Vibration of Structures
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Lecture - 01
Transverse Vibrations of Strings - I

In this lecture, we will discuss transverse vibration of strings. So before we start discussing about vibrations of strings, let us look at what a string is. So here on the view graph, you can see the definition of the string.

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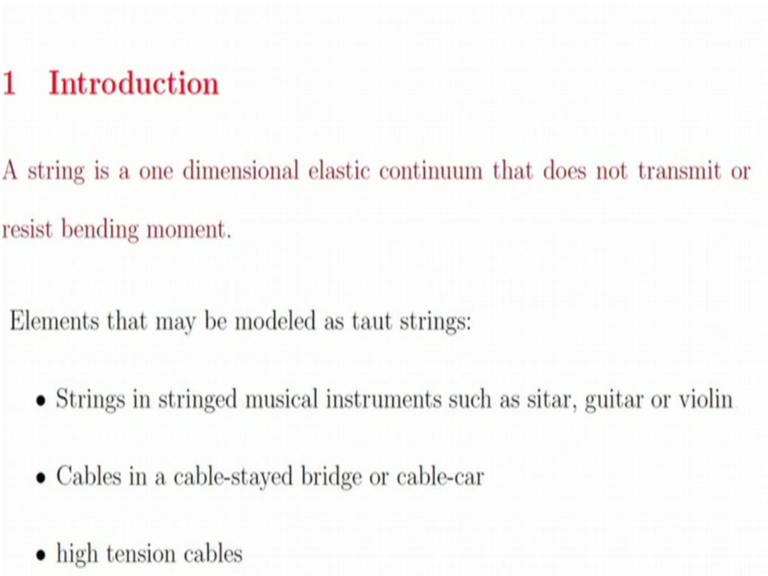

So a string is a one dimensional elastic continuum that does not transmit or resist bending. So this is the definition that we will use for a string. So now I will show you some examples of strings

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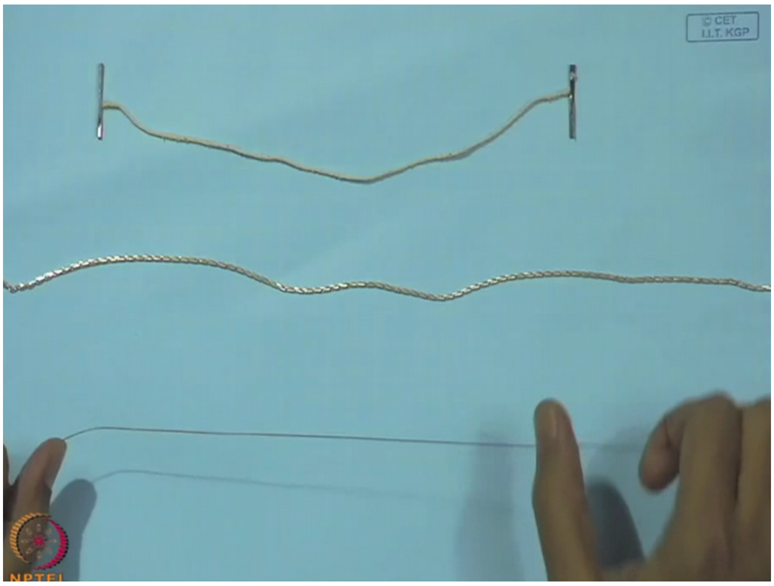

So as you can see here this is an ordinary tag which is like a string because it satisfied the definition that I gave you. That it does not transmit, it is a one dimensional continuum and it does not resist bending in any way. So whatever shape I give it, it will retain that. So the restoring force comes when I make it tight. So this tension that I give to this string acts like the restoring force, the restoring force is produced by the tension in the string

Otherwise the string will take any shape. So it does not resist bending. This is another example of a one dimensional continuum that also does not resist bending. This is a chain and this is a hanging chain. So this also qualifies to be strain. Here is a guitar string, as you can see. This is a guitar string. Now, if I give it some bending, if I bend it, it restores back as you can see here. But still this is called as string.

To understand the reason for this, let us see what happens in a guitar. In a guitar the string is under tremendous amount of tension because of this tension the primary restoring force is because of the tension in the string. Of course there is this bending in the string, in this string which also kind of restores to its original straight shape, but when it is put in a guitar under tremendous amount of tension, the tension becomes the dominant restoring force.

And hence any structural element which is under high tension, qualifies to be analyzed in the first approximation as a string. So where do we find strings? So elements that maybe modeled as taut strings are found in stringed musical instruments such as sitar, guitar, violin, even in the piano. So we have seen such instruments in which the sound is basically reduced by the string.

Then in the cables, in a cable state bridge or in a cable car, so these structures have actually cables which are under tremendous amount of tension and hence they can be analyzed as strengths. In high tension cables which are again under very high tension they may be treated as taut strings.

