that multiplied by this infinitesimal length and integrated from 0 to l would give us the distributed, the total force acting on the string. Now if my distribution gradually becomes narrower and narrower but retaining the same total force. So the total force if fixed. We know this in our example that we are considering at this point, we know the total force that is acting.

Now this distribution happens to be zero from here to here and here to here and it is non zero only on a finite region of the string not on the full domain, so it is non zero only on a finite region. So you can imagine as this nonzero region shrinks to zero as in the case of a concentrated force here that I have shown. Retaining the same total force then this is going to shrink to zero but its amplitude or the magnitude here is going to blow up.

But the integral under this will be finite and that will be given by this quantity. So how do you achieve such a thing. So the total force must be equal to P t that we have here but that must come by integrating out a distribution. So let me write it like this that distribution written like this. So this is our distribution where I have introduced, this is now a distribution, delta x minus a which is known as the Dirac delta distribution.

Now, I mean we can very easily derive the properties of this Dirac delta distribution from what we expect this distribution to do. First of all, this distribution must be zero, everywhere on the string except at x equals to a. So this distribution must be zero, everywhere on the string except at x equal to a and what happens at x equals to a as you realize that this is going to blow up, so we will not write that value.

But what we are going to do is, we are going to write from this integral a second property of this derived delta distribution which is this, so the integral under this distribution over the domain of the string is one, so the area under this distribution is one. So what this distribution achieves, this Dirac delta distribution achieves is that it gives us mathematical representation of a physical idealization.

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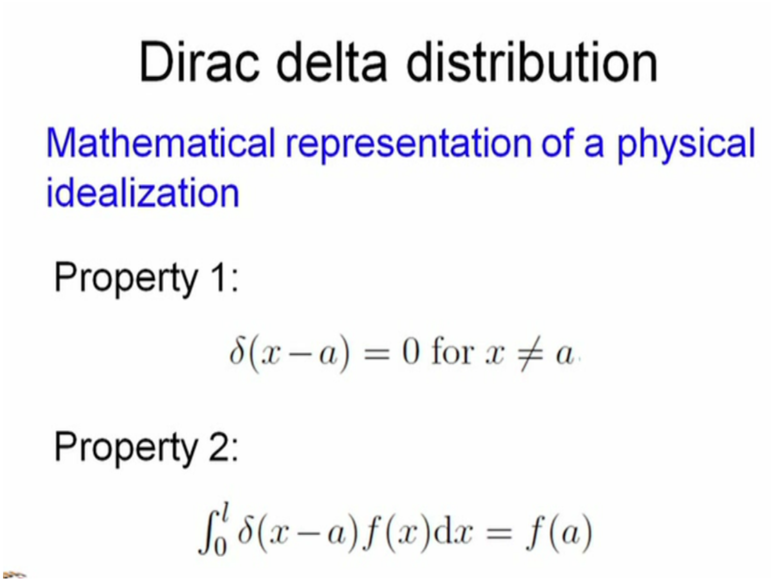

So what is the physical idealization? Now in macroscopic world, you do not have concentrated forces. That means forces acting over zero area. We always have forces acting over a distribution, over distributed area. But then what happens is for example in the violin string on the bridge, the portion on the bridge is much smaller than the length of the string or the damper that damps out the vibration in a piano, the contact maybe very small compared to the total length of the string.

In such cases we can have physical idealization that this is almost a point contact. So this Dirac delta distribution gives us a mathematical description of this idealization. So this is a further generalization of this integral property of Dirac delta distributions. So here I have put together these properties of the Dirac delta distribution that we just now discussed. Now using this let us get back to our strength.