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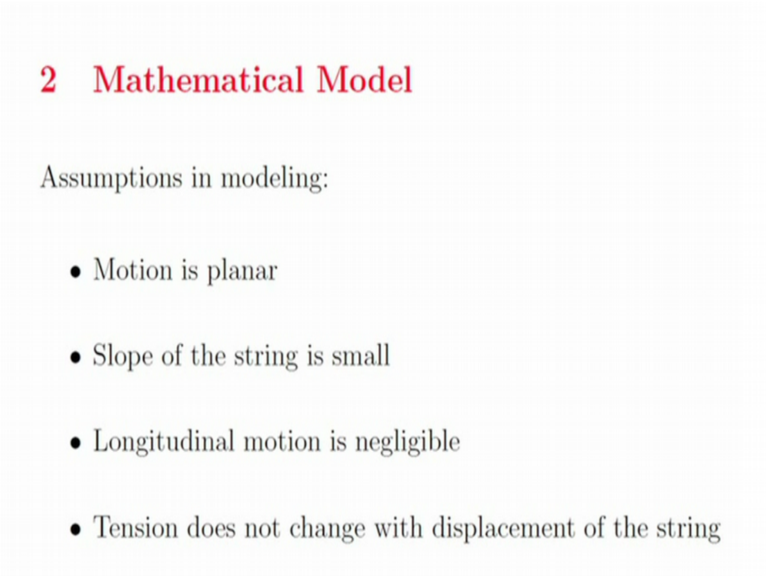

So we start with a mathematical model. So how do we model strings. Now in order to model strings, we will make some assumptions. So here I have listed out some of the assumptions that we make in modeling the string. So the first assumption says that the motion of the string is planar.

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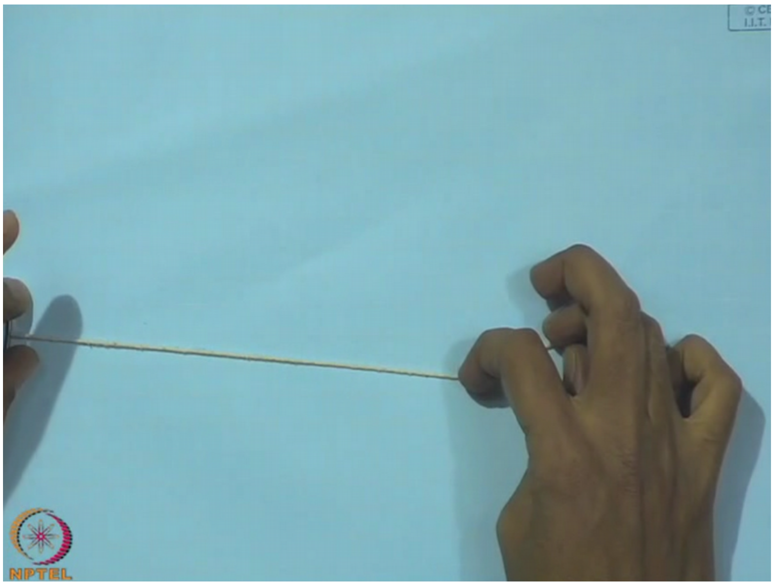

So for example here, what I have, I will assume that the string vibrates only in this plane. Then the slope of the string is small. So when the string deforms the slope at any point of time is small. The third assumption says the longitudinal motion is negligible. So if I make a mark here and trace the motion of this mark, as the string vibrates, you will find most of the time this mark moves transfers to the string. There is hardly any axial motion.

There is hardly any motion in this axial direction of the longitudinal direction. So this is our third assumption. The fourth assumption says that the tension does not change with displacement of the strings. So I have put some tension in this string and as I displace, the tension, the change in tension is negligible. So with these assumptions we can now start modeling our string.

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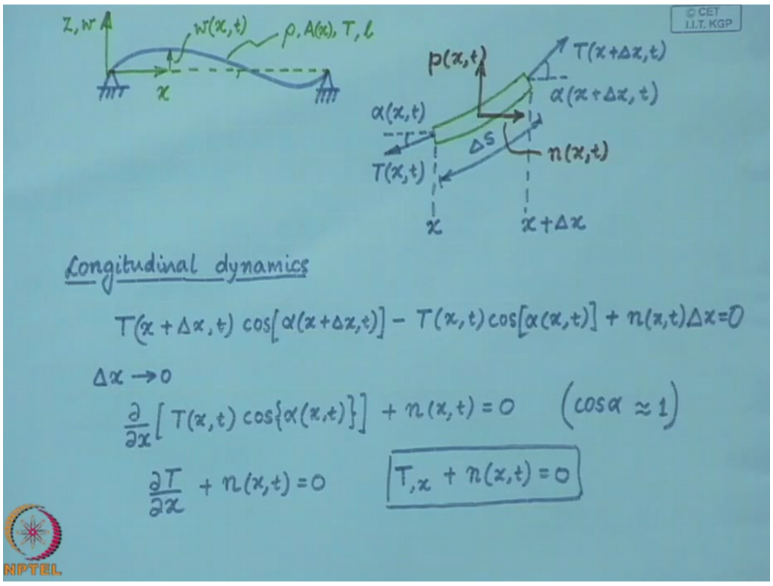

So, consider a string made of a material of density rho has an area of cross section which maybe a function of the position coordinate x. It is under tension T and has a length l. so this transverse motion of the string is measured by this variable w at a location x at a time instant T. So this shows a string, a stretched string, a taut string which has been displaced from its equilibrium position which is the x axis.

Now to write out the equations of motion, we will consider an infinitesimal element as we do in Newtonian mechanics. So we will draw the free body diagram of this infinitesimal element. So this element lies between x, the coordinate x and x plus delta x. On the left, this is under the tension T x, T and it makes an angle alpha. On the right end, the tension is T, x plus delta x plus t and the angle it makes is similarly alpha x, x plus delta x comma t.

The stretched length of this element is delta s. Now to begin this, we are going to write the equations of longitudinal dynamics of this infinitesimal element. Now as we have assumed that the longitudinal motion of this element is negligible, so we will neglect the inertia force, that is the acceleration in the longitudinal direction. So we will neglect that. So then the longitudinal dynamics reduces to just a forced balanced equation in the longitudinal direction.

So let me write out this forced balanced equation. So it is a tension at this right end cos sign of the angle minus cos the tension at the left end times the cosine of the angle. And along with this you may have some external distributed forces, external forces. So you may have some external force distributions as I have indicated here. So these are force per unit length of the string. So in the longitudinal direction, I have for example this n x comma t.

So I will introduce that also in this equation and that must be equal to zero. Now if I divide this whole equation by delta x and take the limit, delta x tends to zero. So that will imply. Now we have assumed that alpha the angle made by the string is very small. So I can safely assume that cos alpha is almost one. So if I make this simplification of this assumption, then this equation simplifies to, now this I will write in a shorter form.

So this comma x in the subscript would indicate a partial derivate with respect to x and we are going to follow this notation throughout this course. So this then is our equation for the longitudinal dynamics which is essentially a force balance equation.

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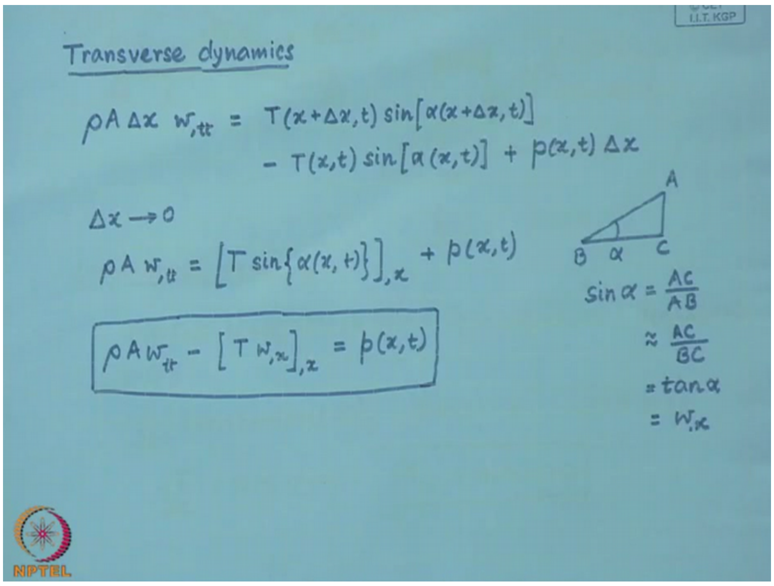

So I am going to use the free body diagram for the transverse dynamics. So if I write out the equation of motion, so mass of this little element can be written as rho times a is the mass per unit length and the length of this element is almost delta x up to a linear approximation. So rho a delta x would be the mass of this little element, times its acceleration. So which I will now write since w is the motion in the transverse direction.

So w comma t t indicates the acceleration, it represents the acceleration of the element in the transverse direction. So mass times acceleration that must be equal to all the forces in the transverse direction. So on the right hand we have, and on the left hand we have minus T x t sine of the angle the left end and plus the distributed force in the transverse direction. Now if I once again divide this whole thing by delta x and take the limit, delta x tends to zero.

So I have partial derivative of this term with respect to x. Now we have again assumed this alpha to be very small. So let us see what sign alpha turns out to be when alpha is small. So sign of alpha is equal to AC over AB. And if alpha is small you can very easily see that this up to a linear approximation is almost equal to AC over BC, since BC is almost equal to AB when alpha is small and this is the tangent of the angle alpha.

And this tangent of alpha can be very easily seen to be del w del x. So by using this approximation, up to the linear order, I can safely write sign alpha, almost equal to del w, del x. So if I make this substitution here and make some rearrangement, I obtain the equation of motion of transfers dynamics of a string.