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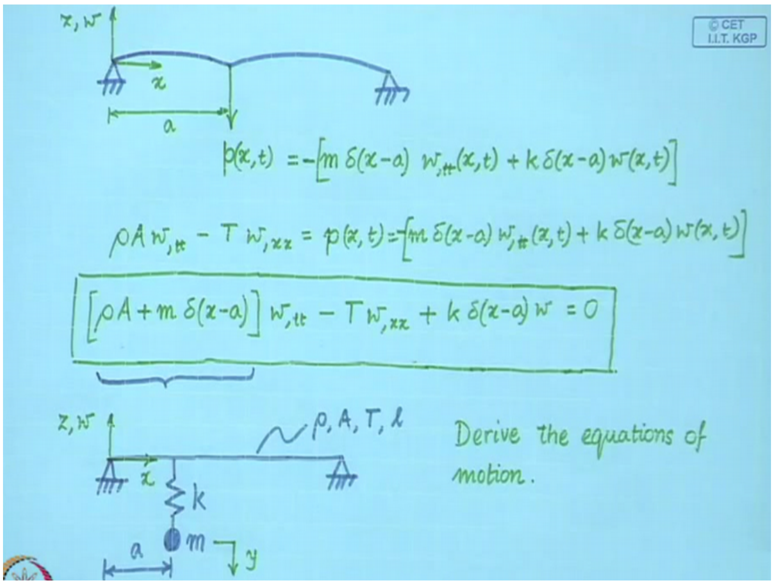

So now I will represent this distribution, the force distribution on the string as the mass times the Dirac delta distribution at x equal to a. Now here I have written the acceleration at x equal to a but from the previous property that I just now discussed, I can write it like this. So this is the forced distribution because of a concentrated force at x equal to a and now I can write out my equation of motion which on slight rearrangement.

So this actually turns out to be, so when I write the distribution on the string, so this comes with a negative sign because of the convention that upward is positive. So this is finally the equation of motion of the string with an oscillator attached to it. Now you can see from this term, this term is nothing but the mass per unit length. So rho a was because of the string and this is because of the oscillator.

Similarly, now you have a stiffness coming into the system at x equal to a, so this completes the description of a string interacting with an oscillator which is coupled to it. As an exercise you can try out an example like this in which you have a string made of material of density rho, area of cross section constant which is a under a tension T and having a length l and there is an absorber connected to the string.

The absorber has a mass m and stiffness k attached at x equal to a. So derive the equations of motion. So here we have to note that there will be two equations, one for the string the other is for the absorber because the absorber is now a separate coordinate. The location of the mass of the absorber is a separate coordinate. So you can try out deriving the equations of motion of this system.

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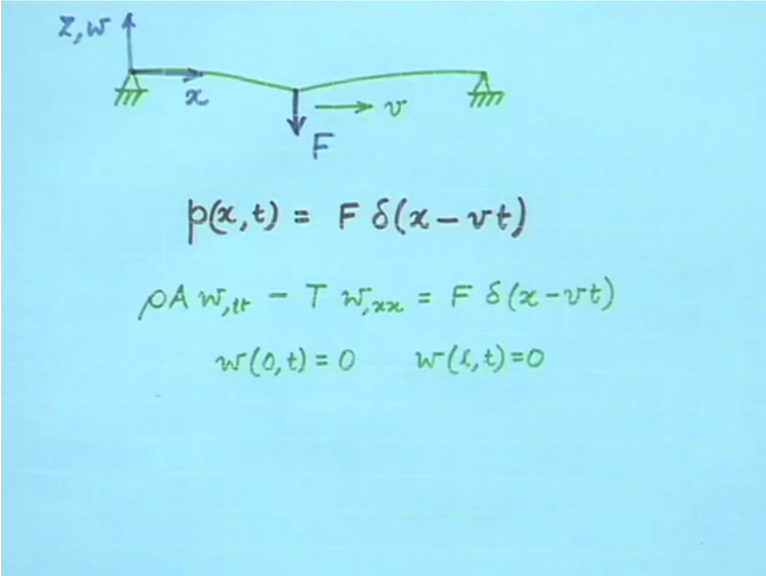

Now, let us look at the cable car problem. So in the cable car problem what you have in the first approximation you may consider this to be a constant force, this is an approximation. So this is not exactly the cable car problem but something analogous to it. And this constant concentrated force is moving at a speed v on the string. So if you have this situation the concentrated force is moving at a constant speed v on the strength.

So this force again has to be represented as a distribution, force distribution which now using the derived delta function, distribution you can very easily write this force distribution now on the string is F which is the magnitude of the force, times the distribution, but this distribution is shifting. So I am assuming that at time t is equal to zero this force was at the origin, that means at x equals to zero.

So this is a representation of the force distribution on the string because of a cable car moving on the string. So therefore we can write the equation of motion for the cable car problem in this form. And with every problem of course comes the boundary conditions. So as we have discussed, geometric boundary conditions in this case, now you can generalize this. Suppose you have a mass moving, a bead moving on a string.