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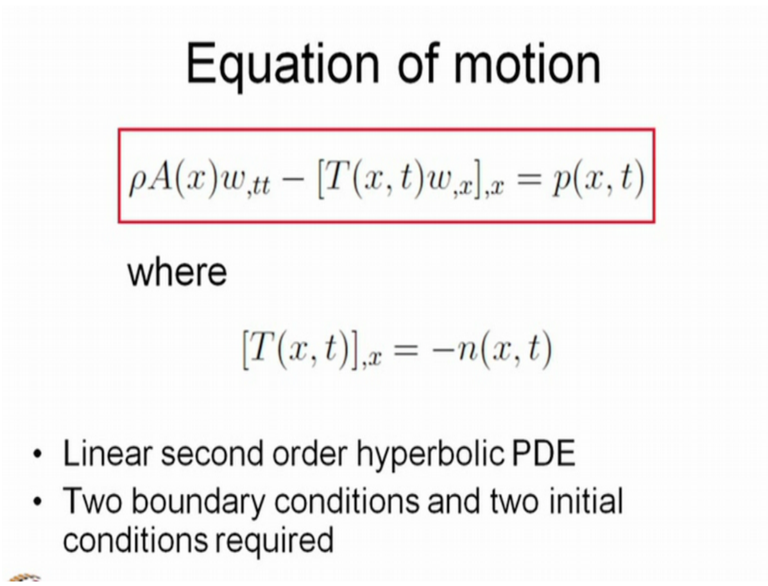

So finally, if I look at this slide. Here, I have put together these two equations that we have just derived, so the boxed equation is the equation of transverse dynamics where this tension which may be a function of space and time if there is an external distributed force n. Now this equation of motion, this is a partial differential equation, it is a linear second order hyperbolic partial differential equation.

Now to solve this, we need boundary conditions and initial conditions. So as you know since we have second order in space and second order in time we will need two boundary conditions and two initial conditions. So let us see, I mean, first why do we need these conditions. So briefly, so this equation of motion as we have seen is derived by considering an infinitesimal element of the string.

This in no way tell us how the string is connected to the ground or if at all it is connected to the ground, so we would need the description to complete the physical description of the whole system. So we will need these boundary conditions at the two ends of the stream. And mathematically if you want to understand this, if you see there are, I mean this equation is second order in space.

So it the space derivative is of second order. So upon integration we will generate two constants of integration and hence to determine these constants of integration, we need the two boundary conditions. In a similar manner when we integrate the time part we will again generate two constants of integration which will be solved from the two initial conditions that are provided.

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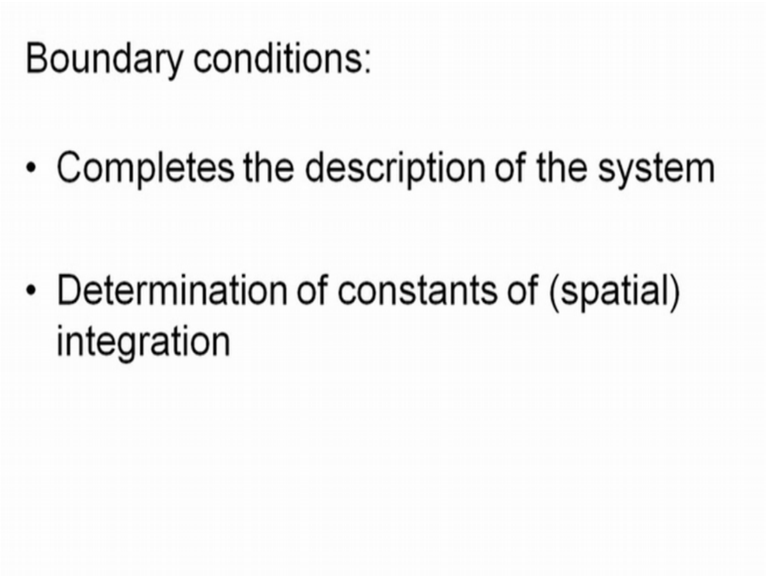

So the boundary condition, therefore, complete the description of the system and they are used for the determination of the constants of this spatial integration.

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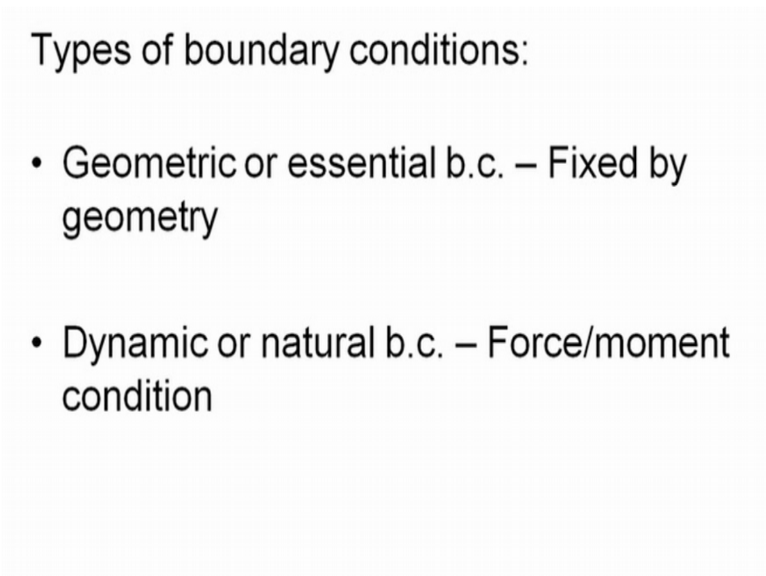

Now this boundary conditions are of two types, the first type is known as the geometric of essential boundary condition, such boundary conditions are fixed by geometry of the problem. The second is the dynamic or natural boundary condition which comes because of some condition on the force or moment, mostly in a string it will be force. So the dynamic boundary condition or natural boundary condition is a result of some force condition on the string. So let us look at some examples.

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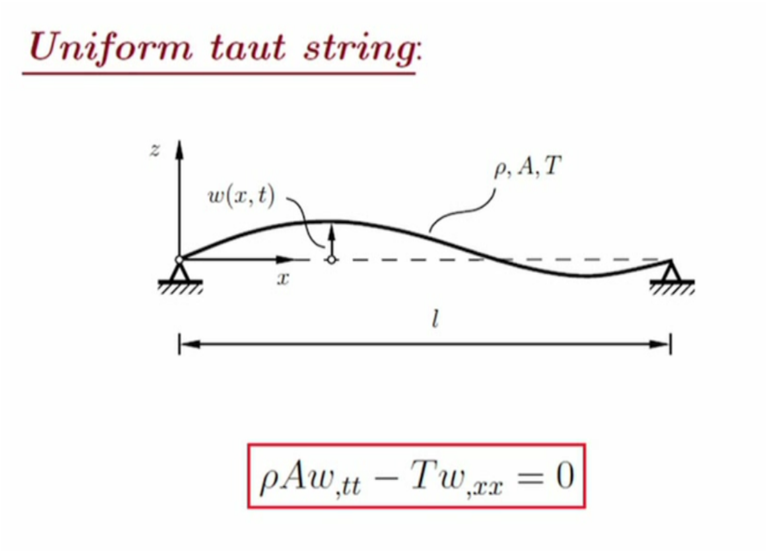

So this shows the taut string which I drew before. Now immediately, so this is a uniform taut string.

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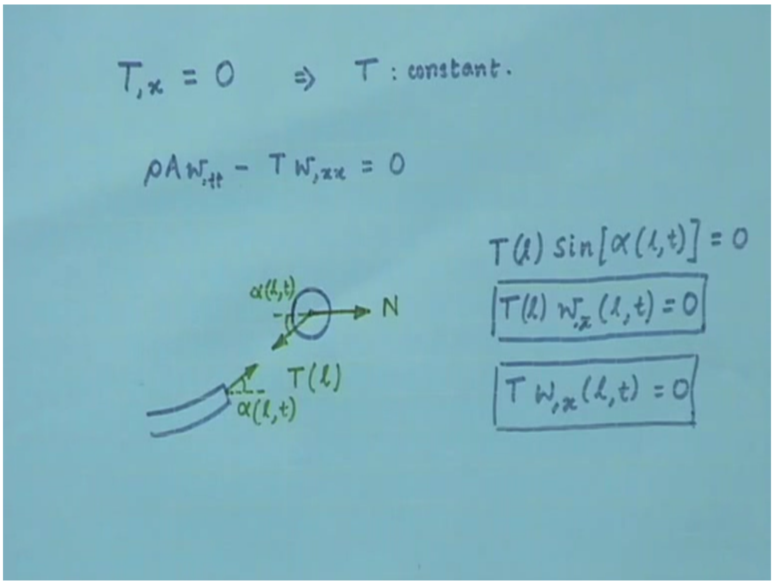

So you can immediately see that the partial derivative of the tension in the string is zero. So there is no external force in the longitudinal direction of the string. So therefore this is our equation for the tension which implies that tension is a constant. So if the tension is a constant then our equation of motion reduced to this. Here we consider that there is no force even in the transverse direction.

So we see in the slide that the equation of motion is given by this box, equation so, this is the equation for a uniform taut string with no external forces.