Then, you find out the corresponding interaction force because now the bead will have the acceleration which comes because of its interaction with the string and that will produce a varying force because of inertia of the bead So you can generalize this to analyze more and more realistic situations.

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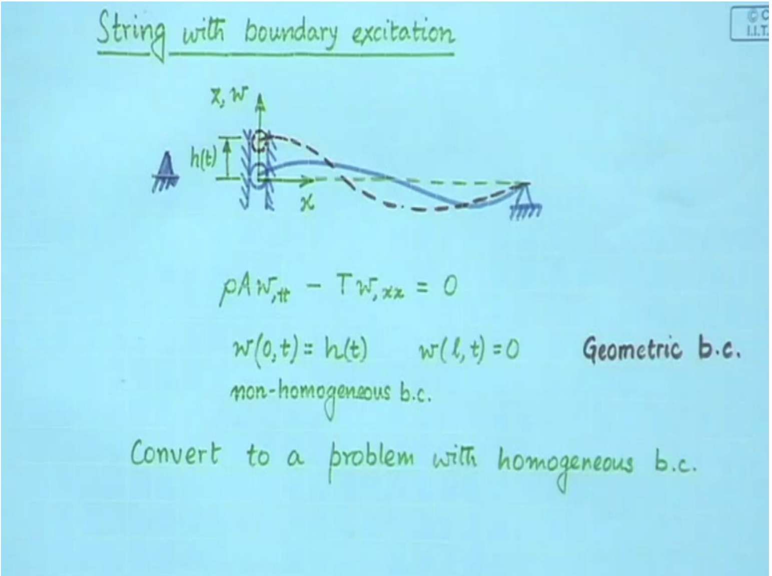

Now let us look at an example of a string with boundary excitation. So we will put a string which is free to slide on this left boundary. I will draw a displaced configuration of the strain. So we will assume that this string is being driven by a robust actuator which actually gives a displacement condition on the boundary. So this string is being excited by a system or an actuator which is robustly driving the left boundary.

So the equation of motion for the string which is made of a material of rho area of cross section A and under tension T, so can be written in this manner. The boundary conditions maybe written out at the left boundary, at x equals to zero. The string is given a displacement which is a function of time. And the right boundary the transverse displacement is zero. So first of all in this case both the boundaries we have a condition on the displacement of the string.

So both these boundary conditions are geometric boundary conditions. Now, here we have a boundary which is being excited, I mean the displacement at this boundary, this sliding boundary is a function of time. Now when you have such a situation, analysis may become little difficult. So what we want to see is whether this problem which is a boundary excitation problem, whether this can be converted to homogenous boundary conditions.

So this boundary condition, for example is non homogenous. So this is a non homogenous boundary condition. So the question arises whether we can convert a problem with non homogenous boundary condition like this to a problem with homogenous boundary condition because that in many ways simplifies the analysis of the problem. So we are interested to convert this problem to a problem with homogenous boundary conditions.

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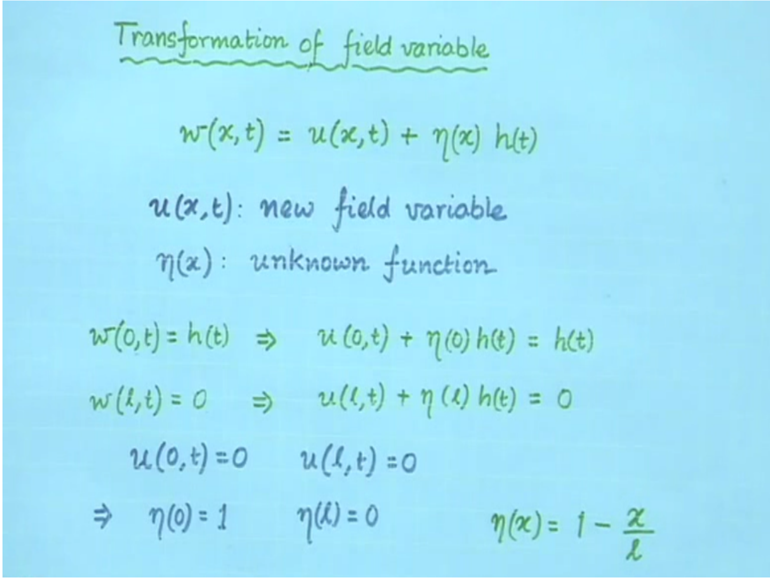

So there are many ways of actually converting this problem with non homogenous boundary conditions to homogenous boundary conditions. We will follow method which realizes on variable transformation. So we want to find transformation of the field variable which was w in this case, we want to transform it to another variable such that the problem can be converted to a problem with homogenous boundary conditions.