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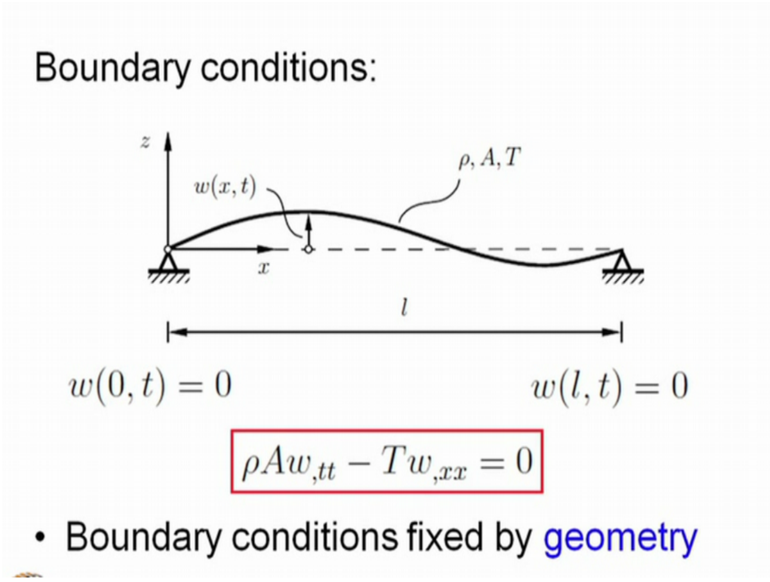

Now the boundary conditions, now it is very simple, from geometry you can easily identify that at the left boundary and right boundary but both the boundaries the displacement of the string must be zero. So these boundary conditions are set by the geometry of the problem. So the geometry of the problem says that the displacement, the transverse displacement of the string at the two boundaries must be zero, so these are the geometric boundary conditions for the string.

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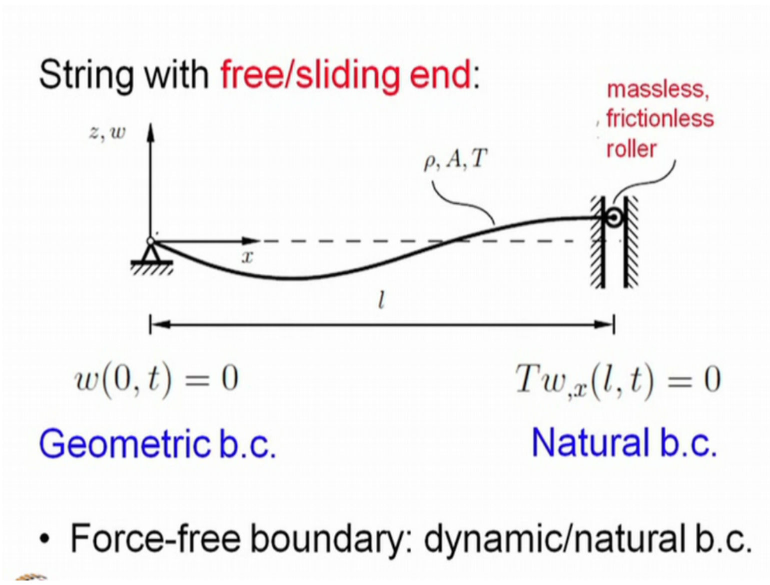

Next we are going to look at a string with a free or sliding end. So here as you can see the equation of motion, so the string is uniform, there are no external forces. So the equation of motion will remain the same as we have discussed before. But the boundary condition at the right boundary is what we want to see. So at the left boundary we again have a geometric boundary condition which the displacement is zero at x equals to zero.

At the right boundary, so to understand the boundary condition of the right boundary let us look at the right boundary, in a little detail. So this is the mass less, friction less pulley to which the string is attached. So now I will draw the free body diagrams of this connection. So at the pulley you have one normal force from the guide, a frictionless guide, so there is only one normal force.

And from the string, we have this tension at x equal to l. So if alpha is the angle, at any instance at this end of the string then we can see that from the free body diagram or the pulley we can write down the equation of equilibrium for this pulley. So if I write out the equilibrium in the transverse direction. So if I write out the equation of equilibrium in the transverse direction for this pulley.

This pulley is massless and frictionless. So we have, this stands out to be zero. So what this essentially tells us is that the force in the transverse direction on the pulley has to be zero. Now if I use the approximation that we have discussed, I obtained this condition. Now it so happens that the tension in this string is uniform, so this can be written even further in this way. So this becomes our boundary condition at the right end.

So this condition comes from a force condition that the force in the transverse direction on the pulley must vanish. And if you take the force balance in the longitudinal direction then you will be able solve for this normal reaction on the wall. So the boundary condition as we can now see on the slide, is given by tension times del w del x at x equal to l and at all time t must vanish.

Such a condition, such a boundary condition is known as a natural boundary condition. So this comes from a forced condition. So the forced free condition, forced free boundary gives us a dynamic or a natural boundary condition.

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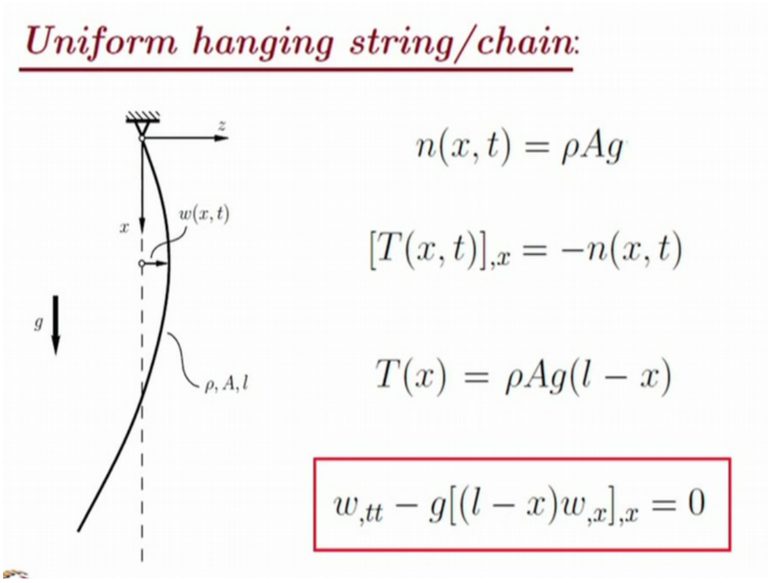

Now let us look at a uniform hanging string or a chain as we have discussed a chain qualifies to be analyzed like a string. So the equation of motion for this chain can be derived like this.

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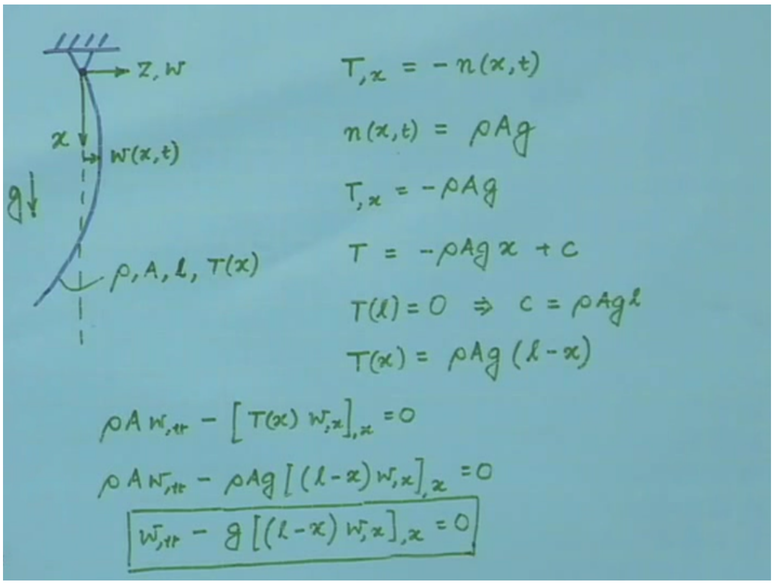

So here I show a hanging chain which is made of a material of density rho, its area of cross section maybe assumed to be uniform for simplicity and its length is l. Now as you can realize this chain will be under varying tension. So the tension in the chain will be a function of the position coordinate x. So this is what we have to now determine. So we already know that this is our equation for the in the longitudinal direction, the force balanced in the longitudinal direction.

So this is force per unit length and n is the force per unit length in the longitudinal direction. So if I express this, so if the density, since the density of the chain is rho and area of cross section assumed to be uniform is A, so this is mass per unit length. So rho A is mass per unit length and if I multiply by the acceleration due to gravity we will assume that this an uniform gravitational field.

Then the transverse force distribution or the force per unit length of the chain is rho Ag. This is weight per unit length, rho A is the mass per unit length, g is the acceleration, so it is weight per unit length, so which means it is force per unit length in the longitudinal direction, vertical direction in this case. So if I substitute this, and integrate out, so that is what I have, the tension as a function of the coordinate.

Now I will use a boundary condition a condition at one of the ends of this chains. So it is convenient to see that to use this condition that the tension at x equal to l is zero, so that gives us c which is, and if I substitute back, this is the tension at any point x in the chain. So now I will substitute this expression in the equation of motion of the transverse dynamics which reads to obtain and which can be simplified further.

So this is the equation of motion of transverse dynamics of a hanging chain in a uniform gravitational field. Now again we will need boundary conditions to complete the description of the problem. So let us look at the boundary conditions for this chain.