So let us look at this transformation, so w x comma t was our field variable. So we introduced a new field variable and an unknown function at this point eta x multiplied by h of t. This was the given motion of the boundary. So here, u is the new field variable and eta is an unknown function which we will find out. Now if I substitute this transformation into the boundary conditions, so what were the boundary conditions?

So this was the boundary condition at the left end, so that would then imply this condition. On the right, boundary, the condition is w at l comma t is zero. So that would imply, so we have these two conditions now. Now we want to have homogenous boundary conditions on the new field variable which means we want to have homogenous boundary conditions on the new field variable which means we want to have and these two conditions therefore from here, we will obtain conditions on eta which are eta at zero must be one and eta at l must be zero.

Now there can be different choices for eta which satisfy these conditions. The simplest choice would be; this could be the simplest choice. Of course you can have powers of this function. For example, one minus x over l whole square can also satisfy this. But will make one choice at this point and we will discuss what are the implications.

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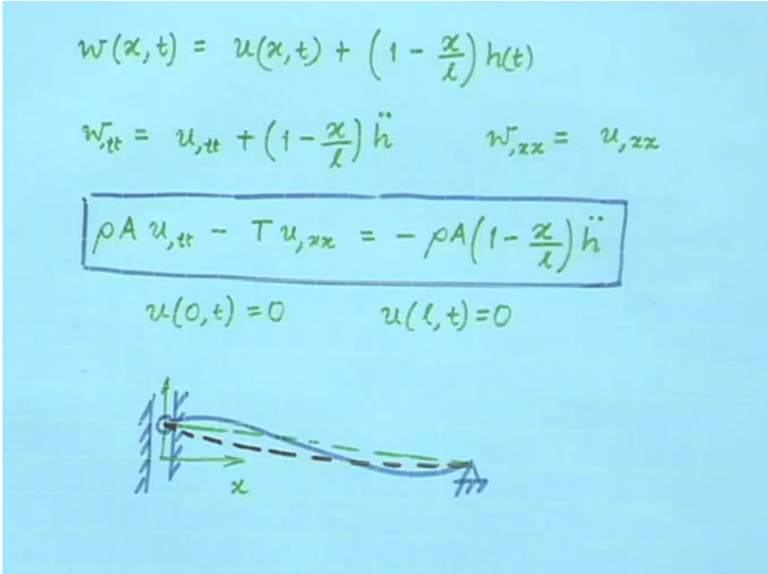

So if you put this back into your transformation you will have this expression for our field variable w in terms of u and if you substitute this in the equations of motion, so we need to calculate the double derivative with time, so that will turn out to be and we also have double derivative with respect to space which will be reduced to only u coma x x.

So therefore our equation with this substitution becomes this along with the boundary conditions, so we have a system with homogeneous boundary conditions. But now our equation of motion has become non homogenous. So if you look carefully this term is result of the transformation and what this transformation does is actually to take us to a non inertial frame.

So with the choice of eta as I have I have done, we have shifted to a frame which is indicated by this green line. So this is moving, I mean if H is periodic or even otherwise, this line would be moving and this will be non inertial frame in which we are expressing the equation of motion. So as soon as we go to a non inertial frame we have the inertia force which is essentially coming on the right hand side of this equation of motion and making it non homogenous.

Now if you have a different form of eta then what happens is for example if you take one minus x over l whole square then it is nothing but a transformation which will take to this kind of a frame, which is quadratic. But still becomes non homogenous. And it still remains non homogenous. So your equation of motion becomes non homogenous because of the inertia force and your boundary condition becomes homogenous.

So with this we come to the end of this discussion on transverse vibrations of strengths. So we have discussed today, strengths with interaction with external oscillators and external systems and we have seen how to represent concentrated forces, how to represent the interaction and how to convert a problem with non homogenous boundary conditions to a problem with homogenous boundary conditions.