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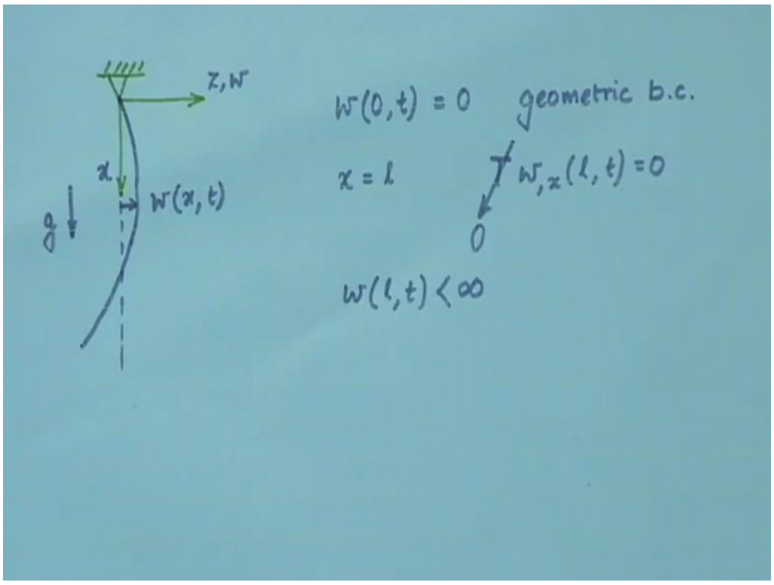

So at x equals to zero, as you can see, so we have a geometric boundary condition. Now what happens at x equals to l. So what is the boundary condition at x equal to l. Now this is a free end of the string and this is free to swing. So one may write in a manner similar to what we have discussed before, so this was the force free condition. So no force in the transverse direction. But then remember this tension at this free end is also zero.

So the tension is also zero. So if you imagine the last particle of this chain, it has almost no restoring force because remember that in the string the restoring force comes because of this tension in the string. Now this free end of the string, the tension goes to zero, so the last particle of the string is, I mean hardly has a restoring force in the transverse direction.

So there is a possibility at least theoretically that this displacement might become infinity because it does not have any restoring force. I mean, the displacement can become very large. But that definitely we know, we have seen a hanging chain or vibrating chain and it does not go to infinity. It remains finite. So from the physical consideration we must have a finite solution at this end. So we write this in this form. So this is an inequality.

So what this says is that the displacement of the string at the free end must be finite. Now this, when we discuss the solution of the vibration problem of the hanging chain, the solution of the equation of motion of a hanging chain, we will see that there is a solution which has an infinity. So what this condition will tell us is that solution should not be present because from the physical considerations we must have a finite displacement of this free end.

So this we will elaborate or discuss in detail when we discuss the solution of vibration of a hanging chain. So let us see what we have discussed in this lecture. So we have started with the motion of modeling the equations of motions of a string, the transverse as well as the longitudinal. So in the longitudinal direction it is a force balance because we have assumed that the motion in the longitudinal direction is negligible.

Then we have derived the equation of motion in the transverse direction of the string assuming it to be planar. Then, we have looked at the boundary conditions that come up in vibrations of, transverse vibrations of strings. There are two kinds of boundary conditions as we have seen. The first one is the geometric boundary condition and second is known as the natural boundary condition or the dynamic boundary condition.

And then we have seen a few examples of strengths of thought strengths and we have looked at the dynamics of a hanging chain and in each of this cases we have looked at the boundary conditions that govern the equations. The total description of the system.

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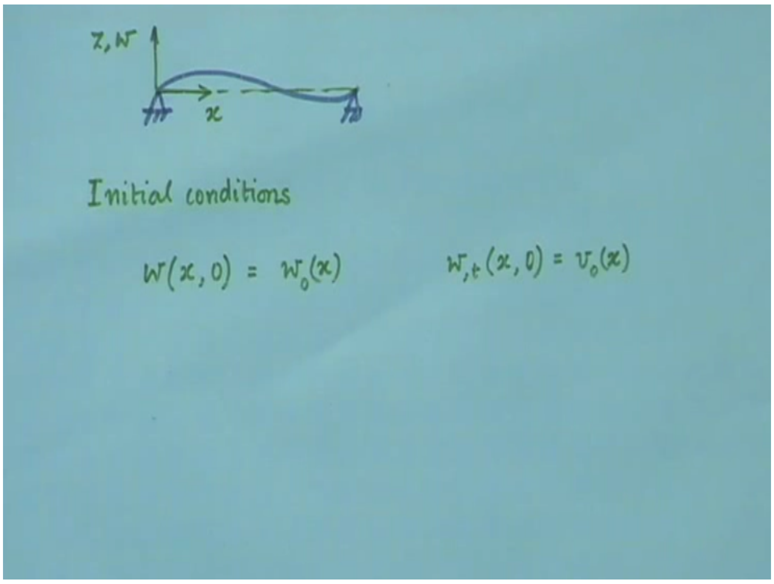

Now along with the boundary conditions we also need the initial conditions which we have not discussed as yet. So as I mentioned that we need two initial conditions. So these initial conditions are usually specified as the initial deformation of the string and the initial velocity distribution over the string. So with these two initial conditions, one on the displacement, other on the velocity.

We can now completely solve or uniquely solve the equation of motion of a vibrating string. So with this we complete this part on the transverse vibration of strings.